CONSTRUCTIVE PROOF FOR THE POSITIVITY OF THE ORBIT POLYNOMIAL $O_{d}^{n,k}(q)$

JAEJIN LEE

Abstract. The cyclic group $C_n = \langle (12\cdots n) \rangle$ acts on the set $\binom{\[n\]}{k}$ of all $k$-subsets of $[n]$. In this action of $C_n$ the number of orbits of size $d$, for $d \mid n$, is

$$O_{d}^{n,k} = \frac{1}{d} \sum_{\frac{n}{d} \mid s \mid n} \mu\left(\frac{ds}{n}\right) \binom{n/s}{k/s}.$$  

Stanton and White [6] generalized the above identity to construct the orbit polynomials

$$O_{d}^{n,k}(q) = \frac{1}{[d]_{q^{n/d}}} \sum_{\frac{n}{d} \mid s \mid n} \mu\left(\frac{ds}{n}\right) \left[\binom{n/s}{k/s}\right]_{q^s}$$

and conjectured that $O_{d}^{n,k}(q)$ have non-negative coefficients. In this paper we give a constructive proof for the positivity of coefficients of the orbit polynomial $O_{d}^{n,2}(q)$.

1. Introduction

When $n$ is a positive integer, we write as $[n] = \{1,2,\ldots,n\}$. Let $C_n$ be the cyclic group generated by a permutation $\sigma = (12\cdots n)$. If $\binom{\[n\]}{k}$ is the set of all $k$-subsets of $[n]$, $C_n$ acts on $\binom{\[n\]}{k}$ via

$$(\tau,\{x_1,x_2,\ldots,x_k\}) \mapsto \{x_{\tau(1)},x_{\tau(2)},\ldots,x_{\tau(k)}\}.$$
The number of orbits in this action of $C_n$ is given

\begin{equation}
O^{n,k} = \frac{1}{n} \sum_{d \mid \gcd(n,k)} \varphi(d) \binom{n/d}{k/d},
\end{equation}

and the number of orbits of size $d$, for $d \mid n$, is

\begin{equation}
O^d_{n,k} = \frac{1}{d} \sum_{\frac{n}{d} \mid s \mid n} \mu \left( \frac{ds}{n} \right) \binom{n/s}{k/s}.
\end{equation}

Here $\varphi$ is the Euler phi-function and $\mu$ is the Möbius function. In preprint [6] Stanton and White constructed the orbit polynomials $O^{n,k}_d(q)$, a $q$-version of (2), and conjectured the following.

**Conjecture 1.1.** Fix $d \mid n$, and any non-negative integer $k$. Polynomials

\begin{equation}
O^d_{n,k}(q) = \frac{1}{[d]_{q^{n/d}}} \sum_{\frac{n}{d} \mid s \mid n} \mu \left( \frac{ds}{n} \right) \left[ \frac{n}{s} \right]_q \left[ \frac{k}{s} \right]_q,
\end{equation}

have non-negative coefficients.

Here $[n]_q = 1 + q + \cdots + q^{n-1}$, $[n]!_q = [1]_q [2]_q \cdots [n]_q$ and

\[ \left[ \frac{n}{k} \right]_q = \frac{[n]!_q}{[k]!_q [n-k]!_q}. \]

Möbius inversion implies

\begin{equation}
\left[ \frac{n}{k} \right]_q = \sum_{d \mid n} [d]_{q^{n/d}} O^d_{n,k}(q).
\end{equation}

Andrews [1] and Haiman [3] independently verified the above conjecture when $(n, k) = 1$. In [4] Reiner, Stanton and White defined the cyclic sieving phenomenon, generalization of Stembridge’s $q = -1$ phenomenon [7], and use it to prove several enumeration problems involving $q$-binomial coefficients, non-crossing partitions, polygon dissections and some finite field $q$-analogues. Drudge [2] has proven that $O^{n,k}(q) = \sum_{d \mid n} O^d_{n,k}(q)$ is the number of orbits of the Singer cycle on the $k$-dimensional subspaces of an $n$-dimensional vector space over a field of order $q$. Recently Sagan [5] gave combinatorial proofs for several theorems appeared in [4].

In this paper we give a new weight for each 2-subset in $\binom{n}{2}$, and show that the sum of weights of all 2-subset in $\binom{n}{2}$ is equal to the $q$-binomial
Constructive proof for the positivity of the orbit polynomial $O_{d}^{n,2}(q)$. This will give a combinatorial proof for the positivity of coefficients of the orbit polynomial $O_{d}^{n,2}(q)$. Finally we suggest a strategy for the constructive proof of the positivity of coefficients of the orbit polynomial $O_{d}^{n,k}(q)$ for any positive integers $n,k$ with $(n,k) = 1$.

