Korean J. Math. **25** (2017), No. 4, pp. 469–481
https://doi.org/10.11568/kjm.2017.25.4.469

GENERAL NONCONVEX SPLIT VARIATIONAL INEQUALITY PROBLEMS

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Abstract. In this paper, we established a *general nonconvex split variational inequality problem*, this is, an extension of *general convex split variational inequality problems* in two different Hilbert spaces. By using the concepts of prox-regularity, we proved the convergence of the iterative schemes for the *general nonconvex split variational inequality problems.* Further, we also discussed the iterative method for the *general convex split variational inequality problems.*

1. Introduction

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces with inner product and norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, *respectively.* Let $\mathcal{C}$ and $\mathcal{Q}$ be nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, *respectively.* For $i \in \{1, 2\}$, let $f_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ and $g_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ be nonlinear mappings and $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a bounded linear operator with its adjoint operator A^*. Consider a problem for finding $x^* \in \mathcal{H}_1$ such that $g_1(x^*) \in \mathcal{C}$ and

$$\langle f_1(x^*),\ x - g_1(x^*) \rangle \geq 0,\ \forall x \in \mathcal{C}, \tag{1.1}$$

Received June 14, 2017. Revised September 25, 2017. Accepted September 27, 2017.

2010 Mathematics Subject Classification: 47J53, 65K10, 49M37, 90C25.

Key words and phrases: General nonconvex split variational inequality problems, general convex split variational inequality problems, relaxed (κ, υ)-cocoercive mappings, inverse strongly monotone mappings, uniform prox-regularity, iterative sequences, adjoint operator, Hilbert spaces.

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and $y^* = Ax^* \in \mathcal{H}_2$ such that $g_2(y^*) \in \mathcal{Q}$ solves

$$\langle f_2(y^*),\ y - g_2(y^*)\rangle \geq 0,\ \forall y \in \mathcal{Q}. \tag{1.2}$$

The problem (1.1)-(1.2) are called *general convex split variational inequality problems.* The split convex variational inequality problem is introduced and studied by Censor *et al.* [6–8]. It is worth mentioning that split convex variational inequality problem is quite general and permits split minimization between two spaces, so the image of a minimizer of a given function, under a bounded linear operator, is a minimizer of another function.

The *general convex split variational inequality problems* (1.1)-(1.2), to take into account of non convexity of subsets $\mathcal{C}$ and $\mathcal{Q}$. This new nonconvex problem is called *general nonconvex split variational inequality problems.*

Poliquin and Rockafellor [17] and Clarke *et al.* [10] have introduced and studied a class of nonconvex sets which are called uniformly prox-regular sets. This class of uniformly prox-regular sets has played an important role in many nonconvex applications such as optimization, dynamic systems and differential inclusions.

Inspired by the recent works going in this fields [1–5,9,12,13,16,21,22], we established the *general nonconvex split variational inequality problems.* By using the concepts of prox-regularity, we proved the convergence of an iterative schemes for the *general nonconvex split variational inequality problems.* Further we also discussed the convergence of an iterative schemes for the *general convex split variational inequality problems.*

Let $\mathcal{C}$ be a nonempty closed subsets of a Hilbert space $\mathcal{H}$, not necessarily convex. Then we have the following:

Definition 1.1. The proximal normal cone of $\mathcal{C}$ at a point $x \in \mathcal{H}$ is given by

$$N_{\mathcal{C}}^P(x) = \{\zeta \in \mathcal{H} : x \in P_{\mathcal{C}}(x + \alpha\zeta)\},$$

where $\alpha > 0$ is a constant and $P_{\mathcal{C}}$ is projection of operator of $\mathcal{H}$ onto $\mathcal{C}$, *that is*

$$P_{\mathcal{C}}(x) = \{x^* \in \mathcal{C} : d_{\mathcal{C}}(x) = \|x - x^*\|\},$$

where $d_{\mathcal{C}}(x)$ or $d(\cdot, \mathcal{C})$ is the usual distance function to the subset of $\mathcal{C}$, *that is*

$$d_{\mathcal{C}}(x) = \inf_{\hat{x} \in \mathcal{C}} \|\hat{x} - x\|.$$

LEMMA 1.2. [10] *Let $\mathcal{C}$ be a nonempty closed subset in $\mathcal{H}$. Then $\zeta \in N_{\mathcal{C}}^P(x)$ if and only if there exists a constant $\alpha = \alpha(\zeta, x) > 0$ such that*

$$\langle \zeta, \hat{x} - x \rangle \leq \alpha \|\hat{x} - x\|^2, \ \forall \hat{x} \in \mathcal{C}.$$

DEFINITION 1.3. [19] The Clarke normal cone denoted by

$$N_{\mathcal{C}}^{cl}(x) = \bar{co}[N_{\mathcal{C}}^P(x)],$$

where $\bar{co}A$ means the closure of the convex hull of A.

