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SOLVABILITY FOR A SYSTEM OF GENERALIZED NONLINEAR ORDERED VARIATIONAL INCLUSIONS IN ORDERED BANACH SPACES

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Abstract. In this paper, we consider a *system of generalized nonlinear ordered variational inclusions* in real ordered Banach spaces and define an iterative algorithm for a solution of our problems. By using the resolvent operator techniques to prove an existence result for the solution of the *system of generalized nonlinear ordered variational inclusions* and discuss convergence of sequences suggested by the algorithms.

1. Introduction

The fundamental concept in the theory of variational inequality is to develop a streamline iterative algorithm for solving a variational inequality and its others forms. These technique include the projection technique and its novel innovative forms, approximation techniques, Newtons methods and the methods derived from the auxiliary principle techniques. As you know that the projection technique cannot be applied to solve variational inclusion problems and thus one has to use resolvent operator techniques to solve them. The beauty of the iterative technique involving the resolvent operator is that the resolvent step involves the maximal monotone operator only, while other parts facilitate the problems decomposition. Most of the problems related to

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variational inclusions and complementarity problems are solved by maximal monotone operators and their generalizations such as H-accretivity, H-monotonicity, (H,η)-monotonicity, (H,η)-accretivity, and (H,η,ϕ)-monotonicity *etc, see* [2, 4, 7–11, 13, 14, 16, 27, 35].

Motivated and inspired by mentioned research works [1, 3, 5, 12, 15, 18, 19, 21, 29–31, 33, 34], we initiate a study of a system of generalized nonlinear ordered variational inclusions in real ordered Banach space. We design an algorithm based on the resolvent operator for solving the system of generalized nonlinear ordered variational inclusion problems. We prove an existence and convergence theorems for our problems.

2. Prelude

Throughout this paper, we assume that X is a real ordered Banach space with norm $\|\cdot\|$, an inner product $\langle\cdot,\cdot\rangle$, a zero element θ and partial order $\leq$ defined by the normal cone C with a normal constant λ_C. The greatest lower bound and least upper bound for the set $\{x,y\}$ with partial order relation $\leq$ are denoted by $glb\{x,y\}$ and $lub\{x,y\}$, *respectively.* Assume that $glb\{x,y\}$ and $lub\{x,y\}$ both exist.

Definition 2.1. Let $C(\neq\emptyset)$ be a closed, convex subset of X. C is said to be a cone if

(i) for $x\in C$ and $\lambda>0$, $\lambda x\in C$;
(ii) if x and $-x\in C$, then $x=\theta$.

Definition 2.2. [6] C is called a normal cone if and only if there exists a constant $\lambda_C>0$ such that $0\leq x\leq y$ implies $\|x\|\leq\lambda_C\|y\|$, where λ_C is called the normal constant of C.

Definition 2.3. For arbitrary elements $x,y\in X$, $x\leq y$ if and only if $x-y\in C$, then the relation $\leq$ is a partial ordered relation in X. The real Banach space X with the ordered relation $\leq$ defined by C is called an ordered real Banach space.

Definition 2.4. [28] For arbitrary elements $x,y\in X$, if $x\leq y$ (or $y\leq x$) holds, then x and y are called comparable to each other and this is denoted by $x\propto y$.

DEFINITION 2.5. [17] A map $A : X \longrightarrow X$ is called β-ordered comparison map, if it is a comparison mapping and

$$A(x) \oplus A(y) \leq \beta(x \oplus y), \text{ for } 0 < \beta < 1.$$

LEMMA 2.6. [6] *If x and y are comparable to each other, then $lub\{x, y\}$ and $glb\{x, y\}$ exist,*

$$x - y \propto y - x, \text{ and } \theta \leq (x - y) \vee (y - x).$$

LEMMA 2.7. [6] *Let C be a normal cone with normal constant λ_C in X, then for each $x, y \in X$, we have the relation:*

(i) $\|\theta \oplus \theta\| = \|\theta\| = \theta$;
(ii) $\|x \wedge y\| \leq \|x\| \wedge \|y\| \leq \|x\| + \|y\|$;
(iii) $\|x \oplus y\| \leq \|x - y\| \leq \lambda_C \|x \oplus y\|$;
(iv) *if $x \propto y$, then $\|x \oplus y\| = \|x - y\|$.*

LEMMA 2.8. [20, 25] *Let $\leq$ be a partial order relation defined by the cone C with a normal constant λ_C in X in Definition 2.3. Then here in after relations survive:*

(i) $x \oplus y = y \oplus x,\ x \oplus x = \theta$;
(ii) $\theta \leq x \oplus \theta$;
(iii) *if, λ to be real, then $(\lambda x) \oplus (\lambda y) = |\lambda| (x \oplus y)$;*
(iv) *if x, y and w can be comparative to each other, then $(x \oplus y) \leq (x \oplus w) + (w \oplus y)$;*
(v) *presume $(x + y) \vee (s + t)$ exists, and if $x \propto s, t$ and $y \propto s, t$, then*

$$(x + y) \oplus (s + t) \leq (x \oplus s + y \oplus t) \wedge (x \oplus t + y \oplus s);$$

(vi) *if x, y, r, w can be compared with each other, then*

$$(x \wedge y) \oplus (r \wedge w) \leq ((x \oplus r) \vee (y \oplus w)) \wedge ((x \oplus w) \vee (y \oplus r));$$

(vii) *if $x \leq y$ and $s \leq t$, then $x + s \leq y + t$;*
(viii) *if $x \propto \theta$, then $-x \oplus \theta \leq x \leq x \oplus \theta$;*
(ix) *if $x \propto y$, then $(x \oplus \theta) \oplus (y \oplus \theta) \leq (x \oplus y) \oplus \theta = x \ominus y$;*
(x) $(x \oplus \theta) - (y \oplus \theta) \leq (x - y) \oplus \theta$;
(xi) *if $\theta \leq x$ and $x \neq \theta$, and $\alpha > 0$, then $\theta \leq \alpha x$ and $\alpha x \neq \theta$, for all $x, y, r, s, w \in X$ and $\alpha, \lambda \in \mathbb{R}$.*

DEFINITION 2.9. [25] Let $A : X \longrightarrow X$ be a single valued map. Then

1. A is called γ-order non-extended mapping if there exists a constant $\gamma > 0$ such that

$$\gamma(x \oplus y) \leq A(x) \oplus A(y), \forall x, y \in X;$$

2. A is called a strongly comparison map if it is a comparison mapping and $A(x) \propto A(y)$ iff $x \propto y$, for all $x, y \in X$.