2. Positivity for the orbit polynomial $O_{d}^{n,2}(q)$

In this section we write as $ij = \{i, j\}$ for convention. We begin with the recurrence relation of $q$-binomial coefficient $\left[\begin{array}{c} n \\ 2 \end{array}\right]_{q}$. Using the recurrence relations
\[
\left[\begin{array}{c} n \\ k \end{array}\right]_{q} = \left[\begin{array}{c} n-1 \\ k \end{array}\right]_{q} + \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_{q}
\]
and
\[
\left[\begin{array}{c} n \\ k \end{array}\right]_{q} = \left[\begin{array}{c} n-1 \\ k \end{array}\right]_{q} + q^{n-k} \left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_{q}
\]
several times, we get the following identity.

**Proposition 2.1.** Let $n \geq 2$ be an integer. Then
\[
\left[\begin{array}{c} n + 2 \\ 2 \end{array}\right]_{q} = q^{2} \left[\begin{array}{c} n \\ 2 \end{array}\right]_{q} + q^{n+2} \left[\begin{array}{c} n-1 \\ 1 \end{array}\right]_{q} + [n + 2]_{q}.
\]

We now describe the representatives $x$ of orbits in the action of of $C_n$ on $\binom{n}{2}$. In each orbit $O$ under $C_n$ we choose $1i \in O$ as the representative of $O$, where

\[
(4) \quad 1 < i \leq \frac{n}{2} + 1.
\]

For example, if $n = 10$, all orbits are given with representatives underlined as follows. Here $a = 10$.

$O_1 = \langle 12 \rangle = \{12, 23, 34, 45, 56, 67, 78, 89, 9a, 1a\}$

$O_2 = \langle 13 \rangle = \{13, 24, 35, 46, 57, 68, 79, 8a, 19, 2a\}$

$O_3 = \langle 14 \rangle = \{14, 25, 36, 47, 58, 69, 7a, 18, 29, 3a\}$

$O_4 = \langle 15 \rangle = \{15, 26, 37, 48, 59, 6a, 17, 28, 39, 4a\}$

$O_5 = \langle 16 \rangle = \{16, 27, 38, 49, 5a\}$. 
Let $1i$ be the representative of an orbit under $C_n$. We define the weight $w_n(1i)$ as
\[
(5) \quad w_n(1i) = \begin{cases} 
q^{n+2-2i} & \text{if } i = \frac{n}{2} + 1 \\
q^{n+1-2i} & \text{else.}
\end{cases}
\]

The weights for the other elements than the representatives are given using the weights of representatives in (5).

Assume first $\gcd(n, 2) = 1$. Note that all orbits are of size $n$ by (1) and (2). If $O_i = \{x_{i1}, x_{i2}, \ldots, x_{i(n-1)}, x_{in}\}$ is an orbit of size $n$ with the representative $x_{i1}$ and with the action
\[
x_{i1} \xrightarrow{\sigma} x_{i2} \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} x_{i(n-1)} \xrightarrow{\sigma} x_{in} \xrightarrow{\sigma} x_{i1},
\]
we define
\[
(6) \quad w_n(x_{ij+1}) = qw_n(x_{ij}) \text{ for } 1 \leq j \leq n - 1.
\]

If $\gcd(n, 2) \neq 1$, there is only one orbit of size $\frac{n}{2}$ and the other orbits are of size $n$ under the action of $C_n$. The weights for elements in an orbit of size $n$ are defined in the same way as (6). On the other hand, if $O_0 = \{x_{01}, x_{02}, \ldots, x_{0\frac{n}{2}}\}$ is the orbit of size $\frac{n}{2}$ with the representative $x_{01}$ and with the action
\[
x_{01} \xrightarrow{\sigma} x_{02} \xrightarrow{\sigma} \cdots \xrightarrow{\sigma} x_{0\frac{n}{2}} \xrightarrow{\sigma} x_{01},
\]
we define
\[
(6) \quad w_n(x_{0j+1}) = q^2w_n(x_{0j}) \text{ for } 1 \leq j \leq \frac{n}{2} - 1.
\]

Then the sum of weights of all elements in $\binom{n}{2}$ is equal to the $q$-binomial coefficient $\left[ \frac{n}{2} \right]_q$ as follows.