LEMMA 1.4. [10] *Let $\mathcal{C}$ be a nonempty closed convex subset in $\mathcal{H}$. Then $\zeta \in N_{\mathcal{C}}^P(x)$ if and only if*

$$\langle \zeta, \hat{x} - x \rangle \leq 0, \ \forall \hat{x} \in \mathcal{C}.$$

DEFINITION 1.5. For any $r \in (0, +\infty]$, a subset $\mathcal{C}_r$ of $\mathcal{H}$ is called the normalized uniformly prox-regular (or uniformly r-prox-regular) if and only if every nonzero proximal normal to $\mathcal{C}_r$ can be realized by an r-ball *that is*, for all $x \in \mathcal{C}_r$ and $0 \neq \zeta \in N_{\mathcal{C}_r}^P(x)$ with $\|\zeta\| = 1$, one has

$$\langle \frac{\zeta}{\|\zeta\|}, \hat{x} - x \rangle \leq \frac{1}{2r} \|\hat{x} - x\|^2, \ \forall \hat{x} \in \mathcal{C}_r.$$

It is known that if $\mathcal{C}_r$ is a uniformly r-prox-regular set, the proximal normal cone $N_{\mathcal{C}_r}^P(x)$ is closed as a set valued mapping. Thus, we have $N_{\mathcal{C}_r}^{cl}(x) = N_{\mathcal{C}_r}^P(x)$. We make the conversion $\frac{1}{r} = 0$ for $r \longrightarrow +\infty$. If $r = +\infty$ then uniformly r-prox-regularity of $\mathcal{C}_r$ reduces to its convexity, *see*, [11, 14, 20].

LEMMA 1.6. [11] *A closed set $\mathcal{C} \subseteq \mathcal{H}$ is convex if and only if it is proximally smooth of radius r for every $r > 0$.*

PROPOSITION 1.7. [18] *For each $r > 0$ and let $\mathcal{C}_r$ be a nonempty closed and uniformly r-prox-regular subset of $\mathcal{H}$. Set*

$$\mathcal{U}(r) = \{x \in \mathcal{H} : 0 \leq d_{\mathcal{C}_r}(x) < r\}.$$

Then the following statements are hold:

(a) for all $x \in \mathcal{U}(r)$, $P_{\mathcal{C}_r}(x) \neq \emptyset$;

(b) for all $r' \in (0, r)$, $P_{\mathcal{C}_r}$ is Lipschitz continuous mapping with constant $\frac{r}{r - r'}$ on

$$\mathcal{U}(r') = \{x \in \mathcal{H} : 0 \leq d_{\mathcal{C}_r}(x) < r'\};$$

(c) the proximal normal cone is closed as a set valued mapping.

LEMMA 1.8. [15]

(i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle,\ \forall x, y \in \mathcal{H}$

(ii) $\|(1-t)x+ty\|^2 = (1-t)\|x\|^2 + t\|y\|^2 - (1-t)t\|x-y\|^2,\ \forall x, y \in \mathcal{H}$ *and for any fixed* $t \in [0,1]$.

2. General nonconvex split variational inequality problems

Throughout this paper, we assume that for given $r, s \in (0, +\infty)$, $\mathcal{C}_r$, $\mathcal{Q}_s$ are uniformly prox regular subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, *respectively.* For each $i = \{1, 2\}$, let $f_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ and $g_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ be nonlinear mappings and $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be a bounded linear operator with its adjoint operator A^*. The *general nonconvex split variational inequality problems* is formulated as follows: find $x^* \in \mathcal{H}_1, g_1(x^*) \in \mathcal{C}_r$ such that

$$\langle f_1(x^*),\ x - g_1(x^*)\rangle + (\frac{\|f_1(x^*)\|}{2r})\|x - g_1(x^*)\|^2 \geq 0,\ \forall x \in \mathcal{C}_r, \tag{2.1}$$

and $y^* = Ax^* \in \mathcal{H}_2$ such that $g_2(y^*) \in \mathcal{Q}_s$ solves

$$\langle f_2(y^*),\ y - g_2(y^*)\rangle + (\frac{\|f_2(y^*)\|}{2s})\|y - g_2(y^*)\|^2 \geq 0,\ \forall y \in \mathcal{Q}_s. \tag{2.2}$$

