ANALYTIC SOLUTIONS FOR AMERICAN PARTIAL BARRIER OPTIONS BY EXPONENTIAL BARRIERS

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Abstract. This paper concerns barrier option of American type where the underlying price is monitored during only part of the option’s life. Analytic valuation formulas of the American partial barrier options are obtained by approximation method. This approximation method is based on barrier options along with exponential early exercise policies. This result is an extension of Jun and Ku [10] where the exercise policies are constant.

1. Introduction

American options are widely traded in the over counter market because American type options give their holders an additional privilege of early exercise. For these reasons, the valuation of the American option price has been very important issue in financial economics.


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They have modeled an American put option as a variational inequality. These numerical methods are very flexible and easy to implement. However, even after employing control variate or convergence extrapolation, they require a very long time.

Many people have tried to reduce the time consuming task. Sullivan [16] approximated the option value function through Chebyshev polynomials and applied a Gaussian quadrature integration scheme at each discrete exercise date. In order to find the value of American put option, Longstaff and Schwartz [14] used the Monte Carlo simulation method. In the Monte Carlo simulation method, optimal stopping is significant problem. They resolved the problem by comparing the conditional expected value of continuing with the value of immediate exercise if the option is currently in the money. Kim et al. [12] introduced a simple iterative method to determine the optimal exercise boundary for American option. They allowed us to compute the values of American options and their Greeks quickly and accurately.


Approximation method of American options is proposed by Ingersoll [7]. He approximated exercise policy employing a simple class of function, and chose a finest policy in that class adopting standard optimization technique. This method is simple and speedy. Concretely, he discussed American put using two types approximate, constant barrier and exponential barrier.

Barrier options have been extensively traded over the counter market since 1967. These options are activated or expired when the price of the underlying asset crosses a barrier level during the option’s life. Heynen and Kat [6] studied partial barrier options where the underlying price is monitored during only part of the option’s lifetime. If barrier is observed from a fixed date after initial starting date until expiration, it is called forward starting barrier option. If barrier is disappeared at
a designated date before the expiry date, it is named early ending barrier option. Heynen and Kat [6] determined pricing formulas for partial barrier options in terms of bivariate normal distribution functions.


This paper concern valuing the American partial barrier option where barrier is observed just from fixed date to expiry date. This paper extend approximation method, derived by Jun and Ku [10], to American partial barrier option as exponential exercise policy. This method includes the case of constant early exercise policy.

Section 2 proposes the approximation of American barrier option based on exponential exercise policies. Section 3 provides the valuation of American partial barrier option using exponential exercise policies.

2. Approximation of American barrier option using exponential barriers

Let $r$ be the risk-free interest rate, $q$ be a dividend rate, and $\sigma > 0$ be a constant. We assume the price of the underlying asset $S$ follows a geometric Brownian motion

$$S_t = S_0 \exp(\mu t + \sigma W_t)$$

where $\mu = r - q - \frac{\sigma^2}{2}$ and $W_t$ is a standard Brownian motion under the risk-neutral probability $P$.

In this section, we consider the partial barrier option of American type as exponential early exercise policies. American option holders gain more benefit than European option on early exercise. An American up-and-in put option can be exercised before the expiration time when it is in the money, but only after the stock price rises above the knock-in barrier. We deal with the up-and-in put where the barrier appear at a specified time $T_1$ strictly after the option’s initiation. If the underlying asset price never hits the up-barrier over the time period between $T_1$ and expiration $T$, payment of option is zero. Otherwise, if the asset
price reaches the up-barrier between $T_1$ and expiration $T$, this option can early exercise.

In order to obtain the approximation to value American partial barrier option using barrier derivatives under exercise policies, Ingergsoll [7] used following digitals: let $D(S, t; A)$ be the value at time $t$ of receiving one dollar at time $T$ if and only in the event $A$ occurs, and $DS(S, t; A)$ be the value at time $t$ of receiving one share of stock at time $T$ if and only if the event $A$ occurs. The $D$ is said to be a digital or binary option and the $DS$ is said to be a digital share. The quantity $E(S, t, K; \tau)$ denotes the value at time $t$ of payment $X - K \tau$ at the first time $\tau$ that the stock price $S$ hits the barrier $K$ provided the event $A$ occurs before time $T$, where $X$ is a strike price. The $E$ is said to be a first-touch digital.

We consider the class of exercise policies, $K_e$, is a set of exponential functions whose elements are in the form of $K_t = K_0 e^{\delta t}$ with constant $K_0$ and $\delta \geq 0$. Since options with exponential barrier have analytical solutions under Black-Scholes conditions, an exponential barrier is a natural choice.

Consider an American up-and-in put expiring $T$ with strike price $X$. Let us denote by $B$ up barrier and by $K_t^*$ the optimal exercise policy. Let $\tau_1$ denote the first time the stock price is equal to $Y_1$ and $\tau_2$ denote the first time after $\tau_1$ that the stock price is equal to $Y_2$.

Let $E_1 = \{t < \tau_B < T, \tau_{BK_t^*} > T, S_T < X\}$ be the event of exercise at maturity under the optimal policy, and $E_2 = \{t < \tau_B, \tau_{BK_t} < T\}$ be the event of early exercise under the optimal policy. Then the value of the up-and-in put can be written as

$$UIP = X \cdot D(S, t; E_1) - DS(S, t; E_1) + E(S, t, K_t^*; E_2)$$

The barrier approximation for this put takes the maximum value within a class of restricted policies. For example, for exponential exercise policies $K_t$,

$$UIP \geq UIP_{\text{exp}} = \max_{k_t \in K_e} [X \cdot D(S, t; E_3) - DS(S, t; E_3) + E(S, t, K_t^*; E_4)]$$