**Theorem 2.2.** Let $n \geq 2$ be an integer and let $T_n$ be the set of all 2-subsets of $[n]$, i.e., $T_n = \binom{[n]}{2}$. If we set $w_n(T_n) = \sum_{x \in T_n} w_n(x)$, then we have
\[
w_n(T_n) = \left[ \frac{n}{2} \right]_q.
\]
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Proof. Computing $w_n(T_n)$ and $\left[ \begin{array}{c} n \\ 2 \end{array} \right]_q$ for $n = 2, 3, 4, 5$ directly, we have

\[
\begin{align*}
w_2(T_2) &= 1 = \left[ \begin{array}{c} 2 \\ 2 \end{array} \right]_q \\
w_3(T_3) &= 1 + q + q^2 = \left[ \begin{array}{c} 3 \\ 2 \end{array} \right]_q \\
w_4(T_4) &= 1 + q + 2q^2 + q^3 + q^4 = \left[ \begin{array}{c} 4 \\ 2 \end{array} \right]_q \\
w_5(T_5) &= 1 + q + 2q^2 + 2q^3 + 2q^4 + q^5 + q^6 = \left[ \begin{array}{c} 5 \\ 2 \end{array} \right]_q.
\end{align*}
\]

We only work out for $n = 2\ell + 1$. The proof for $n = 2\ell$ can be given in the same way with a little modification.

Suppose now $n = 2\ell + 1$ for some $\ell \in \mathbb{N}$ and $w_n(T_n) = \left[ \begin{array}{c} n \\ 2 \end{array} \right]_q$. Since $\gcd(n, 2) = \gcd(n + 2, 2) = 1$, all orbits under $C_n$ are of size $n$ and all orbits under $C_{n+2}$ are of size $n + 2$. Let

\[
x_{11}, x_{21}, \ldots, x_{s1}
\]

be all representatives of orbits in the action of $C_n$, where

\[
s = |T_n|/|\text{orbit}| = \left( \begin{array}{c} n \\ 2 \end{array} \right)/n = \frac{1}{2}(n - 1).
\]

On the other hand, if $t$ is the number of orbits in the action of $C_{n+2}$,

\[
t = \left( \frac{n + 2}{2} \right)/(n + 2) = \frac{1}{2}(n + 1) = s + 1.
\]

Let

\[
x_{11}, x_{21}, \ldots, x_{s1}, x_{(s+1)1}
\]

be all representatives of orbits in the action of $C_{n+2}$. Then all orbits $O_1, O_2, \ldots, O_s$ under the action of $C_n$ are

\[
\begin{align*}
O_1 &= \{x_{11}, x_{12}, \ldots, x_{1(n-1)}, x_{1n}\} \\
O_2 &= \{x_{21}, x_{22}, \ldots, x_{2(n-1)}, x_{2n}\} \\
\vdots \\
O_s &= \{x_{s1}, x_{s2}, \ldots, x_{s(n-1)}, x_{sn}\}
\end{align*}
\]
while

\begin{align*}
O'_1 &= \{x_{11}, x_{12}, \ldots, x_{1n}, x_{1(n+1)}, x_{1(n+2)}\} \\
O'_2 &= \{x_{21}, x_{22}, \ldots, x_{2n}, x_{2(n+1)}, x_{2(n+2)}\} \\
& \vdots \\
O'_s &= \{x_{s1}, x_{s2}, \ldots, x_{sn}, x_{s(n+1)}, x_{s(n+2)}\} \\
O'_{s+1} &= \{x_{(s+1)1}, x_{(s+1)2}, \ldots, x_{(s+1)n}, x_{(s+1)(n+1)}, x_{(s+1)(n+2)}\}
\end{align*}

are all orbits under \(C_{n+2}\). Let \(x\) be the representative of an orbit under the action of \(C_n\). \(x\) can be also the representative of an orbit under the action of \(C_{n+2}\). In this case,

\[ w_{n+2}(x) = q^2 w_n(x). \]

For example, \(x = 12 \in \binom{n}{2}\) is the representative of an orbit under the action of \(C_n\). The weight of \(x\) is

\[ w_n(x) = q^{n+1-2} = q^{n-3}. \]

Also, \(x = 12\) can be considered in \(T_{n+2} = \binom{n+2}{2}\) and the weight \(w_{n+2}(x)\) is

\[ w_{n+2}(x) = q^{(n+2)+1-2} = q^{n-1}, \]

so that \(w_{n+2}(x) = q^2 w_n(x)\). Another 2-subset \(23 = \sigma(12)\) is considered as the element of \(T_{n+2}\) as well as \(T_n\). The weight of 23 is

\[ w_n(23) = q w_n(12) \quad \text{and} \quad w_{n+2}(23) = q w_{n+2}(12) \]

so that \(w_{n+2}(23) = q^2 w_n(23)\). Using this relation we compute \(w_{n+2}(T_{n+2})\).