Definition 2.10. [24] Allow $A : X \longrightarrow X$ and $M : X \longrightarrow 2^X$ to be single-valued and set valued mappings, *respectively.*

(i) M is called a weak-comparison map, if for $t_x \in M(x), x \propto t_x$, and if $x \propto y$, then $\exists t_x \in M(x)$ and $t_y \in M(y)$ such that $t_x \propto t_y$, for all $x, y \in X$;

(ii) M is called an α-weak non-ordinary difference map associated with A, if it is weak comparison and for each $x, y \in X, \exists\ \alpha > 0$ and $t_x \in M(A(x))$ and $t_y \in M(A(y))$ such that
$$(t_x \oplus t_y) \oplus \alpha(A(x) \oplus A(y)) = \theta;$$

(iii) M is called a λ-order different weak-comparison map associated with A if $\exists\ \lambda > 0$, for all $x, y \in X$ and there exist $t_x \in M(A(x)), t_y \in M(A(y))$ such that
$$\lambda(t_x - t_y) \propto x - y;$$

(iv) M (a weak-comparison map) is called an ordered (α_A, λ)-weak-ANODM map, if it is an α-weak non-ordinary difference map and a λ-order different weak comparison map associated with A, and $(A + \lambda M)(X) = X$, for $\alpha, \lambda > 0$.

Definition 2.11. [24] Let $A : X \longrightarrow X$ and $M : X \longrightarrow 2^X$ be γ-order non-extended map and an α-non-ordinary difference mapping with respect to A, *respectively.* The resolvent operator $R^M_{A,\lambda} : X \longrightarrow X$ associated with both A and M is defined by

$$R^M_{A,\lambda}(x) = (A + \lambda M)^{-1}(x), \text{for all } x \in X, \tag{2.1}$$

where $\gamma, \alpha, \lambda > 0$ are constants.

Definition 2.12. [26] A map $A : X \times X \longrightarrow X$ is called (α_1, α_2)-restricted-accretive map, if it is comparison and $\exists$ constants $0 \leq \alpha_1, \alpha_2 \leq 1$ such that

$$(A(x, \cdot) + I(x)) \oplus (A(y, \cdot) + I(y)) \leq \alpha_1(A(x, \cdot) \oplus A(y, \cdot)) + \alpha_2(x \oplus y),$$

for all $x, y \in X$, where I is the identity map on X.

Lemma 2.13. [24] *If $M : X \longrightarrow 2^X$ and $A : X \longrightarrow X$ are an α-weak-non-ordinary difference map associated with A and γ-order non-extended map, respectively, with $\alpha\gamma \neq 1$, then $M_\theta = \{\theta \oplus x \mid x \in M\}$ is an α-weak-non-ordinary difference map associated with A and the resolvent* *operator* $R^{M_\theta}_{A,\lambda} = (A+\lambda M_\theta)^{-1}$ *of* $(A+\lambda M_\theta)$ *is single-valued for* $\alpha, \lambda > 0$, *i.e.,* $R^{M_\theta}_{A,\lambda} : X \longrightarrow X$ *of* M_θ *holds.*

LEMMA 2.14. [24] *Let* $M : X \longrightarrow 2^X$ *and* $A : X \longrightarrow X$ *be* (α_A, λ)-*weak-ANODD set-valued map and strongly comparison map associated with* $R^M_{A,\lambda}$, *respectively. Then the resolvent operator* $R^M_{A,\lambda} : X \longrightarrow X$ *is a comparison map.*

LEMMA 2.15. [24] *Let* $M : X \longrightarrow 2^X$ *be an ordered* (α_A, λ)-*weak-ANODD map and* $A : X \longrightarrow X$ *be* γ-*ordered non-extended map associated with* $R^M_{A,\lambda}$, *for* $\alpha_A > \frac{1}{\lambda}$, *respectively. Then the following relation holds:*

$$R^M_{A,\lambda}(x) \oplus R^M_{A,\lambda}(y) \leq \frac{1}{\gamma(\alpha_A\lambda - 1)}(x \oplus y), \text{for all } x, y \in X. \tag{2.2}$$

3. Formulation of the problem

Allow X to be a real ordered Banach space and C a normal cone having the normal constant λ_C. Let $M_i : X \times X \longrightarrow 2^X (i = 1, 2, 3)$ be set valued mappings. Suppose $f_i, g_i : X \longrightarrow X(i = 1, 2, 3)$ and $F_i : X \times X \longrightarrow X(i = 1, 2, 3)$ are single valued mappings. Now, consider a *system of generalized nonlinear ordered variational inclusion problems* for finding $(x, y, z) \in X \times X \times X$ such that

$$\begin{aligned} &w_1 \in F_1(f_1(x), y) + \rho_1 M_1(g_1(x), y),\\ &w_2 \in F_2(f_2(y), z) + \rho_2 M_2(g_2(y), z),\\ &w_3 \in F_3(f_3(z), x) \oplus M_3(g_3(z), x), \end{aligned} \tag{3.1}$$

where $\rho_1, \rho_2 > 0$ and $(w_1, w_2, w_3) \in X \times X \times X$.

Special Cases:

1. If $(i = 1, 2), \rho_1 = \rho_2 = 1, g_1 = g_2 = f_2 = f_3 = I$ (the identity mapping on X), M_1 and $M_3 = M_2$ are single valued mappings and $M(g_1(x), y) = M(x, y)$, then problem (3.1) reduces to the problem for $w_1, w_2 \in X$, find $x, y \in X$ such that

$$\begin{aligned} &w_1 \leq F_1(f_1(x), y) + M_1(x, y),\\ &w_2 \leq F_2(x, y) \oplus M_2(x, y). \end{aligned} \tag{3.2}$$

Problem (3.2) was variant form of [20].

2. If $i = 1, w_1 = 0, \rho_1 = 1$, M_1 is a single-valued mapping, then problem (3.2) is to find $x, y \in X$ such that

$$0 \leq F_1(f_1(x), y) + M_1(x, y). \tag{3.3}$$

Problem (3.3) was initiated and studied by [32].

3. If $w_2 = w_3 = 0, F_2 = F_3 = f_2 = f_3 = M_2 = M_3 = g_2 = g_3 = 0, g_1 = I, F_1(f_1(x), y) = f(x), \rho_1 = \rho$ and $M_1(g_1(x), y) = M(x)$, then problem (3.1) became the problem to find $x \in X$ such that

$$w_1 \in f(x) + \rho M(x). \tag{3.4}$$

Problem (3.4) was initiated and studied by [24].

4. If $\rho_1 = \rho_2, w_1 = w_2 = 0, F_1 = F_2 = f_1 = f_2 = g_1 = g_2 = M_1 = M_2 = 0, f_3 = g_3 = I, F_3(f_3(x), y) = F(x)$ and $M_3(g_3(x), y) = M(x), w_3 = w$, then problem (3.1) is converted to the problem of finding $x \in X$ such that

$$w \in F(x) \oplus M(x). \tag{3.5}$$

Problem (3.5) was initiated and studied by [23].