where $E_3 = \{t < \tau_B < T, \tau_{BK_t} > T, S_T < X\}$, $E_4 = \{t < \tau_B, \tau_{BK_t} < T\}$, and $\tau_{BK_t}$ is the first time the stock price hits the exponential policy barrier $K_t$ after hitting the barrier $B$. The values for these digitals are
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\[ D(S, t; E_3) = e^{-r(T-t)} \left[ \left( \frac{B}{S_t} \right)^{2\mu} \left\{ N \left( h_2 \left( \frac{B^2}{S_t K_t} \right) \right) - N \left( h_1 \left( \frac{B^2}{S_t X} \right) \right) \right\} \right. \\
\left. + \left( \frac{K_t}{B} \right)^{2\mu} \left( \frac{B^2}{S_t K_t} \right)^{\frac{2\mu}{\sigma^2}} \left\{ N \left( h_1 \left( \frac{S_t K_t^2}{B^2 X} \right) \right) - N \left( h_2 \left( \frac{S_t K_t}{B^2} \right) \right) \right\} \right], \]

\[ DS(S, t; E_3) = S_t e^{-q(T-t)} \left[ \left( \frac{B}{S_t} \right)^{2\mu} \left\{ N \left( h_2 \left( \frac{B^2}{S_t K_t} \right) \right) - N \left( h_1 \left( \frac{B^2}{S_t X} \right) \right) \right\} \right. \\
\left. + \left( \frac{K_t}{B} \right)^{2\mu} \left( \frac{B^2}{S_t K_t} \right)^{\frac{2\mu}{\sigma^2}} \left\{ N \left( h_1 \left( \frac{S_t K_t^2}{B^2 X} \right) \right) - N \left( h_2 \left( \frac{S_t K_t}{B^2} \right) \right) \right\} \right], \]

\[ E(S, t; K_t; E_4) = X \left[ \left( \frac{S_t}{B} \right)^{q-p} \left( \frac{K_t}{B} \right)^{q+p} \left( \frac{B^2}{S_t K_t} \right)^{\frac{2\mu}{(q-p)^2 \sigma^2}} N \left( g_1 \left( \frac{S_t K_t}{B^2} \right) \right) \right. \\
\left. + \left( \frac{B}{K_t} \right)^{q-p} \left( \frac{B}{S_t} \right)^{q+p} N \left( -g_1 \left( \frac{B^2}{S_t K_t} \right) \right) \right] \\
- K_t \left[ \left( \frac{S_t}{B} \right)^{q_1-p_1} \left( \frac{K_t}{B} \right)^{q_1+p_1} \left( \frac{B^2}{S_t K_t} \right)^{\frac{2\mu}{(q_1-p_1)^2 \sigma^2}} N \left( g_1 \left( \frac{S_t K_t}{B^2} \right) \right) \right. \\
\left. + \left( \frac{B}{K_t} \right)^{q_1-p_1} \left( \frac{B}{S_t} \right)^{q_1+p_1} N \left( -g_1 \left( \frac{B^2}{S_t K_t} \right) \right) \right], \]

where \( N \) is the standard normal distribution function,

\[ h_1(z) = \frac{\ln z + \mu(T-t)}{\sigma \sqrt{T-t}}, \quad h_2(z) = \frac{\ln z + (\mu - \delta)(T-t)}{\sigma \sqrt{T-t}}, \]

\[ g_1(z) = \frac{\ln z + (\mu - \delta q)(T-t)}{\sigma \sqrt{T-t}}, \]

\[ \mu = r - q - \frac{1}{2} \sigma^2, \quad \mu = r - q + \frac{1}{2} \sigma^2, \quad p = \mu - \delta, \quad q = \sqrt{p^2 + \frac{2(r - \delta)}{\sigma^2}}. \]

\( h_i \) and \( g_1 \) are the same as \( h_i \), except \( \mu = r - q + \frac{\sigma^2}{2} \) and \( p_1, q_1 \) in replacement of \( \mu, p, q \) for \( i = 1, 2 \) separately.
3. Valuation of American partial barrier option using exponential exercise policies

Now, we present the valuation of American partial barrier option using exponential exercise policies. Let $X_t = \frac{1}{\sigma} \ln \left( \frac{S_t}{S_0} \right)$ and $E^m$ be the expectation operator under $m$-measure. Then $X_t$ is a Brownian motion with drift $\mu \sigma$. Let $k_t = k_0 + \frac{\delta}{\sigma} t$ for $k_0, \delta(\geq 0)$ are constant. Define $\tau_{b(T_1)}$ and $\tau_{bk_t(T_1)}$ by stopping times for this process defined as the first time that $X_t = b > X_0$ after time $T_1$ and the first time after $\tau_{b(T_1)}$ that $X_t = k_t < b$, respectively.

**Lemma 3.1.** For $x \geq k_T$, the probability that the process $X_t$ reaches $b$ after time $T_1$, and then hits $k_t$ before expiration $T$, and $X_T$ is greater than $x$ is

$$P(\tau_{bk_t(T_1)} \leq T, X_T > x | X_0 = 0) = \exp \left( \frac{2\mu}{\sigma} (-b + k_0) \right) \exp \left( -\frac{2\delta}{\sigma} (k_0 - 2b) \right) G_1(x) + \exp \left( \frac{2(\mu - \delta)}{\sigma} k_0 \right) G_2(x)$$

where

$$G_1(x) = N_2 \left( \frac{b - \frac{\mu - 2\delta}{\sigma} T_1}{\sqrt{T_1}}, \frac{2k_0 - 2b - x + \frac{\mu}{\sigma} T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}} \right)$$

$$G_2(x) = N_2 \left( \frac{-b - \frac{\mu - 2\delta}{\sigma} T_1}{\sqrt{T_1}}, \frac{2k_0 - x + \frac{\mu}{\sigma} T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}} \right).$$

