EINSTEIN’S CONNECTION IN 5-DIMENSIONAL
ES-MANIFOLD

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Abstract. The manifold $^{*}g - ESX_{n}$ is a generalized $n$-dimensional
Riemannian manifold on which the differential geometric structure is
imposed by the unified field tensor $^{*}g^{\lambda\nu}$ through the $ES$-connection
which is both Einstein and semi-symmetric. The purpose of the
present paper is to prove a necessary and sufficient condition for a
unique Einstein’s connection to exist in 5-dimensional $^{*}g - ESX_{5}$
and to display a surveyable tensorial representation of 5-dimensional
Einstein’s connection in terms of the unified field tensor, employing
the powerful recurrence relations in the first class.

1. Preliminaries

This paper is a direct continuation of our previous paper [1], which
will be denoted by $I$ in the present paper. All considerations in this
paper are based on the results and symbolism of $I$. Whenever necessary,
they will be quoted in the present paper. In this section, we introduce a
brief collection of basic concepts, notations, and results of $I$, which are
frequently used in the present paper([2],[3],[4]).

(a) $n$-dimensional $^{*}g$-unified field theory
Let \( X_n \) be an \( n \)-dimensional generalized Riemannian manifold referred to a real coordinate system \( x^\nu \), which obeys the coordinate transformations \( x^\nu \rightarrow x'^\nu \) for which

\[
det\left( \frac{\partial x'}{\partial x} \right) \neq 0
\]

In \( n - g - UFT \) the manifold \( X_n \) is endowed with a real nonsymmetric tensor \( g_{\lambda\mu} \), which may be decomposed into its symmetric part \( h_{\lambda\mu} \) and skew-symmetric part \( k_{\lambda\mu} \):

\[
g_{\lambda\mu} = h_{\lambda\mu} + k_{\lambda\mu}
\]

where

\[
g = det(g_{\lambda\mu}) \neq 0, \quad h = det(h_{\lambda\mu}) \neq 0, \quad k = det(k_{\lambda\mu})
\]

In \( n - *g - UFT \) the algebraic structure on \( X_n \) is imposed by the basic real tensor \( *g^{\lambda\nu} \) defined by

\[
g_{\lambda\mu} * g^{\lambda\nu} = g_{\mu\lambda} * g^{\nu\lambda} = \delta_\nu^\mu
\]

It may be also decomposed into its symmetric part \( *h^{\lambda\nu} \) and skew-symmetric part \( *k^{\lambda\nu} \):

\[
*g^{\lambda\nu} = *h^{\lambda\nu} + *k^{\lambda\nu}
\]

Since \( det(*h^{\lambda\nu}) \neq 0 \), we may define a unique tensor \( *h_{\lambda\mu} \) by

\[
*h_{\lambda\mu} * h^{\lambda\nu} = \delta_\nu^\mu
\]

In \( n - *g-UFT \) we use both \( *h^{\lambda\nu} \) and \( *h_{\lambda\mu} \) as tensors for raising and/or lowering indices of all tensors in \( X_n \) in the usual manner. We then have

\[
*k_{\lambda\mu} = *k^{\rho\sigma} h_{\lambda\rho} * h_{\mu\sigma}, \quad g_{\lambda\mu} = *g^{\rho\sigma} h_{\lambda\rho} * h_{\mu\sigma}
\]

so that

\[
*g_{\lambda\mu} = *h_{\lambda\mu} + *k_{\lambda\mu}
\]