By making use of Definition 1.5 and Lemma 1.2, the *general nonconvex split variational inequality problems* can be reformulated as follows: finding $(x^*, y^*) \in \mathcal{C}_r \times \mathcal{Q}_s$ with $y^* = Ax^*, g_1(x^*) \in \mathcal{C}_r, g_2(y^*) \in \mathcal{Q}_s$ such that

$$\begin{aligned} 0 &\in \rho f_1(x^*) + N^P_{\mathcal{C}_r}(g_1(x^*)), \\ 0 &\in \lambda f_2(y^*) + N^P_{\mathcal{Q}_s}(g_2(y^*)) \end{aligned} \tag{2.3}$$

where ρ and λ are parameters with positive values and 0 denotes the zero vector of $\mathcal{H}_1$ and $\mathcal{H}_2$, *respectively.* Since $P_{\mathcal{C}_r} = (I + N^P_{\mathcal{C}_r})^{-1}$ and $P_{\mathcal{Q}_s} = (I + N^P_{\mathcal{Q}_s})^{-1}$ are equivalent to finding $(x^*, y^*) \in \mathcal{C}_r \times \mathcal{Q}_s$ with $y^* = Ax^*$ such that $g_1(x^*) \in \mathcal{C}_r, g_2(y^*) \in \mathcal{Q}_s$ such that

$$\begin{aligned} g_1(x^*) &= P_{\mathcal{C}_r}(g_1(x^*) - \rho f_1(x^*)), \\ g_2(y^*) &= P_{\mathcal{Q}_s}(g_2(y^*) - \lambda f_2(y^*)) \end{aligned} \tag{2.4}$$

where $0 < \rho < \frac{r}{1+\|f_1(x^*)\|}$, $0 < \lambda < \frac{s}{1+\|f_2(y^*)\|}$ and $P_{\mathcal{C}_r}$ and $P_{\mathcal{Q}_s}$ are projection onto $\mathcal{C}_r$ and $\mathcal{Q}_s$, *respectively.*

We note that, for $r, s \longrightarrow +\infty$ we have $\mathcal{C}_r = \mathcal{C}$ and $\mathcal{Q}_s = \mathcal{Q}$, the closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, *respectively*, then *general nonconvex split variational inequality problems* (2.1)-(2.2) reduces to the *general convex* *split variational inequality problems* (1.1)-(1.2) for finding $(x^*, y^*) \in \mathcal{C} \times \mathcal{Q}$ with $y^* = Ax^*$ such that

$$g_1(x^*) = P_{\mathcal{C}}(g_1(x^*) - \rho f_1(x^*)),$$

$$g_2(y^*) = P_{\mathcal{Q}}(g_2(y^*) - \lambda f_2(y^*)), \tag{2.5}$$

where $P_{\mathcal{C}}$ and $P_{\mathcal{Q}}$ are projection onto $\mathcal{C}$ and $\mathcal{Q}$, *respectively.*

Definition 2.1. Let $f : \mathcal{H} \longrightarrow \mathcal{H}$ be a mapping. Then f is said to be:

(i) monotone if

$$\langle f(x) - f(\hat{x}), x - \hat{x} \rangle \geq 0, \ \forall x, \hat{x} \in \mathcal{H},$$

(ii) α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle f(x) - f(\hat{x}), x - \hat{x} \rangle \geq \alpha \|x - \hat{x}\|^2, \ \forall x, \hat{x} \in \mathcal{H},$$

(iii) ξ-inverse strongly monotone if there exists a constant $\xi > 0$ such that

$$\langle f(x) - f(\hat{x}), x - \hat{x} \rangle \geq \xi \|f(x) - f(\hat{x})\|^2, \ \forall x, \hat{x} \in \mathcal{H},$$

(iii) relaxed (κ, υ)-cocoercive mapping if there exist constants $\kappa, \upsilon > 0$ such that

$$\langle f(x) - f(\hat{x}), x - \hat{x} \rangle \geq -\kappa \|f(x) - f(\hat{x})\|^2 + \upsilon \|x - \hat{x}\|^2, \ \forall x, \hat{x} \in \mathcal{H},$$

(iv) β-Lipschitz continuous if there exists a constant $\beta > 0$ such that

$$\|f(x) - f(\hat{x})\| \leq \beta \|x - \hat{x}\|, \ \forall x, \hat{x} \in \mathcal{H}.$$

Remark 2.2. Every λ-inverse strongly monotone mapping f is monotone and $\frac{1}{\lambda}$-Lipschitz continuous.