5. If $F_1 = F_2 = F_3 = f_1 = f_2 = f_3 = M_1 = M_2 = g_1 = g_2 = 0, w_2 = w_3 = 0, g_1 = I$ and $M_1(g_1(x), y) = M(x), \rho_1 = \rho, w_1 = w$, then the problem (3.1) converted to the problem of finding $x \in X$ such that

$$w \in \rho M(x). \tag{3.6}$$

Problem (3.6) was initiated and studied by [22].

Now, we mention the fixed point formulation of (3.1).

LEMMA 3.1. *The set of elements $(x, y, z) \in X \times X \times X$ is a solution of (3.1) if and only if $(x, y, z) \in X \times X \times X$ satisfying the following relations:*

$$\begin{aligned}
x &= R_{A,\lambda_1}^{M_1(g_1(\cdot),y)}[A(x) + \frac{\lambda_1}{\rho_1}(w_1 - F_1(f_1(x), y))], \\
y &= R_{A,\lambda_2}^{M_2(g_2(\cdot),z)}[A(y) + \frac{\lambda_2}{\rho_2}(w_2 - F_2(f_2(y), z))], \\
z &= R_{A,\lambda_3}^{M_3(g_3(\cdot),x)}[A(z) + \lambda_3(w_3 \oplus F_3(f_3(z), x))].
\end{aligned} \tag{3.7}$$

Proof. The proof follows from the definition of the resolvent operator (2.1). □

4. Main results

In this section, we present an existence result for the *system of generalized nonlinear ordered variational inclusions*, under some suitable conditions. Also, discuss convergence of the sequence suggested by an iterative algorithm.

THEOREM 4.1. *Let C be a normal cone having a normal constant λ_C in a real ordered Banach space X. Let $A, f_i, g_i : X \longrightarrow X$ $(i = 1, 2, 3)$ be single-valued mappings such that A is a λ_A-compression mapping, f_i are λ_{f_i}-compression and g_i are comparison mappings for $(i = 1, 2, 3)$,* respectively. *Let $F_i : X \times X \longrightarrow X$ $(i = 1, 2, 3)$ be single-valued mappings such that F_1 is an (α_1, α_2)-restricted-accretive mapping w.r.t. f_1; F_2 is an (β_1, β_2)-restricted accretive mapping w.r.t. f_2 and F_3 is an (σ_1, σ_2)-restricted accretive mapping w.r.t. f_3,* respectively. *Suppose $M_i : X \times X \longrightarrow 2^X$ are the set valued mappings such that M_i is a (α_A, λ_i)-weak-ANODD set valued mapping for $(i = 1, 2, 3)$,* respectively. *In addition, if $x_i \propto y_i, y_i \propto z_i, z_i \propto x_i, R^{M_1}_{A,\lambda_1}(x_i) \propto R^{M_2}_{A,\lambda_2}(y_i), R^{M_2}_{A,\lambda_2}(y_i) \propto R^{M_3}_{A,\lambda_3}(z_i), R^{M_3}_{A,\lambda_3}(z_i) \propto R^{M_1}_{A,\lambda_1}(x_i)(i = 1, 2)$ and for all $\lambda_1, \lambda_2, \lambda_3, \delta_1, \delta_2, \delta_3 > 0$, the following conditions are satisfied:*

$$\begin{aligned} &R^{M_1(g_1(\cdot),y_1)}_{A,\lambda_1}(x_1) \oplus R^{M_1(g_1(\cdot),y_2)}_{A,\lambda_1}(x_1) \le \delta_2(y_1 \oplus y_2);\\ &R^{M_2(g_2(\cdot),z_1)}_{A,\lambda_2}(y_1) \oplus R^{M_2(g_2(\cdot),z_2)}_{A,\lambda_2}(y_1) \le \delta_3(z_1 \oplus z_2)\\ &R^{M_3(g_3(\cdot),x_1)}_{A,\lambda_3}(z_1) \oplus R^{M_3(g_3(\cdot),x_1)}_{A,\lambda_3}(z_2) \le \delta_1(x_1 \oplus x_2), \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\lambda_C(\mu_1\lambda_A + \mu_3\lambda_3\sigma_1 + \delta_1) < 1 - \frac{\lambda_C\mu_1\lambda_1\alpha_1\lambda_{f_1}}{\rho_1};\\ &\lambda_C(\mu_2\lambda_A + \delta_2) < 1 - \frac{\lambda_C(\mu_2\lambda_2\beta_1\rho_1\lambda_{f_2} + \mu_1\lambda_1\alpha_2\rho_2)}{\rho_1\rho_2};\\ &\lambda_C(\mu_3\lambda_A + \mu_3\lambda_3\sigma_1\lambda_{f_3} + \delta_3) < 1 - \frac{\lambda_C\mu_2\lambda_2\beta_2}{\rho_2}. \end{aligned} \tag{4.2}$$

Then the (3.1) *grants a solution $(x, y, z) \in X \times X \times X$.*

Proof. From Lemma 2.15, we know that the resolvent operator $R^{M_1}_{A,\lambda_1}(\cdot)$, $R^{M_2}_{A,\lambda_2}(\cdot)$ and $R^{M_3}_{A,\lambda_3}(\cdot)$ are μ_1-Lipschitz continuous, μ_2-Lipschitz continuous and μ_3-Lipschitz continuous, *respectively.* Here $\mu_1 = \frac{1}{\gamma_1(\alpha_A\lambda_1 - 1)}$,

$\mu_2 = \frac{1}{\gamma_2(\alpha_A \lambda_2 - 1)}$ and $\mu_3 = \frac{1}{\gamma_3(\alpha_A \lambda_3 - 1)}$.
Now, define a map $P : X \times X \times X \longrightarrow X \times X$ by

$$P(x, y, z) = (T(x, y), S(y, z), G(z, x)), \forall (x, y, z) \in X \times X \times X, \tag{4.3}$$

where $T, S, G : X \times X \longrightarrow X$ are mappings defined as

$$T(x, y) = R_{A,\lambda_1}^{M_1(g_1(\cdot),y)}[A(x) + \frac{\lambda_1}{\rho_1}(w_1 - F_1(f_1(x), y))], \tag{4.4}$$

$$S(y, z) = R_{A,\lambda_2}^{M_2(g_2(\cdot),z)}[A(y) + \frac{\lambda_2}{\rho_2}(w_2 - F_2(f_2(y), z))], \tag{4.5}$$

$$G(z, x) = R_{A,\lambda_3}^{M_3(g_3(\cdot),x)}[A(z) + \lambda_3(w_3 \oplus F_3(f_3(z), x))]. \tag{4.6}$$

