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CONSTRUCTIVE PROOF FOR THE POSITIVITY OF THE ORBIT POLYNOMIAL $O_d^{n,2}(q)$

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Abstract. The cyclic group $C_n = \langle(12\cdots n)\rangle$ acts on the set $\binom{[n]}{k}$ of all k-subsets of $[n]$. In this action of C_n the number of orbits of size d, for $d \mid n$, is

$$O_d^{n,k} = \frac{1}{d}\sum_{\frac{n}{d}|s|n} \mu\left(\frac{ds}{n}\right)\binom{n/s}{k/s}.$$

Stanton and White [6] generalized the above identity to construct the orbit polynomials

$$O_d^{n,k}(q) = \frac{1}{[d]_{q^{n/d}}}\sum_{\frac{n}{d}|s|n} \mu\left(\frac{ds}{n}\right)\begin{bmatrix} n/s \\ k/s \end{bmatrix}_{q^s}$$

and conjectured that $O_d^{n,k}(q)$ have non-negative coefficients. In this paper we give a constructive proof for the positivity of coefficients of the orbit polynomial $O_d^{n,2}(q)$.

1. Introduction

When n is a positive integer, we write as $[n] = \{1, 2, \ldots, n\}$. Let C_n be the cyclic group generated by a permutation $\sigma = (12\cdots n)$. If $\binom{[n]}{k}$ is the set of all k-subsets of $[n]$, C_n acts on $\binom{[n]}{k}$ via

$$(\tau, \{x_1, x_2, \ldots, x_k\}) \mapsto \{x_{\tau(1)}, x_{\tau(2)}, \ldots, x_{\tau(k)}\}.$$

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The number of orbits in this action of C_n is given

$$O^{n,k} = \frac{1}{n} \sum_{d \mid \gcd(n,k)} \varphi(d) \binom{n/d}{k/d}, \tag{1}$$

and the number of orbits of size d, for $d \mid n$, is

$$O_d^{n,k} = \frac{1}{d} \sum_{\frac{n}{d} \mid s \mid n} \mu\left(\frac{ds}{n}\right) \binom{n/s}{k/s}. \tag{2}$$

Here φ is the Euler phi-function and μ is the Möbius function. In preprint [6] Stanton and White constructed the orbit polynomials $O_d^{n,k}(q)$, a q-version of (2), and conjectured the following.

Conjecture 1.1. *Fix $d \mid n$, and any non-negative integer k. Polynomials*

$$O_d^{n,k}(q) = \frac{1}{[d]_{q^{n/d}}} \sum_{\frac{n}{d} \mid s \mid n} \mu\left(\frac{ds}{n}\right) \begin{bmatrix} n/s \\ k/s \end{bmatrix}_{q^s}$$

have non-negative coefficients.

Here $[n]_q = 1 + q + \cdots + q^{n-1}$, $[n]!_q = [1]_q[2]_q \cdots [n]_q$ and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!_q}{[k]!_q[n-k]_q!}.$$

Möbius inversion implies

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{d \mid n} [d]_{q^{n/d}} O_d^{n,k}(q). \tag{3}$$

Andrews [1] and Haiman [3] independently verified the above conjecture when $(n, k) = 1$. In [4] Reiner, Stanton and White defined the cyclic sieving phenomenon, generalization of Stembridge's $q = -1$ phenomenon [7], and use it to prove several enumeration problems involving q-binomial coefficients, non-crossing partitions, polygon dissections and some finite field q-analogues. Drudge [2] has proven that $O^{n,k}(q) = \sum_{d \mid n} O_d^{n,k}(q)$ is the number of orbits of the Singer cycle on the k-dimensional subspaces of an n-dimensional vector space over a field of order q. Recently Sagan [5] gave combinatorial proofs for several theorems appeared in [4].

In this paper we give a new weight for each 2-subset in $\binom{[n]}{2}$, and show that the sum of weights of all 2-subset in $\binom{[n]}{2}$ is equal to the q-binomial coefficient $\begin{bmatrix} n \\ 2 \end{bmatrix}_q$. This will give a combinatorial proof for the positivity of coefficients of the orbit polynomial $O_d^{n,2}(q)$. Finally we suggest a strategy for the constructive proof of the positivity of coefficients of the orbit polynomial $O_d^{n,k}(q)$ for any positive integers n, k with $(n,k) = 1$.