Let \(r_n(q)\) be the sum of weights of representatives of all orbits of size \(n\). From (7) and assumption we have

\[ w_n(T_n) = \sum_{i=1}^{s} \sum_{x \in O'_i} w_n(x) = \sum_{i=1}^{s} w_n(x_{i1})[n]_q = r_n(q)[n]_q = \left\lfloor \frac{n}{2} \right\rfloor_q. \]

On the other hand, if we use (8), we have

\[ w_{n+2}(T_{n+2}) = \sum_{i=1}^{s+1} \sum_{x \in O'_i} w_{n+2}(x) = \sum_{i=1}^{s} \sum_{x \in O'_i} w_{n+2}(x) + \sum_{x \in O'_{s+1}} w_{n+2}(x). \]
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Here

$$\sum_{i=1}^{s} \sum_{x \in O'_{i}} w_{n+2}(x) = \sum_{i=1}^{s} \sum_{j=1}^{n+2} w_{n+2}(x_{ij}) = \sum_{i=1}^{s} w_{n+2}(x_{1i})[n+2]_{q}$$

$$= \sum_{i=1}^{s} q^{2}w_{n}(x_{1i})([n]_{q} + q^{n}[2]_{q})$$

$$= q^{2}r_{n}(q)[n]_{q} + q^{n+2}r_{n}(q)[2]_{q}$$

$$= q^{2} \left[ \begin{array}{c} n \\ 2 \end{array} \right]_{q} + q^{n+2} \left[ \begin{array}{c} n \\ 1 \end{array} \right]_{q}.$$

Using (4) we can find the representatives of all orbits under of $C_{n+2}$. $1(\ell + 2)$ is the only one representative of orbit in the action of $C_{n+2}$ which are not in orbits of the action of $C_{n}$. Using the weights given in (5) and (6)

$$\sum_{x \in O'_{i+1}} w_{n+2}(x) = w_{n}(1(\ell + 2))[n+2]_{q}$$

$$= q^{(2\ell+3)+2(\ell+2)}[n+2]_{q} = [n+2]_{q}.$$

Combining (9) and (10), we have

$$w_{n+2}(T_{n+2}) = q^{2} \left[ \begin{array}{c} n \\ 2 \end{array} \right]_{q} + q^{n+2} \left[ \begin{array}{c} n \\ 1 \end{array} \right]_{q} + [n+2]_{q}$$

$$= \left[ \begin{array}{c} n+2 \\ 2 \end{array} \right]_{q}$$

from Proposition 2.1.

Hence we have $w_{n}(T_{n}) = \left[ \begin{array}{c} n \\ 2 \end{array} \right]_{q}$ for $n \geq 2$.

**Theorem 2.3.** Orbit polynomials $O_{n}^{n,2}(q)$ is equal to the sum of weights of representatives of all orbits of size $n$.

**Proof.** Assume first $\gcd(n, 2) = 1$. Then there are only $s$ orbits of size $n$ under $C_{n}$, where $s = \left( \frac{n}{2} \right)/n$. Let $O_{1}, O_{2}, \ldots, O_{s}$ be all orbits of size $n$
under $C_n$. Then from the proof of Theorem 2.2 we know that
\begin{equation}
(11) \quad w_n(T_n) = r_n(q)[n]_q.
\end{equation}

Assume now $\gcd(n,2) \neq 1$. Then there are $s$ orbits $O_1, O_2, \ldots, O_s$ of size $n$ where $s = \left(\left(\frac{n}{2}\right) - \frac{n}{2}\right)/n$, and there is only one orbit
\[ O_0 = \{x_{01}, x_{02}, \ldots, x_{0\frac{n}{2}}\} \]
of size $\frac{n}{2}$. Hence
\begin{equation}
(12) \quad w_n(T_n) = \sum_{x \in \left(\frac{n}{2}\right)} w_n(x) = \sum_{x \in O_0} w_n(x) + \sum_{i=1}^s \sum_{x \in O_i} w_n(x)
\end{equation}
\[ = (1 + q^2 + \cdots + q^{n-2}) + \sum_{i=1}^s w_n(x_{i1})[n]_q \]
\[ = \left[\frac{n}{2}\right] q^2 + r_n(q)[n]_q. \]

From (3), we have
\begin{equation}
(13) \quad \left[\begin{array}{c} n \\ \frac{n}{2} \end{array}\right]_q = \begin{cases} [n]_q O_n^{n,2}(q) & \text{if } \gcd(n,2) = 1 \\ \left[\frac{n}{2}\right] q^2 O_n^{n,2}(q) + [n]_q O_n^{n,2}(q) & \text{if } \gcd(n,2) \neq 1. \end{cases}
\end{equation}