Based on above arguments, we suggest the following iterative algorithm for approximating a solution to (2.1)-(2.2).

Algorithm 2.3. Given $x_0 \in \mathcal{C}_r$, compute the iterative sequence $\{x_n\}$ defined by the iterative schemes:

$$g_1(y_n) = P_{\mathcal{C}_r}[g_1(x_n) - \rho f_1(x_n)], \tag{2.6}$$

$$g_2(z_n) = P_{\mathcal{Q}_s}[g_2(Ay_n) - \lambda f_2(Ay_n)], \tag{2.7}$$

$$x_{n+1} = P_{\mathcal{C}_r}[y_n + \gamma A^*(z_n - Ay_n)] \tag{2.8}$$

for all $n = 0, 1, 2, \cdots, 0 < \rho < \frac{r}{1 + \|f_1(x_n)\|}$, $0 < \lambda < \frac{s}{1 + \|f_2(Ay_n)\|}$ and $0 < \gamma < \frac{r}{1 + \|A^*(z_n - Ay_n)\|}$.

As a particular case of Algorithm 2.3, we have the following algorithm for approximating a solution to (1.1)-(1.2).

ALGORITHM 2.4. Given $x_0 \in \mathcal{C}$, compute the iterative sequence $\{x_n\}$ defined by the iterative schemes:

$$g_1(y_n) = P_{\mathcal{C}}[g_1(x_n) - \rho f_1(x_n)], \tag{2.9}$$

$$g_2(z_n) = P_{\mathcal{Q}}[g_2(Ay_n) - \lambda f_2(Ay_n)], \tag{2.10}$$

$$x_{n+1} = P_{\mathcal{C}}[y_n + \gamma A^*(z_n - Ay_n)] \tag{2.11}$$

for all $n = 0, 1, 2, \cdots, \rho, \lambda, \gamma > 0$.

Let $\{\alpha_n\} \subseteq (0,1)$ be a sequence such that $\sum_{n=1}^{\infty} \alpha_n = +\infty$ and ρ, λ, γ are parameters with positive values. Then we have the following algorithm for approximating a solution to (1.1)-(1.2).

ALGORITHM 2.5. Given $x_0 \in \mathcal{H}_1$, compute the iterative sequence $\{x_n\}$ defined by the iterative schemes:

$$g_1(y_n) = P_{\mathcal{C}}[g_1(x_n) - \rho f_1(x_n)], \tag{2.12}$$

$$g_2(z_n) = P_{\mathcal{Q}}[g_2(Ay_n) - \lambda f_2(Ay_n)], \tag{2.13}$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n[y_n + \gamma A^*(z_n - Ay_n)] \tag{2.14}$$

for all $n = 0, 1, 2, \cdots, \rho, \lambda, \gamma > 0$.

We note that Algorithm 2.4 and Algorithm 2.5 are different form.

3. Main Results

In this section, we discuss the convergence of the iterative sequence generated by algorithms.

THEOREM 3.1. *For given $r, s \in (0, +\infty)$, we assume that $r' \in (0, r)$, $s' \in (0, s)$ and denote $\delta = \frac{r}{r-r'}$ and $\eta = \frac{s}{s-s'}$. Let $\mathcal{C}_r$ and $\mathcal{Q}_s$ be uniformly prox regular subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. For each $i \in \{1, 2\}$, let $f_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ be the relaxed (κ_i, υ_i)-cocoercive mapping with constants $\kappa_i, \upsilon_i > 0$ and β_i-Lipschitz continuous with constant $\beta_i > 0$. Let $g_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ be the ξ_i-inverse strongly monotone with constant $\xi_i > 0$ and σ_i-Lipschitz continuous and let $(g_i - I_i)$ be the ζ_i-strongly monotone with constant $\zeta_i > 0$, where I_i is the identity operator on $\mathcal{H}_i$ $(i = \{1, 2\})$. Let $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be the bounded linear operator such* *that $A(\mathcal{C}_r) \subseteq \mathcal{Q}_s$ and A^* be its adjoint operator. Suppose that $x^* \in \mathcal{C}_r$ is a solution of general nonconvex split variational inequality problems* (2.1)-(2.2). *Then the iterative sequence $\{x_n\}$ generated by Algorithm 2.3 converges strongly to x^* provided that the constants ρ, λ and γ satisfy the following conditions:*

