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ON SOME INEQUALITIES FOR NUMERICAL RADIUS OF OPERATORS IN HILBERT SPACES

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Abstract. By the use of inequalities for nonnegative Hermitian forms some new inequalities for numerical radius of bounded linear operators in complex Hilbert spaces are established.

1. Introduction

Let $\mathbb{K}$ be the field of real or complex numbers, i.e., $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ and X be a linear space over $\mathbb{K}$.

Definition 1. A functional $(\cdot,\cdot) : X\times X\to\mathbb{K}$ is said to be a Hermitian form on X if

(H1) $(ax+by,z)=a(x,z)+b(y,z)$ for $a,b\in\mathbb{K}$ and $x,y,z\in X$;
(H2) $(x,y)=\overline{(y,x)}$ for all $x,y\in X$.

The functional $(\cdot,\cdot)$ is said to be *positive semi-definite* on a subspace Y of X if

(H3) $(y,y)\geq 0$ for every $y\in Y$,

and *positive definite* on Y if it is positive semi-definite on Y and

(H4) $(y,y)=0$, $y\in Y$ implies $y=0$.

The functional $(\cdot,\cdot)$ is said to be *definite* on Y provided that either $(\cdot,\cdot)$ or $-(\cdot,\cdot)$ is positive semi-definite on Y.

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When a Hermitian functional $(\cdot,\cdot)$ is positive-definite on the whole space X, then, as usual, we will call it an *inner product* on X and will denote it by $\langle\cdot,\cdot\rangle$.

We use the following notations related to a given Hermitian form $(\cdot,\cdot)$ on X :

$$X_0 := \{x \in X | \, (x,x) = 0\}, \; K := \{x \in X | \, (x,x) < 0\}$$

and, for a given $z \in X$,

$$X^{(z)} := \{x \in X | \, (x,z) = 0\} \quad \text{and} \quad L(z) := \{az | a \in \mathbb{K}\}.$$

The following fundamental facts concerning Hermitian forms hold:

Theorem 1 (Kurepa, 1968 [28]). *Let X and $(\cdot,\cdot)$ be as above.*

1. *If $e \in X$ is such that $(e,e) \neq 0$, then we have the decomposition*

$$X = L(e) \bigoplus X^{(e)}, \tag{1.1}$$

   where $\bigoplus$ denotes the direct sum of the linear subspaces $X^{(e)}$ and $L(e)$;
2. *If the functional $(\cdot,\cdot)$ is positive semi-definite on $X^{(e)}$ for at least one $e \in K$, then $(\cdot,\cdot)$ is positive semi-definite on $X^{(f)}$ for each $f \in K$;*
3. *The functional $(\cdot,\cdot)$ is positive semi-definite on $X^{(e)}$ with $e \in K$ if and only if the inequality*

$$|(x,y)|^2 \geq (x,x)(y,y) \tag{1.2}$$

   holds for all $x \in K$ and all $y \in X$;
4. *The functional $(\cdot,\cdot)$ is semi-definite on X if and only if the Schwarz's inequality*

$$|(x,y)|^2 \leq (x,x)(y,y) \tag{1.3}$$

   holds for all $x, y \in X$;
5. *The case of equality holds in (1.3) for $x, y \in X$ and in (1.2), for $x \in K$, $y \in X$, respectively; if and only if there exists a scalar $a \in \mathbb{K}$ such that*

$$y - ax \in X_0^{(x)} := X_0 \cap X^{(x)}.$$

Let X be a linear space over the real or complex number field $\mathbb{K}$ and let us denote by $\mathcal{H}(X)$ the class of all positive semi-definite Hermitian forms on X, or, for simplicity, *nonnegative* forms on X.