**Proof.** If $b > k_t$, then $\{X_{T_1} \geq b, \tau_{bk_t(T_1)} \leq T\} = \{X_{T_1} \geq b, \tau_{k_t(T_1)} \leq T\}.$

$$P(\tau_{bk_t(T_1)} \leq T, X_T > x | X_0 = 0) = P(\tau_{bk_t(T_1)} \leq T, X_T > x | X_0 = 0)$$

$$= P(X_{T_1} < b, \tau_{bk_t(T_1)} \leq T, X_T > x | X_0 = 0)$$

$$+ P(X_{T_1} \geq b, \tau_{k_t(T_1)} \leq T, X_T > x | X_0 = 0)$$

$$= \int_{-\infty}^{b} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2} \left( \frac{x_1 - \frac{\mu}{\sigma} T_1}{\sqrt{T_1}} \right)^2} P(\tau_{bk_t(T_1)} \leq T, X_T > x | X_{T_1} = x_1) \, dx_1$$

$$+ \int_{b}^{\infty} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2} \left( \frac{x_2 - \frac{\mu}{\sigma} T_1}{\sqrt{T_1}} \right)^2} P(\tau_{k_t(T_1)} \leq T, X_T > x | X_{T_1} = x_2) \, dx_2.$$
Thus, we calculate $P\left(\tau_{b k_i(T_1)} \leq T, X_T > x | X_{T_1} = x_1\right)$. In order to reflect a path at $\tau_{b(T_1)}$, we define $\tilde{X}_t = \begin{cases} 2b - X_t & \text{if } t \leq \tau_{u(T_1)} \\ X_t & \text{if } t > \tau_{u(T_1)} \end{cases}$.

$$P\left(\tau_{b k_i(T_1)} \leq T, X_T > x | X_{T_1} = x_1\right) = \exp\left(\frac{2\mu}{\sigma}(b - x_1)\right) P\left(\tau_{k_i(T_1)} \leq T, \tilde{X}_T > x | \tilde{X}_{T_1} = 2b - x_1\right).$$

Then we reflect this path before its first touch at $k_0$ again.

$$P\left(\tau_{b k_i(T_1)} \leq T, X_T > x | X_{T_1} = x_1\right) = \exp\left(\frac{2\mu}{\sigma}(b - x_1)\right) \exp\left(\frac{2(\mu - \delta)}{\sigma}(k_0 - 2b + x_1 + \delta T_1)\right) \times N\left(\frac{2k_0 - 2b + x_1 - x + \frac{2\delta}{\sigma} T_1 + \frac{\mu}{\sigma}(T - T_1)}{\sqrt{T - T_1}}\right).$$

Thus,

$$\int_{-\infty}^{b} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2} \left(\frac{t^2 - \frac{2\delta}{\sigma}T_1}{T_1}\right)^2} P\left(\tau_{b k_i(T_1)} \leq T, X_T > x | X_{T_1} = x_1\right) \, dx_1$$

$$= \exp\left(\frac{2\mu}{\sigma}(b - k_0) - \frac{2\delta}{\sigma}(k_0 - 2b)\right) N_2\left(\frac{u - \frac{\mu - 2\delta}{\sigma} T_1}{\sqrt{T_1}}, \frac{2k_0 - 2b - x + \frac{\mu}{\sigma} T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right)$$

and

$$\int_{b}^{\infty} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2} \left(\frac{t^2 - \frac{2\delta}{\sigma}T_1}{T_1}\right)^2} P\left(\tau_{b k_i(T_1)} \leq T, X_T > x | X_{T_1} = x_2\right) \, dx_2$$

$$= \exp\left(\frac{2(\mu - \delta)}{\sigma}k_0\right) N_2\left(\frac{-b - \frac{\mu - 2\delta}{\sigma} T_1}{\sqrt{T_1}}, \frac{2k_0 - x + \frac{\mu}{\sigma} T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}}\right).$$

\[\square\]

**Lemma 3.2.** The probability that the process $X_t$ reaches $b$ after time $T_1$, and then falls below $k_t$ before time $T$ is

$$P(\tau_{b k_i(T)} \leq T | X_0 = 0) = \exp\left(\frac{2\mu}{\sigma}(-b + k_0)\right) \exp\left(-\frac{2\delta}{\sigma}(k_0 - 2b)\right) G_1(k_T)$$

$$+ \exp\left(\frac{2(\mu - \delta)}{\sigma} k_0\right) G_2(k_T) + \exp\left(\frac{2b\mu}{\sigma}\right) G_3(k_T) + G_4(k_T)$$
where
\[ G_3(x) = N_2 \left( \frac{b + \frac{\mu T_1}{\sigma}}{\sqrt{T_1}}, \frac{x - 2b - \frac{\mu}{\sigma} T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}} \right), \]
\[ G_4(x) = N_2 \left( \frac{-b + \frac{\mu T_1}{\sigma}}{\sqrt{T_1}}, \frac{x + \frac{\mu}{\sigma} T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}} \right). \]