2. Positivity for the orbit polynomial $O_d^{n,2}(q)$

In this section we write as $ij = \{i, j\}$ for convention. We begin with the recurrence relation of q-binomial coefficient $\begin{bmatrix} n \\ 2 \end{bmatrix}_q$. Using the recurrence relations

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \text{ and}$$
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q$$

several times, we get the following identity.

Proposition 2.1. *Let $n \geq 2$ be an integer. Then*

$$\begin{bmatrix} n+2 \\ 2 \end{bmatrix}_q = q^2 \begin{bmatrix} n \\ 2 \end{bmatrix}_q + q^{n+2} \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q + [n+2]_q.$$

We now describe the representatives x of orbits in the action of of C_n on $\binom{[n]}{2}$. In each orbit O under C_n we choose $1i \in O$ as the representative of O, where

$$1 < i \leq \frac{n}{2} + 1. \tag{4}$$

For example, if $n = 10$, all orbits are given with representatives underlined as follows. Here $a = 10$.

$$O_1 = \langle \underline{12} \rangle = \{\underline{12}, 23, 34, 45, 56, 67, 78, 89, 9a, 1a\}$$
$$O_2 = \langle \underline{13} \rangle = \{\underline{13}, 24, 35, 46, 57, 68, 79, 8a, 19, 2a\}$$
$$O_3 = \langle \underline{14} \rangle = \{\underline{14}, 25, 36, 47, 58, 69, 7a, 18, 29, 3a\}$$
$$O_4 = \langle \underline{15} \rangle = \{\underline{15}, 26, 37, 48, 59, 6a, 17, 28, 39, 4a\}$$
$$O_0 = \langle \underline{16} \rangle = \{\underline{16}, 27, 38, 49, 5a\}.$$

Let $1i$ be the representative of an orbit under C_n. We define the weight $w_n(1i)$ as

$$w_n(1i) = \begin{cases} q^{n+2-2i} & \text{if } i = \frac{n}{2}+1 \\ q^{n+1-2i} & \text{else.} \end{cases} \tag{5}$$

The weights for the other elements than the representatives are given using the weights of representatives in (5).

Assume first $\gcd(n, 2) = 1$. Note that all orbits are of size n by (1) and (2). If $O_i = \{x_{i1}, x_{i2}, \ldots, x_{i(n-1)}, x_{in}\}$ is an orbit of size n with the representative x_{i1} and with the action

$$x_{i1} \xrightarrow{\sigma} x_{i2} \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} x_{i(n-1)} \xrightarrow{\sigma} x_{in} \xrightarrow{\sigma} x_{i1},$$

we define

$$w_n(x_{ij+1}) = q w_n(x_{ij}) \text{ for } 1 \leq j \leq n-1. \tag{6}$$

If $\gcd(n, 2) \neq 1$, there is only one orbit of size $\frac{n}{2}$ and the other orbits are of size n under the action of C_n. The weights for elements in an orbit of size n are defined in the same way as (6). On the other hand, if $O_0 = \{x_{01}, x_{02}, \ldots, x_{0\frac{n}{2}}\}$ is the orbit of size $\frac{n}{2}$ with the representative x_{01} and with the action

$$x_{01} \xrightarrow{\sigma} x_{02} \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} x_{0\frac{n}{2}} \xrightarrow{\sigma} x_{01},$$

we define

$$w_n(x_{0j+1}) = q^2 w_n(x_{0j}) \text{ for } 1 \leq j \leq \frac{n}{2} - 1.$$

Then the sum of weights of all elements in $\binom{[n]}{2}$ is equal to the q-binomial coefficient $\begin{bmatrix} n \\ 2 \end{bmatrix}_q$ as follows.