If $(\cdot,\cdot) \in \mathcal{H}(X)$, then the functional $\|\cdot\| = (\cdot,\cdot)^{\frac{1}{2}}$ is a *semi-norm* on X and the following equivalent versions of Schwarz's inequality hold:

$$\|x\|^2 \|y\|^2 \geq |(x,y)|^2 \quad \text{or} \quad \|x\| \|y\| \geq |(x,y)| \tag{1.4}$$

for any $x, y \in X$.

Now, let us observe that $\mathcal{H}(X)$ is a *convex cone* in the linear space of all mappings defined on X^2 with values in $\mathbb{K}$, i.e.,

(e) $(\cdot,\cdot)_1, (\cdot,\cdot)_2 \in \mathcal{H}(X)$ implies that $(\cdot,\cdot)_1 + (\cdot,\cdot)_2 \in \mathcal{H}(X)$;
(ee) $\alpha \geq 0$ and $(\cdot,\cdot) \in \mathcal{H}(X)$ implies that $\alpha(\cdot,\cdot) \in \mathcal{H}(X)$.

The following simple result is of interest in itself as well:

LEMMA 1. *Let X be a linear space over the real or complex number field $\mathbb{K}$ and $(\cdot,\cdot)$ a nonnegative Hermitian form on X. If $y \in X$ is such that $(y,y) \neq 0$, then*

$$p_y : H \times H \to \mathbb{K},\; p_y(x,z) = (x,z)\|y\|^2 - (x,y)(y,z) \tag{1.5}$$

is also a nonnegative Hermitian form on X.

We have the inequalities

$$\begin{aligned}&\left(\|x\|^2 \|y\|^2 - |(x,y)|^2\right)\left(\|y\|^2 \|z\|^2 - |(y,z)|^2\right)\\ &\geq \left|(x,z)\|y\|^2 - (x,y)(y,z)\right|^2\end{aligned} \tag{1.6}$$

and

$$\begin{aligned}&\left(\|x+z\|^2 \|y\|^2 - |(x+z,y)|^2\right)^{\frac{1}{2}}\\ &\leq \left(\|x\|^2 \|y\|^2 - |(x,y)|^2\right)^{\frac{1}{2}} + \left(\|y\|^2 \|z\|^2 - |(y,z)|^2\right)^{\frac{1}{2}}\end{aligned} \tag{1.7}$$

for any $x, y, z \in X$.

REMARK 1. *The case when $(\cdot,\cdot)$ is an inner product in Lemma 1 was obtained in 1985 by S. S. Dragomir, [2].*

REMARK 2. *Putting $z = \lambda y$ in (1.7), we get:*

$$0 \leq \|x+\lambda y\|^2 \|y\|^2 - |(x+\lambda y,y)|^2 \leq \|x\|^2 \|y\|^2 - |(x,y)|^2 \tag{1.8}$$

and, in particular,

$$0 \leq \|x \pm y\|^2 \|y\|^2 - |(x \pm y,y)|^2 \leq \|x\|^2 \|y\|^2 - |(x,y)|^2 \tag{1.9}$$

for every $x, y \in H$.

We note here that the inequality (1.8) is in fact equivalent to the following statement

$$\sup_{\lambda\in\mathbb{K}}\left[\|x+\lambda y\|^2\|y\|^2-|(x+\lambda y,y)|^2\right]=\|x\|^2\|y\|^2-|(x,y)|^2 \tag{1.10}$$

for each $x,y\in H$.

The following result holds (see [11, p. 38] for the case of inner product):

Theorem 2. *Let X be a linear space over the real or complex number field $\mathbb{K}$ and $(\cdot,\cdot)$ a nonnegative Hermitian form on X. For any $x,y,z\in X$, the following refinement of the Schwarz inequality holds:*

$$\begin{aligned}\|x\|\,\|z\|\,\|y\|^2 &\geq \left|(x,z)\,\|y\|^2-(x,y)\,(y,z)\right|+|(x,y)\,(y,z)| \\ &\geq |(x,z)|\,\|y\|^2 .\end{aligned} \tag{1.11}$$

Corollary 1. *For any $x,y,z\in X$ we have*

$$\frac{1}{2}\left[\|x\|\,\|z\|+|(x,z)|\right]\|y\|^2\geq|(x,y)\,(y,z)| . \tag{1.12}$$

The inequality (1.12) follows from the first inequality in (1.11) and the triangle inequality for modulus

$$\left|(x,z)\,\|y\|^2-(x,y)\,(y,z)\right|\geq|(x,y)\,(y,z)|-\|y\|^2\,|(x,z)|$$

for any $x,y,z\in X$.