Proof. Since \( \{ \tau_{b_1(T_1)} \leq T, X_T \leq k_T \} = \{ \tau_{b(T_1)} \leq T, X_T \leq k_T \} \)
\[ P(\tau_{b_1(T_1)} \leq T | X_0 = 0) \]
\[ = P(\tau_{b_1(T_1)} \leq T, X_T > k_T | X_0 = 0) + P(\tau_{b_1(T_1)} \leq T, X_T \leq k_T | X_0 = 0) \]
When \( X_{T_1} > b \), the event \( \{ \tau_{b_1(T_1)} \leq T, X_T \leq k_T \} = \{ X_T \leq k_T \} \) and
\[ P(\tau_{b_1(T_1)} \leq T, X_T \leq k_T | X_0 = 0) \]
\[ = \int_{-\infty}^{b} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2} \left( \frac{x_1 - \frac{\mu T_1}{\sigma}}{\sqrt{T_1}} \right)^2} P(\tau_{b(T_1)} \leq T, X_T \leq k_T | X_{T_1} = x_1) \, dx_1 \]
\[ + \int_{b}^{\infty} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2} \left( \frac{x_2 - \frac{\mu T_1}{\sigma}}{\sqrt{T_1}} \right)^2} P(\tau_{b(T_1)} \leq T, X_T \leq k_T | X_{T_1} = x_2) \, dx_2 \]
\[ = \int_{-\infty}^{b} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2} \left( \frac{x_1 - \frac{\mu T_1}{\sigma}}{\sqrt{T_1}} \right)^2} \exp \left( \frac{2\mu \sigma}{\sigma} (b - x_1) \right) \]
\[ \times N \left( \frac{k_0 - 2b + x_1 - \frac{\mu - \delta}{\sigma} T + \frac{\mu}{\sigma} T_1}{\sqrt{T - T_1}} \right) \, dx_1 \]
\[ + \int_{b}^{\infty} \frac{1}{\sqrt{2\pi T_1}} e^{-\frac{1}{2} \left( \frac{x_2 - \frac{\mu T_1}{\sigma}}{\sqrt{T_1}} \right)^2} N \left( \frac{k_0 - x_2 - \frac{\mu - \delta}{\sigma} T + \frac{\mu}{\sigma} T_1}{\sqrt{T - T_1}} \right) \, dx_2 \]
\[ = \exp \left( \frac{2u\mu}{\sigma} \right) N_2 \left( \frac{u + \frac{\mu T_1}{\sigma}}{\sqrt{T_1}}, \frac{l_0 - 2u - \frac{\mu - \delta}{\sigma} T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}} \right) \]
\[ + N_2 \left( \frac{-u + \frac{\mu T_1}{\sigma}}{\sqrt{T_1}}, \frac{l_0 - \frac{\mu - \delta}{\sigma} T}{\sqrt{T}}; -\sqrt{\frac{T_1}{T}} \right). \]

Lemma 3.3. For \( k_T \leq x \leq b \), the probability that the process \( X_t \) reaches \( b \) after time \( T_1 \), and then does not fall below \( k_i \) before expiration \( T \), and its value at time \( T \) is less than \( x \) is
Lemma 3.1. The second and third probabilities are calculated by Lemma 3.2 and

\[ P(D = \tau) = P(T = \tau) + P(S, t) = \exp(2\mu - \delta)k_0[G_2(x) - G_2(l_T)] + \exp(2\mu_k)\left[G_3(x) - G_3(k_T)\right] + G_4(x) - G_4(k_T) \]

Proof.

\[ P(\tau_{b(T_1)} < T, \tau_{b(k(T_1))} > T, X_T \leq x|X_0 = 0) = \exp\left(\frac{2\mu}{\sigma}(b + k_0)\right)\exp\left(-\frac{2\delta}{\sigma}(k_0 - 2u)\right)[G_1(x) - G_1(k_T)] + \exp\left(\frac{2(\mu - \delta)}{\sigma}k_0\right)[G_2(x) - G_2(l_T)] + \exp\left(\frac{2b\mu}{\sigma}\right)[G_3(x) - G_3(k_T)] + G_4(x) - G_4(k_T) \]

\[ P(\tau_{b(T_1)} < T, X_T \leq x|X_0 = 0) = P(\tau_{b(T_1)} < T, X_T \leq x|X_0 = 0) - P(\tau_{b(k(T_1))} \leq T, X_T \leq x|X_0 = 0) \]

\[ = P(\tau_{b(T_1)} < T, X_T \leq x|X_0 = 0) - P(\tau_{b(k(T_1))} \leq T|X_0 = 0) \]

\[ + P(\tau_{b(k(T_1))} \leq T, X_T > x|X_0 = 0) \]

\[ P(\tau_{b(T_1)} < T, X_T \leq x|X_0 = 0) \text{ can be calculated with } k_T = l_0 + \frac{4}{\sigma}T = x. \]

The second and third probabilities are calculated by Lemma 3.2 and Lemma 3.1.

Theorem 3.4. The value of a digital option and a digital share at time t for the event \( E_8 = \{\tau_{BK_i(T_1)} < T\} \) are

\[
\mathcal{D}(S, t; E_8) = e^{-r(T-t)} \left( \frac{K_i}{B} \right)^{\frac{2}{\sigma}} \left( \frac{B^2}{K_iS_t} \right)^{\frac{2\mu}{\sigma}} N_2 \left( h_4 \left( \frac{B}{S_t} \right), h_2 \left( \frac{K_iS_t}{B^2} \right) : -\sqrt{\frac{T_1 - t}{T - t}} \right) + e^{-r(T-t)} \left( \frac{K_i}{S_t} \right)^{2(\mu-\delta)} N_2 \left( h_4 \left( \frac{S_t}{B} \right), h_2 \left( \frac{K_i}{S_t} \right) : -\sqrt{\frac{T_1 - t}{T - t}} \right) + e^{-r(T-t)} \left( \frac{B}{S_t} \right)^{2\mu} N_2 \left( h_3 \left( \frac{B}{S_t} \right), -h_2 \left( \frac{B^2}{K_iS_t} \right) : -\sqrt{\frac{T_1 - t}{T - t}} \right) + e^{-r(T-t)} N_2 \left( h_3 \left( \frac{S_t}{B} \right), -h_2 \left( \frac{S_t}{K_i} \right) : -\sqrt{\frac{T_1 - t}{T - t}} \right) \]