Theorem 2.2. *Let $n \geq 2$ be an integer and let T_n be the set of all 2-subsets of $[n]$, i.e., $T_n = \binom{[n]}{2}$. If we set $w_n(T_n) = \sum_{x \in T_n} w_n(x)$, then we have*

$$w_n(T_n) = \begin{bmatrix} n \\ 2 \end{bmatrix}_q .$$

Proof. Computing $w_n(T_n)$ and $\begin{bmatrix} n \\ 2 \end{bmatrix}_q$ for $n = 2, 3, 4, 5$ directly, we have

$$\begin{aligned}
w_2(T_2) &= 1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}_q \\
w_3(T_3) &= 1 + q + q^2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}_q \\
w_4(T_4) &= 1 + q + 2q^2 + q^3 + q^4 = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q \\
w_5(T_5) &= 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q .
\end{aligned}$$

We only work out for $n = 2\ell + 1$. The proof for $n = 2\ell$ can be given in the same way with a little modification.

Suppose now $n = 2\ell + 1$ for some $\ell \in \mathbb{N}$ and $w_n(T_n) = \begin{bmatrix} n \\ 2 \end{bmatrix}_q$. Since $\gcd(n, 2) = \gcd(n + 2, 2) = 1$, all orbits under C_n are of size n and all orbits under C_{n+2} are of size $n + 2$. Let

$$x_{11}, x_{21}, \dots, x_{s1}$$

be all representatives of orbits in the action of C_n, where

$$s = |T_n|/|\text{orbit}| = \binom{n}{2}/n = \frac{1}{2}(n-1).$$

On the other hand, if t is the number of orbits in the action of C_{n+2},

$$t = \binom{n+2}{2}/(n+2) = \frac{1}{2}(n+1) = s + 1.$$

Let

$$x_{11}, x_{21}, \dots, x_{s1}, x_{(s+1)1}$$

be all representatives of orbits in the action of C_{n+2}. Then all orbits $O_1, O_2, \cdots, O_s$ under the action of C_n are

$$\begin{aligned}
O_1 &= \{x_{11}, x_{12}, \dots, x_{1(n-1)}, x_{1n}\} \\
O_2 &= \{x_{21}, x_{22}, \dots, x_{2(n-1)}, x_{2n}\} \\
&\vdots \\
O_s &= \{x_{s1}, x_{s2}, \dots, x_{s(n-1)}, x_{sn}\}
\end{aligned} \tag{7}$$

while

$$
\begin{aligned}
O'_1 &= \{x_{11}, x_{12}, \ldots, x_{1n}, x_{1(n+1)}, x_{1(n+2)}\}\\
O'_2 &= \{x_{21}, x_{22}, \ldots, x_{2n}, x_{2(n+1)}, x_{2(n+2)}\}\\
&\vdots\\
O'_s &= \{x_{s1}, x_{s2}, \ldots, x_{sn}, x_{s(n+1)}, x_{s(n+2)}\}\\
O'_{s+1} &= \{x_{(s+1)1}, x_{(s+1)2}, \ldots, x_{(s+1)n}, x_{(s+1)(n+1)}, x_{(s+1)(n+2)}\}
\end{aligned} \tag{8}
$$

are all orbits under C_{n+2}. Let x be the representative of an orbit under the action of C_n. x can be also the representative of an orbit under the action of C_{n+2}. In this case,

$$w_{n+2}(x) = q^2 w_n(x).$$

For example, $x = 12 \in \binom{[n]}{2}$ is the representative of an orbit under the action of C_n. The weight of x is

$$w_n(x) = q^{n+1-2\cdot 2} = q^{n-3}.$$

Also, $x = 12$ can be considered in $T_{n+2} = \binom{[n+2]}{2}$ and the weight $w_{n+2}(x)$ is

$$w_{n+2}(x) = q^{(n+2)+1-2\cdot 2} = q^{n-1},$$

so that $w_{n+2}(x) = q^2 w_n(x)$. Another 2-subset $23 = \sigma(12)$ is considered as the element of T_{n+2} as well as T_n. The weight of 23 is

$$w_n(23) = q w_n(12) \quad \text{and} \quad w_{n+2}(23) = q w_{n+2}(12)$$

so that $w_{n+2}(23) = q^2 w_n(23)$. Using this relation we compute $w_{n+2}(T_{n+2})$. Let $r_n(q)$ be the sum of weights of representatives of all orbits of size n. From (7) and assumption we have

$$w_n(T_n) = \sum_{i=1}^{s} \sum_{x \in O_i} w_n(x) = \sum_{i=1}^{s} w_n(x_{i1})[n]_q = r_n(q)[n]_q = \begin{bmatrix} n \\ 2 \end{bmatrix}_q.$$