$$\frac{\upsilon_1 - \kappa_1\beta_1^2}{\beta_1^2} - \Omega < \rho < \min\Big\{\frac{\upsilon_1 - \kappa_1\beta_1^2}{\beta_1^2} + \Omega,\ \frac{r'}{1 + \|f_1(x_n)\|},\ \frac{r'}{1 + \|f_1(x^*)\|}\Big\},$$

$$0 < \lambda < \min\Big\{\frac{s'}{1 + \|f_2(Ay_n)\|},\ \frac{s'}{1 + \|f_2(Ay^*)\|}\Big\},\ \textit{for some}\ r' \in (0, r), s' \in (0, s),$$

$$0 < \gamma < \min\Big\{\frac{2}{\|A\|^2},\ \frac{r'}{1 + \|A^*(z_n - Ay_n)\|}\Big\},\ \Omega = \frac{1}{\beta_1^2}\big(\sqrt{(\upsilon_1 - \kappa_1\beta_1^2)^2 - \beta_1^2(1 - \varrho_1^2)}\big)$$

$$\textit{with}\ \upsilon_1 > \kappa_1\beta_1^2 + \beta_1\sqrt{1 - \varrho_1^2},\ \theta_1 < [\delta(1 + 2\theta_2)]^{-1} = d,$$

$$\varrho_1 = \frac{d\sqrt{2\zeta_1 + 1}}{\delta} - \ell_1,\ \ell_1 = \sqrt{1 - 2\xi_1\sigma_1^2 + \sigma_1^2},\ \theta = \delta\theta_1(1 + 2\theta_2) < 1,$$

$$\theta_2 = \sqrt{1 - 2(\upsilon_2 - \kappa_2\beta_2^2)\lambda + \beta_2^2\lambda^2} + \ell_2,\ \ell_2 = \sqrt{1 - 2\xi_2\sigma_2^2 + \sigma_2^2}. \tag{3.1}$$

Proof. Since $x^* \in \mathcal{C}_r$ is a solution of *general nonconvex split variational inequality problems* (2.1)-(2.2) and the parameters ρ, λ, γ satisfying the conditions (3.1), then we have

$$g_1(x^*) = P_{\mathcal{C}_r}[g_1(x^*) - \rho f_1(x^*)], \tag{3.2}$$

$$g_2(Ax^*) = P_{\mathcal{Q}_s}[g_2(Ax^*) - \lambda f_2(Ax^*)]. \tag{3.3}$$

From Lemma 1.8(i) and since $(g_1 - I_1)$ is ζ_1-strongly monotone, then we have

$$\begin{aligned}\|y_n - x^*\|^2 &\leq \|g_1(y_n) - g_1(x^*)\|^2 - 2\langle (g_1 - I_1)y_n - (g_1 - I_1)x^*, y_n - x^*\rangle \\ &\leq \|g_1(y_n) - g_1(x^*)\|^2 - 2\zeta_1\|y_n - x^*\|^2\end{aligned}$$

which implies that

$$\|y_n - x^*\| \leq \frac{1}{\sqrt{2\zeta_1 + 1}}\|g_1(y_n) - g_1(x^*)\|. \tag{3.4}$$

From (2.6) and conditions (3.1) on ρ, we have

$$\|g_1(y_n) - g_1(x^*)\| = \|P_{\mathcal{C}_r}[g_1(x_n) - \rho f_1(x_n)] - P_{\mathcal{C}_r}[g_1(x^*) - \rho f_1(x^*)]\|$$

$$\leq \delta[\|x_n - x^* - (g_1(x_n) - g_1(x^*))\| + \|x_n - x^* - \rho(f_1(x_n) - f_1(x^*))\|]. \tag{3.5}$$

Since g_1 is ξ_1-inverse strongly monotone with constant $\xi_1 > 0$ and σ_1-Lipschitz continuous with constant $\sigma_1 > 0$, we have

$$\begin{aligned}
&\|x_n - x^* - (g_1(x_n) - g_1(x^*))\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\langle g_1(x_n) - g_1(x^*), x_n - x^*\rangle + \|g_1(x_n) - g_1(x^*)\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\xi_1\sigma_1^2\|x_n - x^*\|^2 + \sigma_1^2\|x_n - x^*\|^2 \\
&\leq (1 - 2\xi_1\sigma_1^2 + \sigma_1^2)\|x_n - x^*\|^2
\end{aligned}$$

$$\Rightarrow \|x_n - x^* - (g_1(x_n) - g_1(x^*))\| \leq \sqrt{1 - 2\xi_1\sigma_1^2 + \sigma_1^2}\|x_n - x^*\|. \tag{3.6}$$