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\[
DS(S, t; E_8) = S_t e^{-q(T-t)} \left( \frac{K_t}{B} \right)^{\frac{2\pi}{\sigma}} \left( \frac{B^2}{K_t S_t} \right)^{\frac{3\pi}{2\sigma}} N_2 \left( \bar{h}_4 \left( \frac{B}{S_t} \right), \bar{h}_2 \left( \frac{K_t S_t}{B^2} \right); -\sqrt{\frac{T_1 - t}{T - t}} \right) \\
+ S_t e^{-q(T-t)} \left( \frac{K_t}{S_t} \right)^{\frac{2\pi-\delta}{\sigma}} N_2 \left( \bar{h}_4 \left( \frac{S_t}{B} \right), \bar{h}_2 \left( \frac{K_t S_t}{B^2} \right); -\sqrt{\frac{T_1 - t}{T - t}} \right) \\
+ S_t e^{-q(T-t)} \left( \frac{B}{S_t} \right)^{\frac{2\pi}{\sigma}} N_2 \left( \frac{S_t}{B}, -h_2 \left( \frac{B^2}{K_t S_t} \right); -\sqrt{\frac{T_1 - t}{T - t}} \right) \\
+ S_t e^{-q(T-t)} N_2 \left( \frac{S_t}{B}, -h_2 \left( \frac{S_t}{K_t} \right); -\sqrt{\frac{T_1 - t}{T - t}} \right)
\]

Proof. Apply Lemma 3.3 with 
\[
b = \frac{1}{\sigma} \ln \frac{B}{S_t}, k_t = \frac{1}{\sigma} \ln \frac{K_t}{S_t} (= \frac{1}{\sigma} \ln K_{\text{pre}})
\]
and 
\[
x = \frac{1}{\sigma} \ln \frac{X}{S_t}
\]
to derive the risk-neutral probability of early exercise.

We calculate each term of 
\[
P(\tau_{BK_i(T_1)} \leq T | S_t)
\]

Thus, the value of digital option at time t
\[
D(S, t; E_8) = e^{-r(T-t)} P(\tau_{BK_i(T_1)} \leq T | S_t)
\]
is obtained.

Lemma 3.5. If the stock does not pay dividends, the value of a first touch digital for the event \(E_8 = \{\tau_{BK_i(T_1)} < T\}\) is
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\[
\mathcal{E}(S, t, K_t; E_8) = \frac{X - K_t}{K_T} S_t \left[ \left( \frac{K_t}{B} \right)^{\frac{2}{\sigma^2} + 1} \left( \frac{B}{K_t S_t} \right)^{\frac{2}{\sigma^2 N_2 \left( \frac{S_t}{K_t} \right) \left( \frac{B^3}{S_t K_t} \right)}; -\frac{\sqrt{T_1 - t}}{T - t} \right) + \left( \frac{B}{S_t} \right)^{\frac{2}{\sigma^2} + 1} \left( \frac{B}{S_t} \right)^{\frac{2}{\sigma^2 N_2 \left( \frac{S_t}{K_t} \right) \left( \frac{B^3}{S_t K_t} \right)}; -\frac{\sqrt{T_1 - t}}{T - t} \right) + N_2 \left( \frac{S_t}{K_t} \right), \frac{B^3}{S_t K_t} \right] 
\]

where \( \tilde{h}_i \) is the same as \( h_i \) except \( \tilde{\mu} = r + \frac{1}{2} \sigma^2 \) in replacement of \( \mu \) for \( i = 2, 3, 4 \).

Proof. The first-touch digital pays \( X - K_t \) at time \( \tau_{BK_0(T_1)} \). This money can be used to purchase \( \frac{X - K_T}{K_T} S_T \) shares of the stock at time. Since the shares do not pay dividends, it is worth \( \frac{X - K_T}{K_T} S_T \) at maturity, i.e.,

\[
\mathcal{E}(S, t, K_t; E_8) = \frac{X - K_T}{K_T} \mathcal{D}(S, t; E_8)
\]

where \( \mathcal{D}(S, t; E_8) \) is the value when \( q = 0 \) in Theorem 3.4.

Let \( K^* \) denote the optimal exercise policy. We denote the \( E_5 = \{ \tau_{BK_0(T_1)} < T, \tau_{BK_{K^*_t}(T_1)} > T, S_T < X \} \) be the event of exercise at maturity under the optimal policy, and \( E_6 = \{ \tau_{BK_{K^*_t}(T_1)} < T \} \) be the event of early exercise under the optimal policy. Then the value of this partial up-and-in put is written as

\[
P_{UIP} = X \cdot \mathcal{D}(S, t; E_5) - \mathcal{D}(S, t; E_5) + \mathcal{E}(S, t, K_t; E_8)
\]

For the barrier approximation of this option, we consider a class of all exponential exercise policies. Let \( E_7 = \{ \tau_{B_{K^*_t}(T_1)} < T, \tau_{B_{K_t}(T_1)} > T, S_T < X \} \) be the event of exercise at maturity under an exponential policy \( K_t \), and \( E_8 = \{ \tau_{B_{K_t}(T_1)} < T \} \) be the event of early exercise under policy \( K_t \). Then we can approximate the option price as

\[
P_{UIP} \geq P_{UIP}^{\text{exp}} = \max_{k_t \in K} [X \cdot \mathcal{D}(S, t; E_7) - \mathcal{D}(S, t; E_7) + \mathcal{E}(S, t, K_t; E_8)].
\]
If the set of policies considered contains all continuous functions, then the resulting put value will be exact. Since the set $K_e$ is the set of all exponential functions, then the resulting value will be an approximation providing a lower bound to the put price.

We first present the digital options in case of barrier greater than strike price for an American partial barrier option.