Remark 3. *We observe that if $(\cdot,\cdot)$ is an inner product, then (1.12) reduces to Buzano's inequality obtained in 1974 [1] in a different way.*

For some inequalities in inner product spaces and operators on Hilbert spaces see [3]- [26] and the references therein.

The *numerical radius* $w(T)$ of an operator T on H is given by [27, p. 8]:

$$w(T)=\sup\{|\lambda|\,,\lambda\in W(T)\}=\sup\{|\langle Tx,x\rangle|\,,\|x\|=1\} . \tag{1.13}$$

It is well known that $w(\cdot)$ is a norm on the Banach algebra $\mathcal{B}(H)$ of all bounded linear operators $T:H\to H$. This norm is equivalent with the operator norm. In fact, the following more precise result holds [27, p. 9]:

Theorem 3 (Equivalent norm). *For any $T\in\mathcal{B}(H)$ one has*

$$w(T)\leq\|T\|\leq 2w(T) . \tag{1.14}$$

Utilising Buzano's inequality we obtained the following inequality for the numerical radius [12] or [13]:

THEOREM 4. *Let* $(H; \langle \cdot, \cdot \rangle)$ *be a Hilbert space and* $T : H \to H$ *a bounded linear operator on* H. *Then*

$$w^2(T) \le \frac{1}{2}\left[w\left(T^2\right) + \|T\|^2\right]. \tag{1.15}$$

The constant $\frac{1}{2}$ *is best possible in (1.15).*

The following general result for the product of two operators holds [27, p. 37]:

THEOREM 5. *If* U, V *are two bounded linear operators on the Hilbert space* $(H, \langle \cdot, \cdot \rangle)$, *then* $w(UV) \le 4w(U)w(V)$. *In the case that* $UV = VU$, *then* $w(UV) \le 2w(U)w(V)$. *The constant* 2 *is best possible here.*

The following results are also well known [27, p. 38].

THEOREM 6. *If* U *is a unitary operator that commutes with another operator* V, *then*

$$w(UV) \le w(V). \tag{1.16}$$

If U *is an isometry and* $UV = VU$, *then (1.16) also holds true.*

We say that U and V *double commute* if $UV = VU$ and $UV^* = V^*U$. The following result holds [27, p. 38].

THEOREM 7. *If the operators* U *and* V *double commute, then*

$$w(UV) \le w(V)\|U\|. \tag{1.17}$$

As a consequence of the above, we have [27, p. 39]:

COROLLARY 2. *Let* U *be a normal operator commuting with* V. *Then*

$$w(UV) \le w(U)w(V). \tag{1.18}$$

For a recent survey of inequalities for numerical radius, see [21] and the references therein.

Motivated by the above facts we establish in this paper some new numerical radius inequalities concerning four operators A, B, C and P on a Hilbert space with P nonnegative in the operator order. Some particular cases of interest that generalize and improve an earlier result are also provided.

2. Main Results

The following result holds for $(H, \langle ., . \rangle)$ a Hilbert space over the real or complex numbers field $\mathbb{K}$.