Again since f_1 is relaxed (κ_1, υ_1)-cocoercive mapping with constants $\kappa_1, \upsilon_1 > 0$ and β_1-Lipschitz continuous with constant $\beta_1 > 0$, we have

$$\begin{aligned}
&\|x_n - x^* - \rho(f_1(x_n) - f_1(x^*))\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\rho\langle f_1(x_n) - f_1(x^*), x_n - x^*\rangle + \rho^2\|f_1(x_n) - f_1(x^*)\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\rho(-\kappa_1\|f_1(x_n) - f_1(x^*)\|^2 + \upsilon_1\|x_n - x^*\|^2) + \rho^2\beta_1^2\|x_n - x^*\|^2 \\
&\leq (1 - 2\rho(\upsilon_1 - \kappa_1\beta_1^2) + \rho^2\beta_1^2)\|x_n - x^*\|^2
\end{aligned}$$

(3.7)

$$\Rightarrow \|x_n - x^* - \rho(f_1(x_n) - f_1(x^*))\| \leq \sqrt{1 - 2\rho(\upsilon_1 - \kappa_1\beta_1^2) + \rho^2\beta_1^2}\|x_n - x^*\|.$$

From (3.5),(3.6) and (3.7), we have

$$\|g_1(y_n) - g_1(x^*)\| \leq (\ell_1 + \sqrt{1 - 2\rho(\upsilon_1 - \kappa_1\beta_1^2) + \rho^2\beta_1^2})\|x_n - x^*\|, \tag{3.8}$$

where $\ell_1 = \sqrt{1 - 2\xi_1\sigma_1^2 + \sigma_1^2}$. Again from (3.4) and (3.8) we obtain

$$\|y_n - x^*\| \leq \theta_1\|x_n - x^*\| \tag{3.9}$$

where

$$\theta_1 = \frac{\delta}{\sqrt{2\zeta_1 + 1}}\{\ell_1 + \sqrt{1 - 2\rho(\upsilon_1 - \kappa_1\beta_1^2) + \rho^2\beta_1^2}\} \text{ and } \ell_1 = \sqrt{1 - 2\xi_1\sigma_1^2 + \sigma_1^2}.$$

Similarly from (2.7), (3.1), (3.3) and condition on parameter λ and using the fact that f_2 is relaxed (κ_2, υ_2)-cocoercive and β_2-Lipschitz continuous mapping; and g_2 is ξ_2-inverse strongly monotone mapping with constant $\xi_2 > 0$ and σ_2-Lipschitz continuous mapping and from $(q_2 - I_2)$ is ζ_2-strongly monotone and $A(\mathcal{C}_r) \subseteq \mathcal{Q}_s$, we have

$$\|g_2(z_n) - g_2(Ax^*)\| = \|P_{\mathcal{Q}_s}[g_2(Ay_n) - \lambda f_2(Ay_n)] - P_{\mathcal{Q}_s}[g_2(Ax^*) - \lambda f_2(Ax^*)]\|$$

$$\leq \eta[\|g_2(Ay_n) - g_2(Ax^*) - \lambda(f_2(Ay_n) - f_2(Ax^*))\|] \tag{3.10}$$

and

$$\|z_n - Ax^*\| \leq \theta_2\|Ay_n - Ax^*\| \tag{3.11}$$

where

$$\theta_2 = \frac{\eta}{\sqrt{2\zeta_2+1}}\{\ell_2+\sqrt{1-2\lambda(\upsilon_2-\kappa_2\beta_2^2)+\lambda^2\beta_2^2}\} \text{ and } \ell_2 = \sqrt{1-2\xi_2\sigma_2^2+\sigma_2^2}.$$

Next from (2.8) and condition (3.1) on γ, we have

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq \|P_{\mathcal{C}_r}[y_n + \gamma A^*(z_n - Ay_n)] - P_{\mathcal{C}_r}[x^* + \gamma A^*(Ax^* - Ax^*)]\| \\ &\leq \delta[\|y_n - x^* - \gamma A^*(Ay_n - Ax^*)\| + \gamma\|A^*(z_n - Ax^*)\|]. \qquad (3.12)\end{aligned}$$