For any $x_i, y_i \in X$ and $x_i \propto y_j, y_i \propto x_j (i, j = 1, 2)$. Using (4.4), Definition 2.5, Definition 2.12, Lemma 2.15 and Lemma 2.8, we have

$$\begin{aligned}
0 &\leq T(x_1, y_1) \oplus T(x_2, y_2) \\
&= R_{A,\lambda_1}^{M_1(g_1(\cdot),y_1)}[A(x_1) + \frac{\lambda_1}{\rho_1}(w_1 - F_1(f_1(x_1), y_1))] \\
&\quad \oplus R_{A,\lambda_1}^{M_1(g_1(\cdot),y_2)}[A(x_2) + \frac{\lambda_1}{\rho_1}(w_1 - F_1(f_1(x_2), y_2))] \\
&\leq R_{A,\lambda_1}^{M_1(g_1(\cdot),y_1)}[A(x_1) + \frac{\lambda_1}{\rho_1}(w_1 - F_1(f_1(x_1), y_1))] \\
&\quad \oplus R_{A,\lambda_1}^{M_1(g_1(\cdot),y_1)}[A(x_2) + \frac{\lambda_1}{\rho_1}(w_1 - F_1(f_1(x_2), y_2))] \\
&\quad \oplus R_{A,\lambda_1}^{M_1(g_1(\cdot),y_1)}[A(x_2) + \frac{\lambda_1}{\rho_1}(w_1 - F_1(f_1(x_2), y_2))] \\
&\quad \oplus R_{A,\lambda_1}^{M_1(g_1(\cdot),y_2)}[A(x_2) + \frac{\lambda_1}{\rho_1}(w_1 - F_1(f_1(x_2), y_2))] \\
&\leq \mu_1[A(x_1) \oplus A(x_2) + \frac{\lambda_1}{\rho_1}(F_1(f_1(x_1), y_1) \oplus F_1(f_1(x_2), y_2))] \oplus \delta_2(y_1 \oplus y_2) \\
&\leq \mu_1[A(x_1) \oplus A(x_2) + \frac{\lambda_1}{\rho_1}(\alpha_1(f_1(x_1) \oplus f_1(x_2)) + \alpha_2(y_1 \oplus y_2))] \oplus \delta_2(y_1 \oplus y_2) \\
&\leq \mu_1[\lambda_A(x_1 \oplus x_2) + \frac{\lambda_1}{\rho_1}(\alpha_1 \lambda_{f_1}(x_1 \oplus x_2) + \alpha_2(y_1 \oplus y_2))] \oplus \delta_2(y_1 \oplus y_2)
\end{aligned}$$

$$\le \mu_1[(\lambda_A + \frac{\lambda_1\alpha_1\lambda_{f_1}}{\rho_1})(x_1 \oplus x_2) + \frac{\lambda_1\alpha_2}{\rho_1}(y_1 \oplus y_2)] \oplus \delta_2(y_1 \oplus y_2)$$

$$\le [\frac{\mu_1(\lambda_A\rho_1 + \lambda_1\alpha_1\lambda_{f_1})}{\rho_1}(x_1 \oplus x_2) + \frac{\mu_1\lambda_1\alpha_2 + \rho_1\delta_2}{\rho_1}(y_1 \oplus y_2)]. \tag{4.7}$$

From Definition 2.2 and Lemma 2.7, we have

$$\begin{aligned}
&\|T(x_1, y_1) \oplus T(x_2, y_2)\| = \|T(x_1, y_1) - T(x_2, y_2)\| \\
&\le \lambda_C\|\frac{\mu_1(\lambda_A\rho_1 + \lambda_1\alpha_1\lambda_{f_1})}{\rho_1}(x_1 \oplus x_2) + \frac{\mu_1\lambda_1\alpha_2 + \rho_1\delta_2}{\rho_1}(y_1 \oplus y_2)\| \\
&\le \lambda_C\|\frac{\mu_1(\lambda_A\rho_1 + \lambda_1\alpha_1\lambda_{f_1})}{\rho_1}(x_1 \oplus x_2)\| + \lambda_C\|\frac{\mu_1\lambda_1\alpha_2 + \rho_1\delta_2}{\rho_1}(y_1 \oplus y_2)\| \\
&\le \lambda_C\mu_1\frac{(\lambda_A\rho_1 + \lambda_1\alpha_1\lambda_{f_1})}{\rho_1}\|x_1 - x_2\| + \lambda_C\frac{(\mu_1\lambda_1\alpha_2 + \rho_1\delta_2)}{\rho_1}\|y_1 - y_2\|.
\end{aligned}$$

That is,

$$\begin{aligned}
\|T(x_1, y_1) - T(x_2, y_2)\| \le \lambda_C\mu_1&\frac{(\lambda_A\rho_1 + \lambda_1\alpha_1\lambda_{f_1})}{\rho_1}\|x_1 - x_2\| \\
&+ \lambda_C\frac{(\mu_1\lambda_1\alpha_2 + \rho_1\delta_2)}{\rho_1}\|y_1 - y_2\|.
\end{aligned} \tag{4.8}$$

For any $y_i, z_i \in X, y_i \propto z_j, z_i \propto y_j (i, j = 1, 2)$. From (4.5), Definition 2.5, Definition 2.12, Lemma 2.8 and Lemma 2.15, we have

$$\begin{aligned}
0 &\le S(y_1, z_1) \oplus S(y_2, z_2) \\
&= R_{A,\lambda_2}^{M_2(g_2(\cdot),z_1)}[A(y_1) + \frac{\lambda_2}{\rho_2}(w_2 - F_2(f_2(y_1), z_1))] \\
&\quad \oplus R_{A,\lambda_2}^{M_2(g_2(\cdot),z_2)}[A(y_2) + \frac{\lambda_2}{\rho_2}(w_2 - F_2(f_2(y_2), z_2))] \\
&\le R_{A,\lambda_2}^{M_2(g_2(\cdot),z_1)}[A(y_1) + \frac{\lambda_2}{\rho_2}(w_2 - F_2(f_2(y_1), z_1))] \\
&\quad \oplus R_{A,\lambda_2}^{M_2(g_2(\cdot),z_1)}[A(y_2) + \frac{\lambda_2}{\rho_2}(w_2 - F_2(f_2(y_2), z_2))] \\
&\quad \oplus R_{A,\lambda_2}^{M_2(g_2(\cdot),z_1)}[A(y_2) + \frac{\lambda_2}{\rho_2}(w_2 - F_2(f_2(y_2), z_2))] \\
&\quad \oplus R_{A,\lambda_2}^{M_2(g_2(\cdot),z_2)}[A(y_2) + \frac{\lambda_2}{\rho_2}(w_2 - F_2(f_2(y_2), z_2))]
\end{aligned}$$