On the other hand, if we use (8), we have

$$w_{n+2}(T_{n+2}) = \sum_{i=1}^{s+1} \sum_{x \in O'_i} w_{n+2}(x) = \sum_{i=1}^{s} \sum_{x \in O'_i} w_{n+2}(x) + \sum_{x \in O'_{s+1}} w_{n+2}(x).$$

Here

$$
\begin{aligned}
\sum_{i=1}^{s}\sum_{x\in O_i'} w_{n+2}(x) &= \sum_{i=1}^{s}\sum_{j=1}^{n+2} w_{n+2}(x_{ij}) = \sum_{i=1}^{s} w_{n+2}(x_{i1})[n+2]_q \\
&= \sum_{i=1}^{s} q^2 w_n(x_{i1})([n]_q + q^n[2]_q) \\
&= q^2 r_n(q)[n]_q + q^{n+2} r_n(q)[2]_q \qquad (9)\\
&= q^2 \begin{bmatrix} n \\ 2 \end{bmatrix}_q + q^{n+2} \frac{\begin{bmatrix} n \\ 2 \end{bmatrix}_q}{[n]_q}[2]_q \\
&= q^2 \begin{bmatrix} n \\ 2 \end{bmatrix}_q + q^{n+2} \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q .
\end{aligned}
$$

Using (4) we can find the representatives of all orbits under of C_{n+2}. $1(\ell+2)$ is the only one representative of orbit in the action of C_{n+2} which are not in orbits of the action of C_n. Using the weights given in (5) and (6)

$$
\begin{aligned}
\sum_{x\in O'_{s+1}} w_{n+2}(x) &= w_n\left(1(\ell+2)\right)[n+2]_q \qquad (10)\\
&= q^{(2\ell+3)+1-2(\ell+2)}[n+2]_q = [n+2]_q.
\end{aligned}
$$

Combining (9) and (10), we have

$$
\begin{aligned}
w_{n+2}(T_{n+2}) &= q^2 \begin{bmatrix} n \\ 2 \end{bmatrix}_q + q^{n+2} \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q + [n+2]_q \\
&= \begin{bmatrix} n+2 \\ 2 \end{bmatrix}_q \text{ from Proposition 2.1.}
\end{aligned}
$$

Hence we have $w_n(T_n) = \begin{bmatrix} n \\ 2 \end{bmatrix}_q$ for $n \geq 2$. □

THEOREM 2.3. *Orbit polynomials $O_n^{n,2}(q)$ is equal to the sum of weights of representatives of all orbits of size n.*

Proof. Assume first $\gcd(n,2)=1$. Then there are only s orbits of size n under C_n, where $s = \binom{n}{2}/n$. Let $O_1, O_2, \ldots, O_s$ be all orbits of size n

under C_n. Then from the proof of Theorem 2.2 we know that

$$w_n(T_n) = r_n(q)[n]_q. \tag{11}$$

Assume now $\gcd(n,2) \neq 1$. Then there are s orbits $O_1, O_2, \ldots, O_s$ of size n where $s = (\binom{n}{2} - \frac{n}{2})/n$, and there is only one orbit

$$O_0 = \{x_{01}, x_{02}, \ldots, x_{0\frac{n}{2}}\}$$

of size $\frac{n}{2}$. Hence

$$\begin{aligned} w_n(T_n) &= \sum_{x \in \binom{[n]}{2}} w_n(x) = \sum_{x \in O_0} w_n(x) + \sum_{i=1}^{s} \sum_{x \in O_i} w_n(x) \\ &= (1 + q^2 + \cdots + q^{n-2}) + \sum_{i=1}^{s} w_n(x_{i1})[n]_q \\ &= \begin{bmatrix} \frac{n}{2} \end{bmatrix}_{q^2} + r_n(q)[n]_q. \end{aligned} \tag{12}$$

From (3), we have

$$\begin{bmatrix} n \\ 2 \end{bmatrix}_q = \begin{cases} [n]_q \, O_n^{n,2}(q) & \text{if } \gcd(n,2) = 1 \\ \left[\frac{n}{2}\right]_{q^2} O_{\frac{n}{2}}^{n,2}(q) + [n]_q \, O_n^{n,2}(q) & \text{if } \gcd(n,2) \neq 1. \end{cases} \tag{13}$$