Further using the definition of A^*, the fact that A^* is a bounded operator with $\|A^*\| = \|A\|$ and condition (3.1), we have

$$\begin{aligned}&\|y_n - x^* - \gamma A^*(Ay_n - Ax^*)\|^2 \\ &= \|y_n - x^*\|^2 - 2\gamma\langle A^*(Ay_n - Ax^*), y_n - x^*\rangle + \gamma^2\|A^*(Ay_n - Ax^*)\|^2 \\ &\leq \|y_n - x^*\|^2 - 2\gamma\|Ay_n - Ax^*\|^2 + \gamma^2\|A\|^2\|Ay_n - Ax^*\|^2 \\ &\leq \|y_n - x^*\|^2 - \gamma(2 - \gamma\|A\|^2)\|Ay_n - Ax^*\|^2\end{aligned}$$

$$\leq \|y_n - x^*\|^2. \tag{3.13}$$

Using (3.11), we have

$$\begin{aligned}\|A^*(z_n - Ax^*)\| &\leq \|A\|\|z_n - Ax^*\| \\ &\leq \theta_2\|A\|\|Ay_n - Ax^*\| \\ &\leq \theta_2\|A\|^2\|y_n - x^*\|. \qquad (3.14)\end{aligned}$$

Combining (3.13) and (3.14) with (3.12), we obtain

$$\|x_{n+1} - x^*\| \leq \delta[\|y_n - x^*\| + \gamma\theta_2\|A\|^2\|y_n - x^*\|],$$
$$\|x_{n+1} - x^*\| \leq \theta\|x_n - x^*\|,$$

where $\theta = \delta\theta_1(1 + \gamma\|A\|^2\theta_2)$.
Thus, we obtain

$$\|x_{n+1} - x^*\| \leq \theta_n\|x_0 - x^*\|. \tag{3.15}$$

Since $\gamma\|A\|^2 < 2$, hence the maximum value of $(1+\gamma\|A\|^2\theta_2)$ is $(1+2\theta_2)$. Further $\theta \in (0, 1)$ if and only if

$$\theta_1 < [\delta(1 + 2\theta_2)]^{-1} = d. \tag{3.16}$$

Since $d \in (0,1)$ and $\delta, \eta > 1$. Finally the inequality (3.15) holds from conditions (3.1). Thus from (3.15) that $\{x_n\}$ strongly converges to x^* as $n \longrightarrow \infty$. Since A is continuous, hence from (3.9) that $y_n \longrightarrow x^*, g_1(y_n) \longrightarrow g_1(x^*),\ Ay_n \longrightarrow Ax^*,\ z_n \longrightarrow Ax^*$ and $g_2(z_n) \longrightarrow g_2(Ax^*)$ as $n \longrightarrow \infty$. This completes the proof. □

In particular case, if $r = +\infty,\ s = +\infty$, one has $\eta = \delta = 1$. Then we have the following convergence result for Algorithm 2.5 to solve (1.1)-(1.2).

THEOREM 3.2. *Let $\mathcal{C}$ and $\mathcal{Q}$ be nonempty closed convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. For each $i \in \{1,2\}$ let $f_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ be the relaxed (κ_i, υ_i)-cocoercive mapping with constants $\kappa_i, \upsilon_i > 0$ and β_i-Lipschitz continuous with constant $\beta_i > 0$. Let $g_i : \mathcal{H}_i \longrightarrow \mathcal{H}_i$ be the ξ_i-inverse strongly monotone with constant $\xi_i > 0$ and σ_i-Lipschitz continuous and let $(g_i - I_i)$ be the ζ_i-strongly monotone with constant $\zeta_i > 0$, where I_i is the identity operator on $\mathcal{H}_i$ $(i = \{1,2\})$. Let $A : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$ be the bounded linear operator and A^* be its adjoint operator. Suppose that $x^* \in \mathcal{C}$ is a solution of general convex split variational inequality problems (1.1)-(1.2). Then the iterative sequence $\{x_n\}$ generated by Algorithm 2.5 converges strongly to x^* provided the constants ρ, η and γ satisfy the following conditions:*

$$
\begin{aligned}
&\frac{\upsilon_1 - \kappa_1\beta_1^2}{\beta_1^2} - \Omega < \rho < \frac{\upsilon_1 - \kappa_1\beta_1^2}{\beta_1^2} + \Omega,\ \gamma \in \Big(0, \frac{2}{\|A\|^2}\Big),\ \textit{where}\\
&\Omega = \frac{1}{\beta_1^2}\Big(\sqrt{(\upsilon_1 - \kappa_1\beta_1^2)^2 - \beta_1^2(1-\varrho_1^2)}\Big)\ \textit{with}\ \upsilon_1 > \kappa_1\beta_1^2 + \beta_1\sqrt{1-\varrho_1^2},\\
&\theta_1 < [\delta(1+2\theta_2)]^{-1} = d,\ \varrho_1 = \frac{d\sqrt{2\zeta_1+1}}{\delta} - \ell_1,\\
&\ell_1 = \sqrt{1 - 2\xi_1\sigma_1^2 + \sigma_1^2},\ \theta = \delta\theta_1(1+2\theta_2) < 1,
\end{aligned}
$$