$$
\begin{aligned}
&\leq \mu_2[A(y_1) \oplus A(y_2) + \frac{\lambda_2}{\rho_2}(F_2(f_2(y_1), z_1) \oplus F_2(f_2(y_2), z_2))] \oplus \delta_3(z_1 \oplus z_2) \\
&\leq \mu_2[A(y_1) \oplus A(y_2) + \frac{\lambda_2}{\rho_2}(\beta_1(f_1(y_1) \oplus f_2(y_2)) + \beta_2(z_1 \oplus z_2))] \oplus \delta_3(z_1 \oplus z_2) \\
&\leq \mu_2[\lambda_A(y_1 \oplus y_2) + \frac{\lambda_2}{\rho_2}(\beta_1 \lambda_{f_2}(y_1 \oplus y_2) + \beta_2(z_1 \oplus z_2))] \oplus \delta_3(z_1 \oplus z_2) \\
&\leq \mu_2[(\lambda_A + \frac{\lambda_2 \beta_1 \lambda_{f_2}}{\rho_2})(y_1 \oplus y_2) + \frac{\lambda_2 \beta_2}{\rho_2}(z_1 \oplus z_2)] \oplus \delta_3(z_1 \oplus z_2) \\
&\leq [\frac{\mu_2(\lambda_A \rho_2 + \lambda_2 \beta_1 \lambda_{f_2})}{\rho_2}(y_1 \oplus y_2) + \frac{\mu_2 \lambda_2 \beta_2 + \rho_2 \delta_3}{\rho_2}(z_1 \oplus z_2)].
\end{aligned}
\tag{4.9}
$$

From Definition 2.2 and Lemma 2.7, we have

$$
\begin{aligned}
&\|S(y_1, z_1) \oplus S(y_2, z_2)\| = \|S(y_1, z_1) - S(y_2, z_2)\| \\
&\leq \lambda_C \|\frac{\mu_2(\lambda_A \rho_2 + \lambda_2 \beta_1 \lambda_{f_2})}{\rho_2}(y_1 \oplus y_2) + \frac{\mu_2 \lambda_2 \beta_2 + \rho_2 \delta_3}{\rho_2}(z_1 \oplus z_2)\| \\
&\leq \lambda_C \|\frac{\mu_2(\lambda_A \rho_2 + \lambda_2 \beta_1 \lambda_{f_2})}{\rho_2}(y_1 \oplus y_2)\| + \lambda_C \|\frac{\mu_2 \lambda_2 \beta_2 + \rho_2 \delta_3}{\rho_2}(z_1 \oplus z_2)\| \\
&\leq \lambda_C \mu_2 \frac{(\lambda_A \rho_2 + \lambda_2 \beta_1 \lambda_{f_2})}{\rho_2}\|y_1 - y_2\| + \lambda_C \frac{(\mu_2 \lambda_2 \beta_2 + \rho_2 \delta_3)}{\rho_2}\|z_1 - z_2\|.
\end{aligned}
$$

That is,

$$
\begin{aligned}
\|S(y_1, z_1) - S(y_2, z_2)\| \leq{}& \lambda_C \mu_2 \frac{(\lambda_A \rho_2 + \lambda_2 \beta_1 \lambda_{f_2})}{\rho_2}\|y_1 - y_2\| \\
&+ \lambda_C \frac{(\mu_2 \lambda_2 \beta_2 + \rho_2 \delta_3)}{\rho_2}\|z_1 - z_2\|.
\end{aligned}
\tag{4.10}
$$

For any $x_i, z_i \in X$ and $x_i \propto z_j, z_i \propto x_j, (i, j = 1, 2)$. Using (4.6), Definition 2.5, Definition 2.9, Lemma 2.15 and Lemma 2.8, we have

$$
\begin{aligned}
0 &\leq G(z_1, x_1) \oplus G(z_2, x_2) \\
&= R_{A,\lambda_3}^{M_3(g_3(\cdot), x_1)}[A(z_1) + \lambda_3(w_3 \oplus F_3(f_3(z_1), x_1))] \\
&\quad \oplus R_{A,\lambda_3}^{M_3(g_3(\cdot), x_2)}[A(z_2) + \lambda_3(w_3 \oplus F_3(f_3(z_2), x_2))]
\end{aligned}
$$

$$
\begin{aligned}
&\leq R_{A,\lambda_3}^{M_3(g_3(\cdot),x_1)}[A(z_1) + \lambda_3(w_3 \oplus F_3(f_3(z_1), x_1))] \\
&\qquad \oplus R_{A,\lambda_3}^{M_3(g_3(\cdot),x_1)}[A(z_2) + \lambda_3(w_3 \oplus F_3(f_3(z_2), x_2))] \\
&\qquad \oplus R_{A,\lambda_3}^{M_3(g_3(\cdot),x_1)}[A(z_2) + \lambda_3(w_3 \oplus F_3(f_3(z_2), x_2))] \\
&\qquad \oplus R_{A,\lambda_3}^{M_3(g_3(\cdot),x_2)}[A(z_2) + \lambda_3(w_3 \oplus F_3(f_3(z_2), x_2))] \\
&\leq \mu_3[A(z_1) \oplus A(z_2) + \lambda_3(F_3(f_3(z_1), x_1) \oplus F_3(f_3(z_2), x_2))] \oplus \delta_1(x_1 \oplus x_2) \\
&\leq \mu_3[A(z_1) \oplus A(z_2) + \lambda_3(\sigma_1(f_3(z_1) \oplus f_3(z_2)) + \sigma_2(x_1 \oplus x_2))] \oplus \delta_1(x_1 \oplus x_2) \\
&\leq \mu_3[\lambda_A(z_1 \oplus z_2) + \lambda_3(\sigma_1\lambda_{f_3}(z_1 \oplus z_2) + \sigma_2(x_1 \oplus x_2))] \oplus \delta_1(x_1 \oplus x_2) \\
&\leq \mu_3[(\lambda_A + \lambda_3\sigma_1\lambda_{f_3})(z_1 \oplus z_2) + \lambda_3\sigma_2(x_1 \oplus x_2)] \oplus \delta_1(x_1 \oplus x_2) \\
&\leq [\mu_3(\lambda_A + \lambda_3\sigma_1\lambda_{f_3})(z_1 \oplus z_2) + (\mu_3\lambda_3\sigma_2 + \delta_1)(x_1 \oplus x_2)].
\end{aligned}
\tag{4.11}
$$

Using Definition 2.2 and Lemma 2.7, we obtain

$$
\begin{aligned}
&\|G(z_1, x_1) \oplus G(z_2, x_2)\| = \|G(z_1, x_1) - G(z_2, x_2)\| \\
&\leq \lambda_C\|\mu_3(\lambda_A + \lambda_3\sigma_1\lambda_{f_3})(z_1 \oplus z_2) + (\mu_3\lambda_3\sigma_2 + \delta_1)(x_1 \oplus x_2)\| \\
&\leq \lambda_C\|\mu_3(\lambda_A + \lambda_3\sigma_1\lambda_{f_3})(z_1 \oplus z_2)\| + \lambda_C\|(\mu_3\lambda_3\sigma_2 + \delta_1)(x_1 \oplus x_2)\| \\
&\leq \lambda_C\mu_3(\lambda_A + \lambda_3\sigma_1\lambda_{f_3})\|z_1 - z_2\| + \lambda_C(\mu_3\lambda_3\sigma_2 + \delta_1)\|x_1 - x_2\|.
\end{aligned}
$$