Note that $O_{\frac{n}{2}}^{n,2}(q) = 1$. Comparing (11) and (12) with (13), we have

$$O_n^{n,2}(q) = r_n(q).$$

□

Corollary 2.4. *Let $d \mid n$. Then orbit polynomials $O_d^{n,2}(q)$ have non-negative coefficients.*

Proof. Since $O_{n/t}^{n,k}(q) = O_{n/t}^{n/t,k/t}(q^t)$, it is sufficient to prove Corollary 2.4 for $d = n$. Then $O_n^{n,2}(q) = r_n(q)$ by Theorem 2.3 and $r_n(q)$ clearly has non-negative coefficients from the definition. □

3. Remark

Let n, k be positive integers with $(n,k) = 1$. In this section we suggest a strategy for the constructive proof of the positivity of coefficients of the orbit polynomial $O_d^{n,k}(q)$.

Question 1. $\begin{bmatrix} n+k \\ k \end{bmatrix}_q$ has recurrence relations similar to Proposition 2.1 for $k = 3, 4, 5$. It would be interesting to find a recurrence relation of $\begin{bmatrix} n+k \\ k \end{bmatrix}_q$ similar to Proposition 2.1 for an arbitrary positive integer k, i.e., to find the polynomial $f_k(q)$ satisfying the equality

$$\begin{bmatrix} n+k \\ k \end{bmatrix}_q = q^{k(k-1)} \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n+k(k-1)} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + f_k(q)[n+k]_q.$$

Let $T_n = \binom{[n]}{k}$ and $T_{n+k} = \binom{[n+k]}{k}$, and let $w_n(x)$ and $w_{n+k}(y)$ be weights of $x \in T_n$ and $y \in T_{n+k}$, respectively. If

$$\begin{aligned} O_1 &= \{x_{11}, x_{12}, \ldots, x_{1(n-1)}, x_{1n}\} \\ O_2 &= \{x_{21}, x_{22}, \ldots, x_{2(n-1)}, x_{2n}\} \\ &\vdots \\ O_s &= \{x_{s1}, x_{s2}, \ldots, x_{s(n-1)}, x_{sn}\} \end{aligned}$$

are all orbits of size n in the action of C_n, and

$$\begin{aligned} O'_1 &= \{x_{11}, x_{12}, \ldots, x_{1n}, x_{1(n+1)}, \ldots, x_{1(n+k)}\} \\ O'_2 &= \{x_{21}, x_{22}, \ldots, x_{2n}, x_{2(n+1)}, \ldots, x_{2(n+k)}\} \\ &\vdots \\ O'_s &= \{x_{s1}, x_{s2}, \ldots, x_{sn}, x_{s(n+1)}, \ldots, x_{s(n+k)}\} \\ O'_{s+1} &= \{x_{(s+1)1}, x_{(s+1)2}, \ldots, x_{(s+1)n}, x_{(s+1)(n+1)}, \ldots, x_{(s+1)(n+k)}\} \\ &\vdots \\ O'_t &= \{x_{t1}, x_{t2}, \ldots, x_{tn}, x_{t(n+1)}, \ldots, x_{t(n+k)}\} \end{aligned}$$

are all orbits of size $n+k$ under C_{n+k}, we have

$$w_{n+k}(T_{n+k}) = \sum_{i=1}^{t} \sum_{x \in O'_i} w_{n+k}(x) = \sum_{i=1}^{s} \sum_{x \in O'_i} w_{n+k}(x) + \sum_{i=s+1}^{t} \sum_{x \in O'_i} w_{n+k}(x).$$

Question 2. Define $w_n(x)$ and $w_{n+k}(y)$ such that

$$\sum_{i=1}^{s}\sum_{x\in O_i'} w_{n+k}(x) = q^{k(k-1)}\begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n+k(k-1)}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \quad \text{and}$$

$$\sum_{i=s+1}^{t}\sum_{x\in O_i'} w_{n+k}(x) = f_k(q)[n+k]_q.$$

The answers for the above Question 1 and 2 will give the constructive proof of the positivity of coefficients of the orbit polynomial $O_d^{n,k}(q)$.

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