THEOREM 8. *Let P be a nonnegative operator on H and A, B, C three bounded operators on H. Then for any $e \in H$ we have the inequalities*

$$\|A^*PCe\| \|B^*PCe\| \leq \frac{1}{2} \left\|P^{1/2}Ce\right\|^2 \left[\left\|P^{1/2}A\right\| \left\|P^{1/2}B\right\| + \|B^*PA\|\right]. \tag{2.1}$$

Moreover, we have

$$w(C^*PAB^*PC) \leq \frac{1}{2} \left\|P^{1/2}C\right\|^2 \left[\left\|P^{1/2}A\right\| \left\|P^{1/2}B\right\| + \|B^*PA\|\right]. \tag{2.2}$$

Proof. We observe that if $P \geq 0$, then the mapping $(., .) : H \times H \to \mathbb{K}$ defined by

$$(x, y)_P := \langle Px, y \rangle$$

is a hermitian form on H and by (1.12) we have the inequality

$$\frac{1}{2} \left[\|x\|_P \|y\|_P + |(x, y)_P|\right] \|e\|_P^2 \geq |(x, e)_P (y, e)_P| \tag{2.3}$$

for any $x, y, e \in H$.

This can be written as

$$\frac{1}{2} \left[\langle Px, x \rangle^{1/2} \langle Py, y \rangle^{1/2} + |\langle Px, y \rangle|\right] \langle Pe, e \rangle \geq |\langle Px, e \rangle \langle Py, e \rangle| \tag{2.4}$$

for any $x, y, e \in H$.

Now if we replace x by Ax, y by By and e by Ce we get

$$\begin{aligned} &\frac{1}{2} \left[\langle PAx, Ax \rangle^{1/2} \langle PBy, By \rangle^{1/2} + |\langle PAx, By \rangle|\right] \langle PCe, Ce \rangle \\ &\geq |\langle PAx, Ce \rangle \langle PBy, Ce \rangle| \end{aligned} \tag{2.5}$$

for any $x, y, e \in H$, which is equivalent to

$$\begin{aligned} &\frac{1}{2} \left[\langle A^*PAx, x \rangle^{1/2} \langle B^*PBy, y \rangle^{1/2} + |\langle B^*PAx, y \rangle|\right] \langle C^*PCe, e \rangle \\ &\geq |\langle x, A^*PCe \rangle \langle y, B^*PCe \rangle| \end{aligned} \tag{2.6}$$

for any $x, y, e \in H$.

Taking the supremum over $x, y \in H$ with $\|x\| = \|y\| = 1$ we have

(2.7)

$$
\begin{aligned}
&\|A^* PCe\| \|B^* PCe\| \\
&= \sup_{\|x\|=1} |\langle x, A^* PCe\rangle| \sup_{\|y\|=1} |\langle y, B^* PCe\rangle| \\
&= \sup_{\|x\|=\|y\|=1} \{|\langle x, A^* PCe\rangle \langle y, B^* PCe\rangle|\} \\
&\le \frac{1}{2} \langle C^* PCe, e\rangle \\
&\times \sup_{\|x\|=\|y\|=1} \left[\langle A^* PAx, x\rangle^{1/2} \langle B^* PBy, y\rangle^{1/2} + |\langle B^* PAx, y\rangle|\right] \\
&\le \frac{1}{2} \langle C^* PCe, e\rangle \\
&\times \left[\sup_{\|x\|=1} \langle A^* PAx, x\rangle^{1/2} \sup_{\|y\|=1} \langle B^* PBy, y\rangle^{1/2} + \sup_{\|x\|=\|y\|=1} |\langle B^* PAx, y\rangle|\right] \\
&= \frac{1}{2} \langle C^* PCe, e\rangle \left[\|A^* PA\|^{1/2} \|B^* PB\|^{1/2} + \|B^* PA\|\right]
\end{aligned}
$$

for any $e \in H$.

Since

$$A^* PA = \left|P^{1/2} A\right|^2, \; B^* PB = \left|P^{1/2} B\right|^2$$

and

$$C^* PC = \left|P^{1/2} C\right|^2$$

then by (2.7) we get the desired inequality in (2.1).