That is,

$$
\begin{aligned}
\|G(z_1, x_1) - G(z_2, x_2)\| &\leq \lambda_C\mu_3(\lambda_A + \lambda_3\sigma_1\lambda_{f_3})\|z_1 - z_2\| \\
&\quad + \lambda_C(\mu_3\lambda_3\sigma_2 + \delta_1)\|x_1 - x_2\|.
\end{aligned}
\tag{4.12}
$$

From (4.8), (4.10) and (4.12), we have

$$
\begin{aligned}
&\|T(x_1, y_1) - T(x_2, y_2)\| + \|S(y_1, z_1) - S(y_2, z_2)\| + \|G(z_1, x_1) - G(z_2, x_2)\| \\
&\leq \lambda_C\mu_1\frac{(\lambda_A\rho_1 + \lambda_1\alpha_1\lambda_{f_1})}{\rho_1}\|x_1 - x_2\| + \lambda_C\frac{(\mu_1\lambda_1\alpha_2 + \rho_1\delta_2)}{\rho_1}\|y_1 - y_2\| \\
&\quad + \lambda_C\mu_2\frac{(\lambda_A\rho_2 + \lambda_2\beta_1\lambda_{f_2})}{\rho_2}\|y_1 - y_2\| + \lambda_C\frac{(\mu_2\lambda_2\beta_2 + \rho_2\delta_3)}{\rho_2}\|z_1 - z_2\| \\
&\quad + \lambda_C\mu_3(\lambda_A + \lambda_3\sigma_1\lambda_{f_3})\|z_1 - z_2\| + \lambda_C(\mu_3\lambda_3\sigma_2 + \delta_1)\|x_1 - x_2\|
\end{aligned}
$$

$$\leq \lambda_C[\frac{\mu_1(\lambda_A\rho_1+\lambda_1\alpha_1\lambda_{f_1})}{\rho_1}+(\mu_3\lambda_3\sigma_2+\delta_1)]\|x_1-x_2\|$$
$$+\lambda_C[\frac{\mu_2(\lambda_A\rho_2+\lambda_2\beta_1\lambda_{f_2})}{\rho_2}+\frac{(\mu_1\lambda_1\alpha_2+\rho_1\delta_2)}{\rho_1}]\|y_1-y_2\|$$
$$+\lambda_C[\mu_3(\lambda_A+\lambda_3\sigma_1\lambda_{f_3})+\frac{(\mu_2\lambda_2\beta_2+\rho_2\delta_3)}{\rho_2}]\|z_1-z_2\|$$
$$\leq \frac{\lambda_C(\mu_1(\lambda_A\rho_1+\lambda_1\alpha_1\lambda_{f_1})+\rho_1(\mu_3\lambda_3\sigma_1+\delta_1))}{\rho_1}\|x_1-x_2\|$$
$$+\frac{\lambda_C(\mu_2\rho_1(\lambda_A\rho_2+\lambda_2\beta_1\lambda_{f_2})+\rho_2(\mu_1\lambda_1\alpha_2+\delta_2\rho_1))}{\rho_1\rho_2}\|y_1-y_2\|$$
$$+\frac{\lambda_C(\mu_3\rho_2(\lambda_A+\lambda_3\sigma_1\lambda_{f_3})+(\mu_2\lambda_2\beta_2+\delta_3\rho_2))}{\rho_2}\|z_1-z_2\|$$
$$=\Omega_1\|x_1-x_2\|+\Omega_2\|y_1-y_2\|+\Omega_3\|z_1-z_2\|$$
$$\leq \max\{\Omega_1,\Omega_2,\Omega_3\}(\|x_1-x_2\|+\|y_1-y_2\|+\|z_1-z_2\|), \tag{4.13}$$

where
$$\Omega_1=\frac{\lambda_C(\mu_1(\lambda_A\rho_1+\lambda_1\alpha_1\lambda_{f_1})+\rho_1(\mu_3\lambda_3\sigma_1+\delta_1))}{\rho_1};$$
$$\Omega_2=\frac{\lambda_C(\mu_2\rho_1(\lambda_A\rho_2+\lambda_2\beta_1\lambda_{f_2})+\rho_2(\mu_1\lambda_1\alpha_2+\delta_2\rho_1))}{\rho_1\rho_2}$$
and
$$\Omega_3=\frac{\lambda_C(\mu_3\rho_2(\lambda_A+\lambda_3\sigma_1\lambda_{f_3})+(\mu_2\lambda_2\beta_2+\delta_3\rho_2))}{\rho_2}.$$
Now, we define $\|(x,y,z)\|_*$ on $X\times X\times X$ by
$$\|(x,y,z)\|_*=\|x\|+\|y\|+\|z\|, \forall (x,y,z)\in X\times X\times X. \tag{4.14}$$
One can easily show that $(X\times X\times X, \|\cdot\|)$ is a Banach space. Hence from (4.3), (4.13) and (4.14), we have
$$\|P(x_1,y_1,z_1)-P(x_2,y_2,z_2)\|_* \leq \max\{\Omega_1,\Omega_2,\Omega_3\}(\|x_1-x_2\|+\|y_1-y_2\|+\|z_1-z_2\|). \tag{4.15}$$
From (4.2), we know that $\max\{\Omega_1,\Omega_2,\Omega_3\}<1$. It follows from (4.15) that P is a contraction mapping. Hence there exists unique $(x,y,z)\in X\times X\times X$ such that
$$P(x,y,z)=(x,y,z).$$

This leads to

$$x = R^{M_1(g_1(\cdot),y)}_{A,\lambda_1}[A(x) + \frac{\lambda_1}{\rho_1}(w_1 - F_1(f_1(x), y))],$$

$$y = R^{M_2(g_2(\cdot),z)}_{A,\lambda_2}[A(y) + \frac{\lambda_2}{\rho_2}(w_2 - F_2(f_2(y), z))],$$

$$z = R^{M_3(g_3(\cdot),x)}_{A,\lambda_3}[A(z) + \lambda_3(w_3 \oplus F_3(f_3(z), x))].$$

It is determined by Lemma 3.1 that (x, y, z) is a solution of (3.1). □

Now, we construct an iterative scheme for problem (3.1).