**Theorem 3.6.** For $X \leq B$, the values of a digital option and a digital share at time $t < T_1$ for the event $E_7 = \{ \tau_{B(T_1)} < T, \tau_{B_2(T_1)} > T, S_T \leq X \}$ are

$$
D(S, t; E_7) = e^{-r(T-t)} \left[ \left( \frac{K_t}{B} \right)^{\frac{2\mu}{\sigma^2}} \left( \frac{B^2}{K_tS_t} \right)^{\frac{2\delta}{\sigma^2}} \left( F_1 \left( \frac{B}{S_t}, \frac{K^2 S_t}{B^2 X} \right) - F_2 \left( \frac{B}{S_t}, \frac{K_t S_t}{B^2} \right) \right) 
+ \left( \frac{K_t}{S_t} \right)^{\frac{2\mu}{\sigma^2}} \left( F_1 \left( \frac{S_t}{B}, \frac{K^2}{S_t} \right) - F_2 \left( \frac{S_t}{B}, \frac{K_t}{S_t} \right) \right) 
+ \left( \frac{B}{S_t} \right)^{\frac{2\mu}{\sigma^2}} \left( F_3 \left( \frac{B}{S_t}, \frac{B^2}{S_t X} \right) - F_4 \left( \frac{B}{S_t}, \frac{B^2}{S_t K_t} \right) \right) 
+ \left( F_3 \left( \frac{S_t}{B}, \frac{S_t}{X} \right) - F_4 \left( \frac{S_t}{B}, \frac{S_t}{K_t} \right) \right) \right],
$$

$$
DS(S, t; E_7) = S_t e^{-q(T-t)} \left[ \left( \frac{K_t}{B} \right)^{\frac{2\mu}{\sigma^2}} \left( \frac{B^2}{K_tS_t} \right)^{\frac{2\delta}{\sigma^2}} \left( \overline{F}_1 \left( \frac{B}{S_t}, \frac{K^2 S_t}{B^2 X} \right) - \overline{F}_2 \left( \frac{B}{S_t}, \frac{K_t S_t}{B^2} \right) \right) 
+ \left( \frac{K_t}{S_t} \right)^{\frac{2\mu}{\sigma^2}} \left( \overline{F}_1 \left( \frac{S_t}{B}, \frac{K^2}{S_t} \right) - \overline{F}_2 \left( \frac{S_t}{B}, \frac{K_t}{S_t} \right) \right) 
+ \left( \frac{B}{S_t} \right)^{\frac{2\mu}{\sigma^2}} \left( \overline{F}_3 \left( \frac{B}{S_t}, \frac{B^2}{S_t X} \right) - \overline{F}_4 \left( \frac{B}{S_t}, \frac{B^2}{S_t K_t} \right) \right) 
+ \left( \overline{F}_3 \left( \frac{S_t}{B}, \frac{S_t}{X} \right) - \overline{F}_4 \left( \frac{S_t}{B}, \frac{S_t}{K_t} \right) \right) \right],
$$

where

$$
F_1(x, y) = N_2 \left( h(x), h(y); -\sqrt{\frac{T_1 - t}{T - t}} \right),
$$

and

$$
\overline{F}_1(x, y) = N_2 \left( h(x), h(y); -\sqrt{\frac{\tau_{B(T_1)} - t}{T - t}} \right).
$$
F_2(x, y) = N_2\left(h_4(x), h_2(y); -\sqrt{\frac{T_1 - t}{T - t}} \right),
F_3(x, y) = N_2\left(h_3(x), -h_1(y); -\sqrt{\frac{T_1 - t}{T - t}} \right),
F_4(x, y) = N_2\left(h_4(x), -h_2(y); -\sqrt{\frac{T_1 - t}{T - t}} \right),

and
\begin{align*}
h_3(z) &= \frac{\ln z + \mu(T_1 - t)}{\sigma \sqrt{T_1 - t}}, \\
h_4(z) &= \frac{\ln z - (\mu - 2\delta)(T_1 - t)}{\sigma \sqrt{T_1 - t}}.
\end{align*}

\(\overline{F}_i(x, y)\) and \(\overline{h}_j(z)\) are the same as \(F_i(x, y)\), \(h_j(z)\) except \(\mu = r - q + \frac{\sigma^2}{2}\) in replacement of \(\mu\) for \(i = 1, 2, 3, 4, j = 3, 4\) separately.

**Proof.** Apply Lemma 3.3 with letting \(b = \frac{1}{\sigma} \ln \frac{B}{S_t}, k_t = \frac{1}{\sigma} \ln \frac{K_t}{S_t}(= \frac{1}{\sigma} \ln \frac{K_{\text{std}}}{S_t})\) and \(x = \frac{1}{\sigma} \ln \frac{X}{S_t}\) to derive the risk-neutral probability of exercise at maturity.

Then
\[
\begin{align*}
P(\tau_{B(T_1)} < T, \tau_{BK_i(T_1)} > T, S_T \leq X|S_t) &= \left( \frac{K_t}{B} \right)^{\frac{2\mu}{\sigma^2}} \left( \frac{B^2}{K_t S_t} \right)^{\frac{2\sigma^2}{\sigma^2}} \left( N_2\left(h_4\left(\frac{B}{S_t}\right), h_1\left(\frac{K_t^2 S_t}{B^2 X}\right); -\sqrt{\frac{T_1 - t}{T - t}} \right) - N_2\left(h_4\left(\frac{B}{S_t}\right), h_2\left(\frac{K_t S_t}{B^2}\right); -\sqrt{\frac{T_1 - t}{T - t}} \right) \right) \\
&+ \left( \frac{K_t}{S_t} \right)^{\frac{2(\mu-\delta)}{\sigma^2}} \left( N_2\left(h_4\left(\frac{B}{S_t}\right), h_1\left(\frac{K_t^2}{S_t X}\right); -\sqrt{\frac{T_1 - t}{T - t}} \right) - N_2\left(h_4\left(\frac{S_t}{B}\right), h_2\left(\frac{K_t}{X}\right); -\sqrt{\frac{T_1 - t}{T - t}} \right) \right) \\
&+ \left( \frac{B}{S_t} \right)^{\frac{2\mu}{\sigma^2}} \left( N_2\left(h_3\left(\frac{B}{S_t}\right), -h_1\left(\frac{B^2}{S_t X}\right); -\sqrt{\frac{T_1 - t}{T - t}} \right) - N_2\left(h_3\left(\frac{B}{S_t}\right), -h_2\left(\frac{B^2}{S_t K_t}\right); -\sqrt{\frac{T_1 - t}{T - t}} \right) \right) \\
&+ N_2\left(h_3\left(\frac{S_t}{B}\right), -h_1\left(\frac{S_t}{X}\right); -\sqrt{\frac{T_1 - t}{T - t}} \right) - N_2\left(h_3\left(\frac{S_t}{B}\right), -h_2\left(\frac{S_t}{K_t}\right); -\sqrt{\frac{T_1 - t}{T - t}} \right)
\end{align*}
\]
Thus