By Schwarz inequality we have

$$|\langle C^* PBA^* PCe, e\rangle| \le \|A^* PCe\| \|B^* PCe\| \tag{2.8}$$

for any $e \in H$.

Using inequality (2.1) we then have

(2.9)

$$|\langle C^* PBA^* PCe, e\rangle| \le \frac{1}{2} \left\|P^{1/2} Ce\right\|^2 \left[\left\|P^{1/2} A\right\| \left\|P^{1/2} B\right\| + \|B^* PA\|\right]$$

for any $e \in H$.

Taking the supremum over $e \in H$, $\|e\| = 1$ in (2.9) we get

$$w\left(C^* PBA^* PC\right) \le \frac{1}{2} \left\|P^{1/2} C\right\|^2 \left[\left\|P^{1/2} A\right\| \left\|P^{1/2} B\right\| + \|B^* PA\|\right] \tag{2.10}$$

and since

$$w\left(C^{*}PBA^{*}PC\right)=w\left(C^{*}PAB^{*}PC\right)$$

then by (2.10) we get the desired result (2.2). □

The following result also holds.

THEOREM 9. *Let P be a nonnegative operator on H and A,B,C three bounded operators on H such that $B^{*}PC=C^{*}PA$, then*

$$w^{2}\left(C^{*}PA\right)\leq\frac{1}{2}\left\|P^{1/2}C\right\|^{2}\left[\left\|P^{1/2}A\right\|\left\|P^{1/2}B\right\|+w\left(B^{*}PA\right)\right] \tag{2.11}$$

and

$$w^{2}\left(C^{*}PA\right)\leq\frac{1}{2}\left\|P^{1/2}C\right\|^{2}\left[\left\|\frac{\left|P^{1/2}A\right|^{2}+\left|P^{1/2}B\right|^{2}}{2}\right\|+w\left(B^{*}PA\right)\right]. \tag{2.12}$$

Proof. From the inequality (2.6) we have

$$\begin{aligned}&\frac{1}{2}\left[\left\langle A^{*}PAe,e\right\rangle^{1/2}\left\langle B^{*}PBe,e\right\rangle^{1/2}+\left|\left\langle B^{*}PAe,e\right\rangle\right|\right]\left\langle C^{*}PCe,e\right\rangle\\&\geq\left|\left\langle e,A^{*}PCe\right\rangle\left\langle e,B^{*}PCe\right\rangle\right|\end{aligned} \tag{2.13}$$

for any $e\in H$.

Since

$$B^{*}PC=C^{*}PA=\left(A^{*}PC\right)^{*}$$

then

$$\begin{aligned}\left|\left\langle e,A^{*}PCe\right\rangle\left\langle e,B^{*}PCe\right\rangle\right|&=\left|\left\langle e,A^{*}PCe\right\rangle\left\langle e,\left(A^{*}PC\right)^{*}e\right\rangle\right|\\&=\left|\left\langle A^{*}PCe,e\right\rangle\right|^{2}=\left|\left\langle C^{*}PAe,e\right\rangle\right|^{2}\end{aligned} \tag{2.14}$$

for any $e\in H$.

By (2.13) and (2.14) we then have

$$\begin{aligned}&\left|\left\langle C^{*}PAe,e\right\rangle\right|^{2}\\&\leq\frac{1}{2}\left[\left\langle A^{*}PAe,e\right\rangle^{1/2}\left\langle B^{*}PBe,e\right\rangle^{1/2}+\left|\left\langle B^{*}PAe,e\right\rangle\right|\right]\left\langle C^{*}PCe,e\right\rangle\end{aligned} \tag{2.15}$$

for any $e\in H$. This inequality is of interest in itself.