ALGORITHM 4.2. Let C be a normal cone with a normal constant λ_C in a real ordered Banach space X. Let $M_i : X \times X \longrightarrow 2^X$ $(i = 1, 2, 3)$ be set valued mappings. Assume that $f_i, g_i : X \longrightarrow X$ and $F_i : X \times X \longrightarrow X$ are single-valued mappings for $i = 1, 2, 3$. For the initial guess $(x_0, y_0, z_0) \in X \times X \times X$, assume that $x_0 \propto x_1, y_0 \propto y_1, z_0 \propto z_1$. We define an iterative sequence $\{(x_n, y_n, z_n)\}$ and let $x_{n+1} \propto x_n, y_{n+1} \propto y_n, z_{n+1} \propto z_n$. such that

(4.16)
$$x_{n+1} = \pi_n x_n + (1 - \pi_n)R^{M_1(g_1(\cdot),y_n)}_{A,\lambda_1}[A(x_n) + \frac{\lambda_1}{\rho_1}(w_1 - F_1(f_1(x_n), y_n))],$$

(4.17)
$$y_{n+1} = \pi_n y_n + (1 - \pi_n)R^{M_2(g_2(\cdot),z_n)}_{A,\lambda_2}[A(y_n) + \frac{\lambda_2}{\rho_2}(w_2 - F_2(f_2(y_n), z_n))],$$

(4.18)
$$z_{n+1} = \pi_n z_n + (1 - \pi_n)R^{M_3(g_3(\cdot),x_n)}_{A,\lambda_3}[A(z_n) + \lambda_3(w_3 \oplus F_3(f_3(z_n), x_n))].$$

For $n = 0, 1, 2, ...,$ where $0 \leq \pi_n < 1$ with $\limsup_n \pi_n < 1$.

LEMMA 4.3. *[10] Allow $\{\vartheta_n\}$ and ς_n to be sequences of nonnegative real numbers such that they satisfy*

(i) $0 \leq \varsigma_n < 1, n = 0, 1, 2, \cdots$ *and* $\limsup_n \varsigma_n < 1$;
(ii) $\vartheta_{n+1} \leq \varsigma_n \vartheta_n, n = 0, 1, 2, 3, \cdots$.

Then $\{\vartheta_n\}$ approaches zero as n tends to ∞.

THEOREM 4.4. *Allow X, C, M_i, f_i, g_i, F_i $(i = 1, 2, 3)$ to be as in Theorem 4.1 such that all the assertions of Theorem 4.1 are valid. Then the sequence $\{(x_n, y_n, z_n)\}$ formulated by Algorithm 4.2, converges strongly to the unique solution $\{(x, y, z)\}$ of (3.1).*

Proof. From Theorem 4.1, the system (3.1) admits a unique solution (x, y, z). It follows from Lemma 3.1 that

$$x = \pi_n x + (1-\pi_n) R_{A,\lambda_1}^{M_1(g_1(\cdot),y)}[A(x) + \frac{\lambda_1}{\rho_1}(w_1 - F_1(f_1(x), y))], \tag{4.19}$$

$$y = \pi_n y + (1-\pi_n) R_{A,\lambda_2}^{M_2(g_2(\cdot),z)}[A(y) + \frac{\lambda_2}{\rho_2}(w_2 - F_2(f_2(y), z))], \tag{4.20}$$

and

$$z = \pi_n z + (1-\pi_n) R_{A,\lambda_3}^{M_3(g_3(\cdot),x)}[A(z) + \lambda_3(w_3 \oplus F_3(f_3(z), x))]. \tag{4.21}$$

From (4.16), (4.19) and Lemma 2.8, we get

$$\begin{aligned}
0 &\le x_{n+1} \oplus x \\
&= \pi_n x_n + (1-\pi_n) R_{A,\lambda_1}^{M_1(g_1(\cdot),y_n)}[A(x_n) + \frac{\lambda_1}{\rho_1}(w_1 - F_1(f_1(x_n), y_n))] \\
&\quad \oplus \pi_n x + (1-\pi_n) R_{A,\lambda_1}^{M_1(g_1(\cdot),y)}[A(x) + \frac{\lambda_1}{\rho_1}(w_1 - F_1(f_1(x), y))] \\
&= \pi_n (x_n \oplus x) + (1-\pi_n)[R_{A,\lambda_1}^{M_1(g_1(\cdot),y_n)}[A(x_n) + \frac{\lambda_1}{\rho_1}(w_1 - F_1(f_1(x_n), y_n))] \\
&\quad \oplus R_{A,\lambda_1}^{M_1(g_1(\cdot),y)}[A(x) + \frac{\lambda_1}{\rho_1}(w_1 - F_1(f_1(x), y))]].
\end{aligned} \tag{4.22}$$

Using the same argument as in Theorem 4.1, for (4.7), we have

$$\begin{aligned}
\|x_{n+1} \oplus x\| &= \|x_{n+1} - x\| \\
&\le \pi_n \|x_n - x\| + (1-\pi_n)[\lambda_C \mu_1 \frac{(\lambda_A \rho_1 + \lambda_1 \alpha_1 \lambda_{f_1})}{\rho_1} \|x_n - x\| \\
&\quad + \lambda_C \frac{(\mu_1 \lambda_1 \alpha_2 + \rho_1 \delta_2)}{\rho_1} \|y_n - y\|] \\
&\le [\pi_n + \frac{(1-\pi_n)\lambda_C \mu_1 (\lambda_A \rho_1 + \lambda_1 \alpha_1 \lambda_{f_1})}{\rho_1}] \|x_n - x\| \\
&\quad + [\frac{(1-\pi_n)\lambda_C (\mu_1 \lambda_1 \alpha_2 + \rho_1 \delta_2)}{\rho_1}] \|y_n - y\|.
\end{aligned} \tag{4.23}$$

Similarly, it follows from (4.17) and (4.20) that

$$
\begin{aligned}
0 &\le y_{n+1} \oplus y \\
&= (\pi_n y_n + (1-\pi_n) R_{A,\lambda_2}^{M_2(g_2(\cdot),z_n)}[A(y_n) + \frac{\lambda_2}{\rho_2}(w_2 - F_2(f_2(y_n), z_n))] \\
&\qquad \oplus \pi_n y + (1-\pi_n) R_{A,\lambda_2}^{M_2(g_2(\cdot),z)}[A(y) + \frac{\lambda_2}{\rho_2}(w_2 - F_2(f_2(y), z))]) \\
&= \pi_n(y_n \oplus y) + (1-\pi_n)(R_{A,\lambda_2}^{M_2(g_2(\cdot),z_n)}[A(y_n) + \frac{\lambda_2}{\rho_2}(w_2 - F_2(f_2(y_n), z_n))] \\
&\qquad \oplus R_{A,\lambda_2}^{M_2(g_2(\cdot),z)}[A(y) + \frac{\lambda_2}{\rho_2}(w_2 - F_2(f_2(y), z))])
\end{aligned}
\tag{4.24}
$$

.