(3.17)

$$\theta_2 = \sqrt{1 - 2(\upsilon_2 - \kappa_2\beta_2^2)\lambda + \beta_2^2\lambda^2} + \ell_2,\ \ell_2 = \sqrt{1 - 2\xi_2\sigma_2^2 + \sigma_2^2},\ \lambda > 1.$$

Proof. Since $x^* \in \mathcal{C}$ is a solution of *general convex split variational inequality problems* (1.1)-(1.2), then $\rho, \lambda > 0$ such that

$$g_1(x^*) = P_{\mathcal{C}}[g_1(x^*) - \rho f_1(x^*)], \tag{3.18}$$

$$g_2(Ax^*) = P_{\mathcal{Q}}[g_2(Ax^*) - \lambda f_2(Ax^*)]. \tag{3.19}$$

Using the same argument used in proof of Theorem 3.1, we have

$$\|y_n - x^*\| \le \theta_1 \|x_n - x^*\| \tag{3.20}$$

where

$$\theta_1 = \frac{\delta}{\sqrt{2\zeta_1 + 1}}\{\ell_1 + \sqrt{1 - 2\rho(\upsilon_1 - \kappa_1\beta_1^2) + \rho^2\beta_1^2}\} \text{ and } \ell_1 = \sqrt{1 - 2\xi_1\sigma_1^2 + \sigma_1^2}.$$

Again

$$\|z_n - Ax^*\| \le \theta_2 \|Ay_n - Ax^*\| \tag{3.21}$$

where

$$\theta_2 = \frac{\eta}{\sqrt{2\zeta_2 + 1}}\{\ell_2 + \sqrt{1 - 2\lambda(\upsilon_2 - \kappa_2\beta_2^2) + \lambda^2\beta_2^2}\} \text{ and } \ell_2 = \sqrt{1 - 2\xi_2\sigma_2^2 + \sigma_2^2}.$$

Next from Algorithm 2.5 (2.14), we have

$$\begin{aligned} \|x_{n+1} - x^*\| \le (1-\alpha_n)\|x_n - x^*\| + \alpha_n[\|y_n - x^* - \gamma A^*(Ay_n - Ax^*)\| \\ + \gamma\|A^*(z_n - Ax^*)\|]. \end{aligned} \tag{3.22}$$

Since A^* is a bounded linear operator with $\|A^*\| = \|A\|$ and given condition on γ, we have

$$\begin{aligned} &\|y_n - x^* - \gamma A^*(Ay_n - Ax^*)\|^2 \\ &= \|y_n - x^*\|^2 - 2\gamma\langle A^*(Ay_n - Ax^*), y_n - x^*\rangle + \gamma^2\|A^*(Ay_n - Ax^*)\|^2 \\ &\le \|y_n - x^*\|^2. \end{aligned} \tag{3.23}$$

Using (3.21) we have

$$\begin{aligned} \|A^*(z_n - Ax^*)\| &\le \|A\|\|z_n - Ax^*\| \\ &\le \theta_2\|A\|\|Ay_n - Ax^*\| \\ &\le \theta_2\|A\|^2\|y_n - x^*\|. \end{aligned} \tag{3.24}$$

Combining (3.23) and (3.24) with (3.22), we obtain

$$\|x_{n+1} - x^*\| \le [1 - \alpha_n(1-\theta)]\|x_n - x^*\|,$$

where $\theta = \theta_1(1 + \gamma\|A\|^2\theta_2)$.
Thus, we obtain

$$\|x_{n+1} - x^*\| \le \prod_{j=1}^{n}[1 - \alpha_j(1-\theta)]\|x_0 - x^*\|. \tag{3.25}$$

It follows from condition on $\rho, \lambda \in (0,1)$. Since $\sum_{n=1}^{\infty} a_n = +\infty$ and $\theta \in (0,1)$ and from [23], we have

$$\lim_{n\longrightarrow\infty} \prod_{j=1}^{n} [1 - \alpha_j(1-\theta)] = 0.$$

The rest of the proof is same as the proof of Theorem 3.1. This completes the proof. □

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