The differential geometric structure on \( X_n \) is imposed by the tensor \( *g^{\lambda\nu} \) by means of a connection \( \Gamma_{\lambda\nu}^\mu \) defined by a system of equations

\[
D_\omega * g^{\lambda\nu} = -2S_{\omega\alpha}^{\nu} * g^{\lambda\alpha}
\]

where \( D_\omega \) denotes the symbol of the covariant derivative with respect to \( \Gamma_{\lambda\nu}^\mu \) and \( S_{\lambda\mu}^{\nu} \) is the torsion tensor of \( \Gamma_{\lambda\nu}^\mu \). Under certain conditions the system (1.9) admits a unique solutions \( \Gamma_{\lambda\nu}^\mu \).
It has been shown in [5] that if the system (1.9) admits $\Gamma^\nu_{\lambda\mu}$, it must be of the form

$$\Gamma^\nu_{\lambda\mu} = \star \left\{ \frac{\nu}{\lambda\mu} \right\} + U^\nu_{\lambda\mu} + S_{\lambda\mu}^\nu. \quad (1.10)$$

where

$$U_{\nu\lambda\mu} = \frac{100}{S_{(\lambda\mu)\nu} + 2 S_{\nu(\lambda\mu)}}. \quad (1.11)$$

(b) Some notations and results

The following quantities are frequently used in our further considerations:

$$\star g = \det(\star g_{\lambda\mu}), \quad \star h = \det(\star h_{\lambda\mu}), \quad \star k = \det(\star k_{\lambda\mu}) \quad (1.12)$$

$$\star g = \star g \star h, \quad \star k = \star k \star h. \quad (1.13)$$

$$K_p = \star k_{\alpha_1}^{\alpha_2} \cdots \star k_{\alpha_p}^{\alpha_1}, \quad (p = 0, 1, 2, \cdots). \quad (1.14)$$

$$\star k^\nu_{\lambda} = \delta^\nu_{\lambda}, \quad (p)^*k^\nu_{\lambda} = \star k^{\alpha}_{\lambda} (p-1)^*k^{\nu}_{\alpha} \quad (p = 1, 2, \cdots). \quad (1.15)$$

$$K_{\omega\mu\nu} = \nabla^\nu \star k_{\omega\mu} + \nabla^\omega \star k_{\nu\mu} + \nabla^\mu \star k_{\omega\nu} \quad (1.16)$$

where $\nabla_\omega$ is the symbolic vector of the covariant derivative with respect to the christoffel symbols $\star \left\{ \frac{\nu}{\lambda\mu} \right\}$ defined by $\star h_{\lambda\mu}$ in the usual way.

In $X_n$ it was proved in [5] that

$$K_0 = 1, \quad K_n = \star k \text{ if } n \text{ is even, and } K_n = 0 \text{ if } n \text{ is odd}. \quad (1.17)$$

$$\star g = 1 + K_2 + \cdots + K_{n-\sigma}. \quad (1.18)$$

$$\sum_{s=0}^{n-\sigma} K_s (n-s)^*k_{\lambda}^\nu = 0 \quad (p = 0, 1, 2, \cdots). \quad (1.19)$$

We also use the following useful abbreviations, denoting an arbitrary tensor $T_{\omega\mu\nu}$ skew-symmetric in the first two indices by $T$:

$$\frac{pqr}{T} = T_{\omega\mu\lambda} = T_{\alpha\beta\gamma} (p)^*k_{\omega}^{\alpha(q)} k_{\mu}^{\beta(r)} k_{\lambda}^{\gamma}. \quad (1.20)$$
and for an arbitrary tensor $T_{\cdots}$ for $p = 1, 2, 3, \cdots$:

\begin{equation}
(p)T_{\cdots \alpha \cdots} = (p-1) \ast k_{\nu \alpha} T_{\cdots \nu \cdots}.
\end{equation}

On the other hand, it has shown in [6] that the tensor $S_{\lambda \mu \nu}$ satisfies

\begin{equation}
S = B - 3 S
\end{equation}

where

\begin{equation}
2B_{\omega \mu \nu} = K_{\omega \mu \nu} + 3K_{\alpha [\mu \beta} \ast k_{\omega] \alpha} \ast k_{\nu \beta}
\end{equation}