Taking the supremum over $e \in H$, $\|e\| = 1$ in (2.15) we have

$$
\begin{aligned}
& w^2 (C^* PA) \\
& = \sup_{\|e\|=1} |\langle C^* PAe, e\rangle|^2 \\
& \leq \frac{1}{2} \sup_{\|e\|=1} \left\{ \left[\langle A^* PAe, e\rangle^{1/2} \langle B^* PBe, e\rangle^{1/2} + |\langle B^* PAe, e\rangle| \right] \langle C^* PCe, e\rangle \right\} \\
& \leq \frac{1}{2} \sup_{\|e\|=1} \left[\langle A^* PAe, e\rangle^{1/2} \langle B^* PBe, e\rangle^{1/2} + |\langle B^* PAe, e\rangle| \right] \sup_{\|e\|=1} \langle C^* PCe, e\rangle \\
& \leq \frac{1}{2} \left[\sup_{\|e\|=1} \langle A^* PAe, e\rangle^{1/2} \sup_{\|e\|=1} \langle B^* PBe, e\rangle^{1/2} + \sup_{\|e\|=1} |\langle B^* PAe, e\rangle| \right] \\
& \times \sup_{\|e\|=1} \langle C^* PCe, e\rangle \\
& = \frac{1}{2} \left[\|A^* PA\|^{1/2} \|B^* PB\|^{1/2} + w (B^* PA) \right] \|C^* PC\|,
\end{aligned}
$$

which proves the inequality (2.11).

Using the *arithmetic mean - geometric mean* inequality we also have

$$
\begin{aligned}
\langle A^* PAe, e\rangle^{1/2} \langle B^* PBe, e\rangle^{1/2} & \leq \frac{1}{2} [\langle A^* PAe, e\rangle + \langle B^* PBe, e\rangle] \\
& = \left\langle \frac{A^* PA + B^* PB}{2} e, e \right\rangle
\end{aligned}
$$

for any $e \in H$.

By (2.15) we then have

(2.16)

$$
|\langle C^* PAe, e\rangle|^2 \leq \frac{1}{2} \left[\left\langle \frac{A^* PA + B^* PB}{2} e, e \right\rangle + |\langle B^* PAe, e\rangle| \right] \langle C^* PCe, e\rangle
$$

for any $e \in H$.

Taking the supremum over $e \in H$, $\|e\| = 1$ in (2.16) we obtain the desired result (2.12). □

3. Some Particular Inequalities

In this section we explore some particular inequalities of interest that can be obtained from the main results stated above.

If we take in (2.1) and (2.2) $B = A^*$, then we get

$$\|A^*PCe\| \|APCe\| \le \frac{1}{2} \left\|P^{1/2}Ce\right\|^2 \left[\left\|P^{1/2}A\right\| \left\|AP^{1/2}\right\| + \|APA\|\right] \tag{3.1}$$

for any $e \in H$ and

$$w\left(C^*PA^2PC\right) \le \frac{1}{2} \left\|P^{1/2}C\right\|^2 \left[\left\|P^{1/2}A\right\| \left\|AP^{1/2}\right\| + \|APA\|\right], \tag{3.2}$$

where A, C are bounded operators on H and P is a nonnegative operator on H.

If we put in (2.1) and (2.2) $P = 1_H$, then we have

$$\|A^*Ce\| \|B^*Ce\| \le \frac{1}{2} \|Ce\|^2 \left[\|A\| \|B\| + \|B^*A\|\right] \tag{3.3}$$

for any $e \in H$ and

$$w\left(C^*AB^*C\right) \le \frac{1}{2} \|C\|^2 \left[\|A\| \|B\| + \|B^*A\|\right] \tag{3.4}$$

where A, B, C are bounded operators on H.