Note that $O_n^{n,2}(q) = 1$. Comparing (11) and (12) with (13), we have
\[ O_n^{n,2}(q) = r_n(q). \]

**Corollary 2.4.** Let $d \mid n$. Then orbit polynomials $O_d^{n,2}(q)$ have non-negative coefficients.

**Proof.** Since $O_{n/t}^{n,k}(q) = O_{n/t}^{n,t,k/t}(q^{t})$, it is sufficient to prove Corollary 2.4 for $d = n$. Then $O_n^{n,2}(q) = r_n(q)$ by Theorem 2.3 and $r_n(q)$ clearly has non-negative coefficients from the definition.

3. **Remark**

Let $n, k$ be positive integers with $(n, k) = 1$. In this section we suggest a strategy for the constructive proof of the positivity of coefficients of the orbit polynomial $O_d^{n,k}(q)$. 

\[ \square \]
Question 1. \[ \binom{n+k}{k}_q \] has recurrence relations similar to Proposition 2.1 for \( k = 3, 4, 5 \). It would be interesting to find a recurrence relation of \( \binom{n+k}{k}_q \) similar to Proposition 2.1 for an arbitrary positive integer \( k \), i.e., to find the polynomial \( f_k(q) \) satisfying the equality

\[
\binom{n+k}{k}_q = q^{k(k-1)} \binom{n}{k}_q + q^{n+k(k-1)} \binom{n-1}{k-1}_q + f_k(q)[n+k]_q.
\]

Let \( T_n = \binom{n}{k} \) and \( T_{n+k} = \binom{n+k}{k} \), and let \( w_n(x) \) and \( w_{n+k}(y) \) be weights of \( x \in T_n \) and \( y \in T_{n+k} \), respectively. If

\[
O_1 = \{x_{11}, x_{12}, \ldots, x_{1(n-1)}, x_{1n}\} \\
O_2 = \{x_{21}, x_{22}, \ldots, x_{2(n-1)}, x_{2n}\} \\
\vdots \\
O_s = \{x_{s1}, x_{s2}, \ldots, x_{s(n-1)}, x_{sn}\}
\]

are all orbits of size \( n \) in the action of \( C_n \), and

\[
O'_1 = \{x_{11}, x_{12}, \ldots, x_{1n}, x_{1(n+1)}, \ldots, x_{1(n+k)}\} \\
O'_2 = \{x_{21}, x_{22}, \ldots, x_{2n}, x_{2(n+1)}, \ldots, x_{2(n+k)}\} \\
\vdots \\
O'_s = \{x_{s1}, x_{s2}, \ldots, x_{sn}, x_{s(n+1)}, \ldots, x_{s(n+k)}\} \\
O'_{s+1} = \{x_{(s+1)1}, x_{(s+1)2}, \ldots, x_{(s+1)n}, x_{(s+1)(n+1)}, \ldots, x_{(s+1)(n+k)}\} \\
\vdots \\
O'_t = \{x_{t1}, x_{t2}, \ldots, x_{tn}, x_{t(n+1)}, \ldots, x_{t(n+k)}\}
\]

are all orbits of size \( n + k \) under \( C_{n+k} \), we have

\[
w_{n+k}(T_{n+k}) = \sum_{i=1}^{t} \sum_{x \in O'_i} w_{n+k}(x) = \sum_{i=1}^{s} \sum_{x \in O'_i} w_{n+k}(x) + \sum_{i=s+1}^{t} \sum_{x \in O'_i} w_{n+k}(x). 
\]
**Question 2.** Define $w_n(x)$ and $w_{n+k}(y)$ such that
\[
\sum_{i=1}^{s} \sum_{x \in O'_i} w_{n+k}(x) = q^{k(k-1)} \left[ \binom{n}{k} \right]_q + q^{n+k(k-1)} \left[ \binom{n-1}{k-1} \right]_q \quad \text{and}
\]
\[
\sum_{i=s+1}^{t} \sum_{x \in O'_i} w_{n+k}(x) = f_k(q)[n+k]_q.
\]

The answers for the above Question 1 and 2 will give the constructive proof of the positivity of coefficients of the orbit polynomial $O^{\alpha,k}_{\alpha}(q)$.

**Acknowledgements**

The author thanks Professor Bruce Sagan for encouraging discussions.

**References**


**Jaejin Lee**

Department of Mathematics
Hallym University
Chunchon 24252, Korea
E-mail: jjlee@hallym.ac.kr