In our subsequent chapter, we start with the relation (1.22) to solve the system (1.9). Furthermore, for the first class, the nonholonomic solution of (1.22) may be given by

\begin{equation}
MS_{xyz} = B_{xyz}
\end{equation}

or equivalently

\begin{equation}
4MS_{xyz} = (2 + MM + MM)K_{xyz} + M(M + M)K_{zxy} + M(M + M)K_{yzx}
\end{equation}

where

\begin{equation}
M_{xyz} = 1 + MM + MM + MM
\end{equation}

Therefore, in virtue of (1.24), we see that a necessary and sufficient condition for the system (1.9) to have a unique solution in the first class is

\begin{equation}
M_{xyz} \neq 0 \text{ for all } x, y, z
\end{equation}

\textbf{Definition 1.1.} A connection $\Gamma_{\lambda \nu}^{\mu}$ is said to be \textit{semi-symmetric} if its torsion tensor $S_{\lambda \mu \nu}$ is of the form

\begin{equation}
S_{\lambda \mu \nu} = 2\delta_{(\lambda}^{\nu} X_{\mu]}.
\end{equation}

for an arbitrary non-null vector $X_{\mu}$. 
A connection which is both semi-symmetric and Einstein is called an ES connection. An \( n \)-dimensional generalized Riemannian manifold \( X_n \), on which the differential geometric structure is imposed by \( *g^{\mu\nu} \) by means of an ES connection, is called an \( n \)-dimensional \( *g – ES \) manifold. We denote this manifold by \( *g – ESX_n \) in our further considerations.

In \( *g – ESX_5 \), the following theorems were proved in \( I \).

**Theorem 1.2.** The basic scalars in \( *g – ESX_5 \) may be given by

\[
\begin{align*}
M_1 &= -M_2 = \sqrt{-L - K} \neq 0 \\
M_3 &= -M_4 = \sqrt{L - K} \neq 0, \quad M_5 = 0
\end{align*}
\]

where

\[
K = \frac{K_2}{2}, \quad L = \sqrt{(\frac{K_2}{2})^2 - K_4}
\]

**Theorem 1.3.** The main recurrence relation in the first class is

\[
(p+5)k_\lambda^\nu = -K_2(p+3)k_\lambda^\nu - K_4(p+1)k_\lambda^\nu, \quad (p = 0, 1, 2, \ldots)
\]

**Theorem 1.4.** The basic scalars \( M \) satisfy

\[
\begin{align*}
M_1 + M_2 &= M_3 + M_4 = 0 \\
M_1M_2M_3M_4 &= M_3M_4 = M_5M = 0 \\
M_1M_2M_3M_4 &= M_1M_2M_3M_4 = K_4 \\
M_1^2 + M_2^2 &= M_3^2 + M_4^2 = M_2^2 + M_4^2 = M_2^2 + M_4^2 = -K_2
\end{align*}
\]

In virtue of the above theorem, we have

**Theorem 1.5.** In the first class, the following identities hold for all values of \( x \) and \( y \) when \( x \neq y \)

\[
\begin{align*}
M_x^{(4)M^{1}} &= -M_x^{(3)M^{2}} - K_2M_x^{(2)M^{1}} \\
M_x^{(4)M^{3}} &= K_4M_x^{(2)M^{1}}
\end{align*}
\]
\[
M^4 M^1 = K_4^2 M^2 M^2 + K_2 M^3 M^3 + 2 K_4 M^3 M^1
\]
\[
2 M^4 M^2 = - M^3 M^3 - K_2 M^2 M^2 + K_4 M^3 M^1
\]

**Theorem 1.6.** (Recurrence relations in the first class) If \( T_{\omega \mu \nu} \) is a tensor skew-symmetric in the first two indices, then the following recurrence relations hold in the first class of \( 5 - g - ESX_5 \):

\[
(1.38) \quad (41)^r T = - T - K_2 T
\]
\[
(1.39) \quad (32)^r T = K_4 T
\]
\[
(1.40) \quad (41)^r T = - T - K_2 T
\]
\[
(1.41) \quad (33)^r T = K_4 T
\]
\[
(1.42) \quad 2 T = - T - K_2 T + K_4 T
\]

2. Einstein’s connection \( \Gamma^\nu_\lambda^\mu \) in the first class

In this section, we shall derive surveyable representations of \( \Gamma^\nu_\lambda^\mu \) in terms of \( *g^{\lambda \nu} \), employing the recurrence relations.