Importing the same logic as in Theorem 4.1 for (4.9), we have

$$
\begin{aligned}
\|y_{n+1} \oplus y\| = \|y_{n+1} - y\| &\le [\pi_n + \frac{(1-\pi_n)\lambda_C \mu_2(\lambda_A \rho_2 + \lambda_2 \beta_1 \lambda_{f_2})}{\rho_2}]\|y_n - y\| \\
&\quad + \frac{(1-\pi)\lambda_C(\mu_2 \lambda_2 \beta_2 + \rho_2 \delta_3)}{\rho_2}\|z_n - z\|.
\end{aligned}
\tag{4.25}
$$

Similarly, it follows from (4.18) and (4.21) that

$$
\begin{aligned}
0 &\le z_{n+1} \oplus z \\
&= \pi_n z_n + (1-\pi_n) R_{A,\lambda_3}^{M_3(g_3(\cdot),x_n)}[A(z_n) + \lambda_3(w_3 \oplus F_3(f_3(z_n), x_n))] \\
&\qquad \oplus \pi_n z + (1-\pi_n) R_{A,\lambda_3}^{M_3(g_3(\cdot),x)}[A(z) + \lambda_3(w_3 \oplus F_3(f_3(z), x))] \\
&= \pi_n(z_n \oplus z) + (1-\pi_n)(R_{A,\lambda_3}^{M_3(g_3(\cdot),x_n)}[A(z_n) + \lambda_3(w_3 \oplus F_3(f_3(z_n), x_n))] \\
&\qquad \oplus R_{A,\lambda_3}^{M_3(g_3(\cdot),x)}[A(z) + \lambda_3(w_3 \oplus F_3(f_3(z), x))]).
\end{aligned}
\tag{4.26}
$$

Importing the same logic as in Theorem 4.1 for (4.11), we have

$$
\begin{aligned}
\|z_{n+1} \oplus z\| = \|z_{n+1} - z\| &\le [\pi_n + (1-\pi_n)\lambda_C \mu_3(\lambda_A + \lambda_3 \sigma_1 \lambda_{f_3})\|z_n - z\| \\
&\quad + (1-\pi_n)\lambda_C(\mu_3 \lambda_3 \sigma_2 + \delta_1)\|x_n - x\|].
\end{aligned}
\tag{4.27}
$$

From (4.23), (4.25) and (4.27), we have

$$\begin{aligned}
&\|x_{n+1}-x\|+\|y_{n+1}-y\|+\|z_{n+1}-z\| \\
&\leq [\pi_n+\frac{(1-\pi_n)\lambda_C\mu_1(\lambda_A\rho_1+\lambda_1\alpha_1\lambda_{f_1})}{\rho_1}]\|x_n-x\| \\
&\quad +[\frac{(1-\pi_n)\lambda_C(\mu_1\lambda_1\alpha_2+\rho_1\delta_2)}{\rho_1}]\|y_n-y\| \\
&\quad +[\pi_n+\frac{(1-\pi_n)\lambda_C\mu_2(\lambda_A\rho_2+\lambda_2\beta_1\lambda_{f_2})}{\rho_2}]\|y_n-y\| \\
&\quad +[\frac{(1-\pi)\lambda_C(\mu_2\lambda_2\beta_2+\rho_2\delta_3)}{\rho_2}]\|z_n-z\| \\
&\quad +[\pi_n+(1-\pi_n)\lambda_C\mu_3(\lambda_A+\lambda_3\sigma_1\lambda_{f_3})]\|z_n-z\| \\
&\quad +[(1-\pi_n)\lambda_C(\mu_3\lambda_3\sigma_2+\delta_1)]\|x_n-x\| \\
&\leq \pi_n(\|x_n-x\|+\|y_n-y\|+\|z_n-z\|) \\
&\quad +(1-\pi_n)[\frac{\lambda_C\mu_1(\lambda_A\rho_1+\lambda_1\alpha_1\lambda_{f_1})}{\rho_1}+\lambda_C(\mu_3\lambda_3\sigma_2+\delta_1)]\|x_n-x\| \\
&\quad +(1-\pi_n)[\frac{\lambda_C\mu_2(\lambda_A\rho_2+\lambda_2\beta_1\lambda_{f_2})}{\rho_2}+\frac{\lambda_C(\mu_1\lambda_1\alpha_2+\rho_1\delta_2)}{\rho_1}]\|y_n-y\| \\
&\quad +(1-\pi_n)[\lambda_C\mu_3(\lambda_A+\lambda_3\sigma_1\lambda_{f_3})+\frac{\lambda_C(\mu_2\lambda_2\beta_2+\rho_2\delta_3)}{\rho_2}]\|z_n-z\| \\
&= \pi_n(\|x_n-x\|+\|y_n-y\|+\|z_n-z\|) \\
&\quad +(1-\pi_n)(\Omega_1\|x_n-x\|+\Omega_2\|y_n-y\|+\Omega_3\|z_n-z\|).
\end{aligned} \tag{4.28}$$

From (4.2), we know that $\max\{\Omega_1,\Omega_2,\Omega_3\}<1$. Then (4.28) becomes

$$\begin{aligned}
&\|x_{n+1}-x\|+\|y_{n+1}-y\|+\|z_{n+1}-z\| \\
&\leq \pi_n(\|x_n-x\|+\|y_n-y\|+\|z_n-z\|) \\
&\quad +(1-\pi_n)\Omega(\|x_n-x\|+\|y_n-y\|+\|z_n-z\|),
\end{aligned} \tag{4.29}$$

where $\Omega=\max\{\Omega_1,\Omega_2,\Omega_3\}$ and

$$\Omega_1=\frac{\lambda_C}{\rho_1}[\mu_1(\lambda_A\rho_1+\lambda_1\alpha_1\lambda_{f_1})+\rho_1(\mu_3\lambda_3\sigma_1+\delta_1)],$$

$$\Omega_2=\frac{\lambda_C}{\rho_1\rho_2}[\mu_2\rho_1(\lambda_A\rho_2+\lambda_2\beta_1\lambda_{f_2})+\rho_2(\mu_1\lambda_1\alpha_2+\delta_2\rho_1)],$$

and

$$\Omega_3 = \frac{\lambda_C}{\rho_2}[\mu_3\rho_2(\lambda_A + \lambda_3\sigma_1\lambda_{f_3}) + (\mu_2\lambda_2\beta_2 + \delta_3\rho_2)].$$

Let $\vartheta_n = (\|x_n - x\| + \|y_n - y\| + \|z_n - z\|)$ and $\varsigma_n = \Omega + (1-\Omega)\pi_n$, then (4.29) can be rewritten as

$$\vartheta_{n+1} \leq \varsigma_n\vartheta_n, n = 0, 1, 2, \cdots .$$

Choosing ς_n, we know that $\limsup_n \varsigma_n < 1$. It follows from Lemma 4.3 that $0 \leq \varsigma_n < 1$. Therefore, $\{(x_n, y_n, z_n)\}$ converges strongly to the unique solution $\{(x, y, z)\}$ of (3.1). □

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