Choosing $B = A^*$ in (3.3) and (3.4), we get

$$\|A^*Ce\| \|ACe\| \le \frac{1}{2} \|Ce\|^2 \left[\|A\|^2 + \left\|A^2\right\|\right] \tag{3.5}$$

for any $e \in H$ and

$$w\left(C^*A^2C\right) \le \frac{1}{2} \|C\|^2 \left[\|A\|^2 + \left\|A^2\right\|\right]. \tag{3.6}$$

If we take in (2.1) and (2.2) $C = 1_H$, then we get

$$\|A^*Pe\| \|B^*Pe\| \le \frac{1}{2} \left\|P^{1/2}e\right\|^2 \left[\left\|P^{1/2}A\right\| \left\|P^{1/2}B\right\| + \|B^*PA\|\right] \tag{3.7}$$

for any $e \in H$ and

$$w\left(PAB^*P\right) \le \frac{1}{2} \|P\| \left[\left\|P^{1/2}A\right\| \left\|P^{1/2}B\right\| + \|B^*PA\|\right], \tag{3.8}$$

where A, B are bounded operators on H and P is a nonnegative operator on H. Moreover, if in (3.7) and (3.8) we take $B = A^*$, then we get the inequalities

$$\|A^*Pe\| \|APe\| \le \frac{1}{2} \left\|P^{1/2}e\right\|^2 \left[\left\|P^{1/2}A\right\| \left\|AP^{1/2}\right\| + \|APA\|\right] \tag{3.9}$$

for any $e \in H$ and

$$w\left(PA^2P\right) \le \frac{1}{2} \|P\| \left[\left\|P^{1/2}A\right\| \left\|AP^{1/2}\right\| + \|APA\|\right]. \tag{3.10}$$

Further, if we assume that $APC = C^*PA$, then by taking $B = A^*$ in (2.11) and (2.12) we get

$$w^2(APC) \le \frac{1}{2}\left\|P^{1/2}C\right\|^2\left[\left\|P^{1/2}A\right\|\left\|AP^{1/2}\right\| + w(APA)\right] \tag{3.11}$$

and

$$w^2(APC) \le \frac{1}{2}\left\|P^{1/2}C\right\|^2\left[\left\|\frac{\left|P^{1/2}A\right|^2 + \left|P^{1/2}A^*\right|^2}{2}\right\| + w(APA)\right]. \tag{3.12}$$

If $AC = C^*A$, then by taking $P = 1_H$ in (3.11) and (3.12) we have

$$w^2(AC) \le \frac{1}{2}\|C\|^2\left[\|A\|^2 + w\left(A^2\right)\right] \tag{3.13}$$

and

$$w^2(AC) \le \frac{1}{2}\|C\|^2\left[\left\|\frac{|A|^2 + |A^*|^2}{2}\right\| + w\left(A^2\right)\right]. \tag{3.14}$$

Since

$$\left\|\frac{|A|^2 + |A^*|^2}{2}\right\| \le \frac{1}{2}\left[\left\||A|^2\right\| + \left\||A^*|^2\right\|\right] = \|A\|^2,$$

then the inequality (3.14) is better than (3.13).

If $AP = PA$, then by taking $C = 1_H$ in (3.11) and (3.12) we also have

$$w^2(AP) \le \frac{1}{2}\|P\|\left[\left\|P^{1/2}A\right\|\left\|AP^{1/2}\right\| + w\left(PA^2\right)\right] \tag{3.15}$$

and

$$w^2(AP) \le \frac{1}{2}\|P\|\left[\left\|\frac{\left|P^{1/2}A\right|^2 + \left|P^{1/2}A^*\right|^2}{2}\right\| + w\left(PA^2\right)\right]. \tag{3.16}$$

Taking into account the above results, we can state the following two inequalities for an operator T, namely

$$w^2(T) \le \frac{1}{2}\left[\|T\|^2 + w\left(T^2\right)\right], \text{ see (1.15),} \tag{3.17}$$

and

$$w^2(T) \le \frac{1}{2}\left[\left\|\frac{|T|^2 + |T^*|^2}{2}\right\| + w\left(T^2\right)\right]. \tag{3.18}$$

The inequality (3.18) is better than (3.17).

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