In the following theorem, we shall prove two relations in \( X_n \). These relations will be used in our subsequent theorem when we are concerned with the solution of (1.9).

**Theorem 2.1.** We have

\[
\begin{align*}
\frac{(pq)^r}{2 B_{\omega \mu \nu}} &= (p^q)^r \left( \frac{K_{\omega \mu \nu}}{\omega \mu \nu} + \frac{r'}{(pq)^r} \frac{(p^q)^r}{\omega \mu \nu} + \frac{r''}{(pq)^r} \frac{(p^q)^r}{\omega \mu \nu} \right) \\
&+ \frac{1}{2} \left( \frac{(pq)^r}{\omega \mu \nu} + \frac{(p^q)^r}{\omega \mu \nu} + \frac{r'}{(pq)^r} \frac{(p^q)^r}{\omega \mu \nu} + \frac{r''}{(pq)^r} \frac{(p^q)^r}{\omega \mu \nu} \right)
\end{align*}
\]

where

\[
p' = p + 1, \quad q' = q + 1, \quad r' = r + 1, \quad r'' = r + 2
\]
\textbf{Proof.} In virtue of (1.22) and (1.20), the first relation (2.1) is obtained as in the following way:

\begin{equation}
\frac{(pq)^r}{B} = B_{\omega \mu \nu} = \frac{1}{2} B_{\omega \beta \gamma} ((p)^*k_\omega \alpha(q)^*k_\mu \beta + (q)^*k_\omega \alpha(p)^*k_\mu \beta)^{(r)} k_\nu \gamma
\end{equation}

\begin{equation}
= \frac{1}{2} (S_{\alpha \beta \gamma} + S_{\epsilon \eta \gamma} k_\alpha \epsilon \eta \beta \eta + S_{\alpha \beta \eta} k_\alpha \epsilon \beta \eta \gamma + S_{\alpha \epsilon \eta} k_\beta \epsilon \beta \eta \gamma) \times (p)^*k_\omega \alpha(q)^*k_\mu \beta + (q)^*k_\omega \alpha(p)^*k_\mu \beta)^{(r)} k_\nu \gamma
\end{equation}

(2.4)

After a lengthy calculation, we note that the right-hand side of the above equation is equal to that of (2.1). Similarly, we verify (2.2) using (1.20) and (1.23). \qed

\textbf{Theorem 2.2.} A necessary and sufficient condition for the system (1.9) to admit a unique solution $\Gamma^{\nu}_{\mu}$ in the first class is that

\begin{equation}
g_{AB}(C^2 - 4K_4D^2) \neq 0
\end{equation}

where

\begin{equation}
A = 1 - K_2 + K_4, \\
B = 1 - K_4, \\
C = 1 - K_2 + 5K_4, \\
D = K_2 - 2
\end{equation}

(2.6)

\textbf{Proof.} In virtue of (1.29) and (1.30), the symmetric scalars $M_{xyz}$ by (1.26) takes values as in the following 3 cases:

If two of the indices $x, y, z$ are 1, 2 or 3, 4, then

\begin{equation}
M_{xyz} = 1 + K + L, \quad 1 + K - L
\end{equation}

(2.7)

If at least one of $x, y, z$ is 5 and no two take the values 1, 2 nor 3, 4, then

\begin{equation}
M_{xyz} = 1 - K + L, \quad 1 - K - L, \quad 1 + \sqrt{K_4}, \quad 1 - \sqrt{K_4}, \quad 1
\end{equation}

(2.8)

In the remaining cases,

\begin{equation}
M_{xyz} = 1 - K - L - 2\sqrt{K_4}, \quad 1 - K + L - 2\sqrt{K_4}
\end{equation}

\begin{equation}
1 - K - L + 2\sqrt{K_4}, \quad 1 - K + L + 2\sqrt{K_4}
\end{equation}

(2.9)