$$\mathcal{D}(S, t; E_7) = e^{-r(T-t)} P(\tau_{B(T_1)} < T, \tau_{BK_1(T_1)} > T, S_T \leq X | S_t)$$

is obtained as desired. The digital share $$\mathcal{DS}(S, t; E_7)$$ can be valued by changing $$\mu$$ to $$\mu = r - q + \frac{\sigma^2}{2}$$ and replacing the discount factor $$e^{-r(T-t)}$$ to $$S_t e^{-q(T-t)}$$. 

\[ \square \]

**Theorem 3.7.** The value of the first-touch digital for the event $$E_8 = \{ \tau_{BK_1(T_1)} < T \}$$ is

$$\mathcal{E}(S, t, K_t; E_8)$$

$$= X \left[ \left( \frac{S_tK_t}{BK_t} \right)^{q-p} \left( \frac{K_t}{B} \right)^{q+p} \left( \frac{B^2}{K_tS_t} \right)^{\frac{2\delta}{\sigma^2}} H_1 \left( \frac{B}{S_t}, S_tK_t \right) \right]

+ \left( \frac{K_t}{B} \right)^{q-p} \left( \frac{K_t}{S_t} \right)^{q+p} \left( \frac{K_t}{S_t} \right)^{\frac{-2\delta}{\sigma^2}} H_1 \left( \frac{S_t}{B}, S_t \right)

+ \left( \frac{B}{K_t} \right)^{q-p} \left( \frac{B}{S_t} \right)^{q+p} \left( \frac{B^2}{K_tS_t} \right) + \left( \frac{S_t}{K_t} \right)^{q-p} \left( \frac{S_t}{S_t} \right)

- K_t \left[ \left( \frac{S_tK_t}{BK_t} \right)^{q_1-p_1} \left( \frac{K_t}{B} \right)^{q_1+p_1} \left( \frac{B^2}{K_tS_t} \right)^{\frac{2\delta}{\sigma^2}} H_1 \left( \frac{B}{S_t}, S_t \right) \right]

+ \left( \frac{K_t}{B} \right)^{q_1-p_1} \left( \frac{K_t}{S_t} \right)^{q_1+p_1} \left( \frac{K_t}{S_t} \right)^{\frac{-2\delta}{\sigma^2}} H_1 \left( \frac{S_t}{B}, S_t \right)

+ \left( \frac{B}{K_t} \right)^{q_1-p_1} \left( \frac{B}{S_t} \right)^{q_1+p_1} \left( \frac{B^2}{K_tS_t} \right) + \left( \frac{S_t}{K_t} \right)^{q_1-p_1} \left( \frac{S_t}{S_t} \right),

where

$$H_1(x, y) = N_2 \left( g_2(x), g_1(y); -\frac{T_1 - t}{T - t} \right),$$

$$H_2(x, y) = N_2 \left( g_3(x), -g_1(y); -\frac{T_1 - t}{T - t} \right),$$

and
\[
g_2(z) = \frac{\ln z - \left( q_1 \sigma^2 - q_2 \rho \right) (T_1 - t)}{\sigma \sqrt{T_1 - t}}, \quad \tilde{g}_2(z) = \frac{\ln z - \left( q_1 \sigma^2 - q_2 \rho \right) (T_1 - t)}{\sigma \sqrt{T_1 - t}}.
\]

\[
g_3(z) = \frac{\ln z + q_2 \sigma^2 (T_1 - t)}{\sigma \sqrt{T_1 - t}}, \quad \tilde{g}_3(z) = \frac{\ln z + q_2 \sigma^2 (T_1 - t)}{\sigma \sqrt{T_1 - t}}.
\]

\[\tilde{H}_i(x, y)\] is the same as \(H_i(x, y)\) except \(r - \delta\) in replacement of \(r\) for \(i = 1, 2\).

**Proof.** By Lemma 3.5, we note that \(\mathcal{E}(S, t, K; E_k) = \frac{X - \text{K}^2}{K^2}DS(S, t; E_k)\) for \(0 < t < T_1\) and \(T_1 < \tau < T\). i.e.

\[
\mathcal{E}(S, t, K; E_k) = X S_t \frac{K_i}{K} \left[ \left( \frac{K_i}{B} \right)^{2 \tau - 1} \left( \frac{B^2}{K_i S_t} \right)^{\frac{2 \alpha}{\sigma^2}} N_2 \left( \frac{\bar{h}_4}{\bar{S}_t}, \bar{h}_2 \left( \frac{S_t K_i}{B^2} \right) ; -\sqrt{\frac{T_1 - t}{T - t}} \right) + \left( \frac{K_i}{S_t} \right)^{2 \left( r - \delta \right) \tau - 1} N_2 \left( \frac{\bar{h}_4}{\bar{S}_t}, \bar{h}_2 \left( \frac{K_i}{S_t} \right) ; -\sqrt{\frac{T_1 - t}{T - t}} \right) + \left( B \frac{S_t}{S_i} \right)^{2 \tau + 1} N_2 \left( \frac{\bar{h}_3}{\bar{S}_t}, \bar{h}_2 \left( \frac{B^2}{S_t K_i} \right) ; -\sqrt{\frac{T_1 - t}{T - t}} \right) + N_2 \left( \frac{\bar{h}_3}{\bar{S}_t}, -\bar{h}_2 \left( \frac{S_t}{K} \right) ; -\sqrt{\frac{T_1 - t}{T - t}} \right) \right] - K \tau \frac{S_t}{K} \left[ \left( \frac{K_i}{B} \right)^{2 \tau - 1} \left( \frac{B^2}{K_i S_t} \right)^{\frac{2 \alpha}{\sigma^2}} N_2 \left( \frac{\bar{h}_4}{\bar{S}_t}, \bar{h}_2 \left( \frac{S_t K_i}{B^2} \right) ; -\sqrt{\frac{T_1 - t}{T - t}} \right) + \left( \frac{K_i}{S_t} \right)^{2 \left( r - \delta \right) \tau - 1} N_2 \left( \frac{\bar{h}_4}{\bar{S}_t}, \bar{h}_2 \left( \frac{K_i}{S_t} \right) ; -\sqrt{\frac{T_1 - t}{T - t}} \right) + \left( B \frac{S_t}{S_i} \right)^{2 \tau + 1} N_2 \left( \frac{\bar{h}_3}{\bar{S}_t}, \bar{h}_2 \left( \frac{B^2}{S_t K_i} \right) ; -\sqrt{\frac{T_1 - t}{T - t}} \right) + N_2 \left( \frac{\bar{h}_3}{\bar{S}_t}, -\bar{h}_2 \left( \frac{S_t}{K} \right) ; -\sqrt{\frac{T_1 - t}{T - t}} \right) \right].
\]

When the stock price pays dividends, the asset price follow the continuous diffusion process \(dS_t = (r - q)S_t dt + \sigma S_t dW\). In order to calculate the first term with constant payment \(X\), we set

\[
V_t = S_t^{q - p}
\]
where
\[
p = \frac{\mu}{\sigma^2} \quad \text{and} \quad q = \sqrt{p^2 + \frac{2r}{\sigma^2}}.
\]

Then, by Itô’s lemma,
\[
dV_t = rV_t dt + (q - p)\sigma V_t dW_t
\]
(1)
We may apply Lemma 3.5 to the process \(V_t\) since (1) does not contain the dividend term. The barriers for \(V_t\) corresponding to \(B\) and \(K_t\) are \(B_t^{q-p}\) and \(K_t^{q-p}\). Furthermore, the volatility \(\sigma\) is replaced by \((q - p)\sigma\). Then \(\frac{X}{K_t} \frac{dS_t}{S_t}\) is
\[
\begin{align*}
X \frac{V_t}{K_t^{q-p}} & \left[ \left( \frac{K_t^{(q-p)}}{B(q-p)} \right)^{2r} \frac{(q-p)^{2q+1}}{(q-p)^{q+1}} + \frac{B(q-p)}{K_t^{(q-p)}V_t} \right] \\
& \times N_2 \left( \tilde{h}_4 \left( \frac{B(q-p)}{V_t} \right), \tilde{h}_2 \left( \frac{V_tK_t^{(q-p)}}{B^{2(q-p)}} \right) ; -\sqrt{\frac{T_1 - t}{T - t}} \right) \\
& + \left( \frac{K_t^{(q-p)}}{V_t} \right)^{2(r-\delta)} \left( \frac{(q-p)^{2q+1}}{(q-p)^{q+1}} + \frac{B(q-p)}{V_tK_t^{(q-p)}} \right) \\
& + \left( \frac{B(q-p)}{V_t} \right)^{2q} \left( \tilde{h}_4 \left( \frac{B^{2(q-p)}}{V_t} \right), \tilde{h}_2 \left( \frac{V_tK_t^{(q-p)}}{B^{2(q-p)}} \right) ; -\sqrt{\frac{T_1 - t}{T - t}} \right) \\
& + N_2 \left( \tilde{h}_3 \left( \frac{V_t}{K_t^{(q-p)}} \right), \tilde{h}_2 \left( \frac{V_t}{K_t^{(q-p)}} \right) ; -\sqrt{\frac{T_1 - t}{T - t}} \right)
\end{align*}
\]
\[
= X \left[ \left( \frac{S_t K_t}{B K_t} \right)^{(q-p)} \left( \frac{K_t}{B} \right)^{(q+p)} \left( \frac{B^2}{K_t S_t} \right)^{2q \frac{2\delta}{(q-p)^2}} H_1 \left( \frac{S_t}{B}, \frac{S_t K_t}{B} \right) \\
+ \left( \frac{K_t}{K_t} \right)^{q-p} \left( \frac{K_t}{S_t} \right)^{(q+p)} \left( \frac{K_t}{S_t} \right)^{-2q \frac{2\delta}{(q-p)^2}} H_1 \left( \frac{S_t}{B}, \frac{S_t K_t}{B} \right) \\
+ \left( \frac{B}{K_t} \right)^{q-p} \left( \frac{B}{S_t} \right)^{(q+p)} H_2 \left( \frac{S_t}{K_t S_t}, \frac{B^2}{K_t S_t} \right) + \left( \frac{S_t}{K_t} \right)^{(q-p)} H_2 \left( \frac{S_t}{B}, \frac{S_t}{K_t} \right) \right].
\]

For the second term with exponential payment \(K_t\), we note when a payment grows exponentially at rate \(\delta\), discounting the payment at the interest rate \(r\) is equivalent to discounting a constant payment at the rate \(r - \delta\), therefore, set
\[
V_t = S_t^{q(1-p)}
\]
where
Analytic solutions for American partial barrier options

\[ p_1 = \frac{u-\delta}{\sigma^2} \] \text{and} \quad \[ q_1 = \sqrt{p_1^2 + \frac{2(r-\delta)}{\sigma^2}}. \]

Then, by Ito’s lemma,

\[ dV_t = (r-\delta)V_t dt + (q_1 - p_1)\sigma V_t dW_t. \]

References


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