It may easily verified that the product of two factors in the right of (2.7) is $g$, that of five factors in the right of (2.8) is $(1 - K_2 + K_4)(1 - K_4)$, and that of four factors in the right of (2.9) is $(1 - K_2 + 5K_4)^2 - 4K_4(K_2 - 2)^2$ . Hence we have proved our assertion (2.5) in virtue of (1.27) and (2.6). \qed
Theorem 2.3. The system of equations (1.22) in the first class is reduced to the following 25 equations:

\[(2.10)\]

\[
\begin{align*}
B &= S + S + 2S \\
(10)^1 & (10)^1 (21)^1 (20)^2 (20)^2 112 \\
B &= S + S + S + S \\
(12)^1 & (12)^1 (23)^1 222 (13)^2 \\
B &= S + S + S + S \\
(20)^2 & (20)^2 (31)^2 (30)^3 (21)^3 \\
B &= S + S + S + S \\
(23)^1 & (23)^1 (21)^1 332 112 222 (13)^2 \\
B &= S - K_2 S - S + S + S \\
(30)^3 & (30)^3 (21)^3 (32)^3 (40)^4 (31)^4 \\
B &= S - K_2 S - S + S + S \\
(21)^3 & (21)^3 (32)^3 (31)^4 224 \\
B &= S - K_2 S - S + S + S \\
(32)^1 & (32)^1 (21)^1 114 224 334 \\
B &= S - K_2 S - K_4 S - K_2 S + S \\
(40)^4 & (40)^4 (31)^4 114 (30)^3 (10)^3 (30)^1 \\
B &= S - K_2 S - K_4 S + K_2 S + K_2 K_4 S + K_4 S + S + K_4 S + K_2 K_4 S + K_4 S + S + S \\
(10)^1 & (21)^3 (21)^1 (32)^3 (32)^1 \\
B &= S - K_2 S - K_4 S + K_2 S + K_2 K_4 S + K_4 S + S + S + S + S \\
(31)^4 & (31)^4 (21)^3 (21)^1 114 224 334 \\
B &= S - K_2 S - K_4 S - K_2 S + K_2 S + K_4 S - K_2 S - S \\
(10)^3 & (10)^3 (21)^3 (20)^4 114 \\
B &= S + S + S + S \\
(30)^1 & (30)^1 (21)^1 (32)^1 (40)^2 (31)^2 \\
B &= S - K_2 S - S + S + S \\
(20)^4 & (20)^4 (31)^4 (30)^3 (30)^1 (21)^3 (21)^1 \\
B &= S - S - K_2 S - K_4 S - K_2 S + K_4 S - K_2 S - K_4 S - K_2 S - S \\
(40)^2 & (40)^2 (31)^2 112 (30)^3 (10)^3 (21)^3 (32)^3 \\
B &= S - K_2 S - K_4 S - K_2 S + K_4 S - K_2 S - K_4 S - K_2 S - S \\
(110) & 110 220 (21)^1 \\
B &= S + S + 2S \\
112 & 112 222 (21)^3 (32)^2 222 332 (32)^3 \\
B &= S + S + 2S B = S + S + 2S \\
332 & 332 222 (21)^3 (31)^2 \\
B &= (1 + K_2) S + K_4 S + 2K_4 S + 2K_4 S \\
224 & 224 334 (32)^3 (32)^1 \\
B &= S + 2K_2 S - 2K_4 S \\
\end{align*}
\]
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\[
\begin{align*}
B &= S + S - 2K_2 S - 2K_4 S \\
\bar{B} &= (1 + K_2) S + K_4 S + 2K_4 S - 2K_2 K_4 S - 2K_2^2 S \\
\hat{B} &= (1 + K_2) S + K_4 S + 2K_4 S - 2K_2 K_4 S - 2K_2^2 S \\
\tilde{B} &= 2 S + K_4 S - K_2 S - S - 2K_2 S
\end{align*}
\]

Proof. This assertion follows from (2.1) using (1.31) and (1.40)-(1.43).

References


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