SOME PROPERTIES OF GR-MULTIPLICATION MODULES

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Abstract. In this paper, we provide the necessary and sufficient conditions for a faithful graded module to be a graded multiplication module and for a graded submodule of a faithful gr-multiplication to be gr-essential.

1. Introduction

Let $R$ be a commutative ring with identity $1 \neq 0$ and $M$ a unital $R$-module. $M$ is called a multiplication module provided for each submodule $N$ of $M$, there exists an ideal $I$ of $R$ such that $N = IM$ [2]. Let $G$ be a multiplicative group with identity $e$. A ring $R$ is said to be a graded ring of type $G$ if there is a family of additive subgroups of $R$, say $\{R_i \mid i \in G\}$, such that $R = \bigoplus_{i \in G} R_i$ and $R_i R_j \subseteq R_{ij}$ for all $i, j \in G$, where $R_i R_j$ is the set of all finite sums of products $r_i r_j$ with $r_i \in R_i$ and $r_j \in R_j$. The elements of $h(R) = \bigcup_{i \in G} R_i$ are called the homogeneous elements of $R$. Any nonzero $r \in R$ has a unique expression as a sum of homogeneous elements, that is, $r = \sum_{i \in G} r_i$ where $r_i$ is nonzero for a finite number of $i$ in $G$. The nonzero elements $r_i$ in the decomposition of $r$ are called the homogeneous components of $r$. Let $R$ be a graded ring of type $G$ then $R$-module $M$ is said to be a graded $R$-module if there is a family $\{M_i \mid i \in G\}$ of additive subgroups of $M$ such that $M = \bigoplus_{i \in G} M_i$ and $R_i M_j \subseteq M_{ij}$ for all $i, j \in G$. Elements of $h(M) = \bigcup_{i \in G} M_i$ are called the homogeneous elements of $M$. A submodule $N$ of $M$ is a graded submodule if $N = \bigoplus_{i \in G} (N \cap M_i)$, or equivalently, if for any $x \in N$, the homogeneous components of $x$ are again in $N$. Properties


2010 Mathematics Subject Classification: 16W50, 13A02.

Key words and phrases: gr-multiplication module, multiplication module.

This Research was supported by the Sookmyung Women’s University Research Grants 2012.
of multiplication module have been studied by many mathematicians [1], [2], [3], [5], [6], [7], [8], [9], [10]. In this paper, we generalize some of the properties of the multiplication modules to graded multiplication modules.

2. Gr-multiplication modules

In this Section we state the definition of the gr-multiplication module and introduce a basic theorem which will be a main tool used to provide proofs of the theorems in the following sections.

**Definition 2.1.** Let \( R \) be a graded ring and let \( M \) be a graded \( R \)-module. Then \( M \) is called a gr-multiplication module if for any graded submodule \( N \) of \( M \), there exists a graded ideal \( I \) of \( R \) such that \( N = IM \).

For any graded submodule \( N \) of \( M \), we denote \((N : M)_g\) the graded ideal of \( R \) generated by \((h(N) : h(M)) = \{r \in h(R) \mid rh(M) \subseteq h(N)\}\). Note that \((N : M)_g\) is the graded ideal of \( R \) generated by \((N : M) \cap h(R)\) and that \((N : M)_g = (N : M)\), where \((N : M) = \{r \in R \mid rM \subseteq N\}\). Note that if \( M \) is a graded \( R \)-module and \( N \) is a submodule of \( M \), then \((N : M)\) is a graded ideal of \( R \) [4].

**Proposition 2.2.** Let \( R \) be a graded ring and let \( M \) be a graded \( R \)-module. Then \( M \) is a gr-multiplication \( R \)-module if and only if for any graded submodule \( N \) of \( M \), \( N = (N : M)_gM \).

**Proof.** Suppose that \( M \) is a gr-multiplication module and let \( N \) be a graded submodule. Then \( N = IM \) for some graded ideal \( I \) of \( R \). Since \( I \subseteq (N : M) = (N : M)_g \), \( N = IM \subseteq (N : M)_gM \subseteq N \). Thus \( N = (N : M)_gM \). The other direction of the proof is clear by taking \((N : M)_g = I\). This completes the proof.

**Remark.** If \( M \) is a graded module and a multiplication module, then \( M \) is a gr-multiplication module. However, a gr-multiplication module may not be a multiplication module. An example of a gr-multiplication module which is not a multiplication module is given in [4].

**Proposition 2.3.** Let \( R \) be a graded ring and let \( M \) be a graded \( R \)-module. Then \( M \) is a gr-multiplication \( R \)-module if and only if for each \( m \in h(M) \), there exists a graded ideal \( I \) of \( R \) such that \( Rm = IM \).
Suppose that $\mathcal{M}$ is a gr-multiplication module. Let $m \in h(M)$. Since $Rm \simeq R$ as an $R$-module, $Rm$ is a graded submodule of $\mathcal{M}$. Hence there exists a graded ideal $I$ of $R$ such that $Rm = IM$. 

Conversely, suppose that for each $m \in h(M)$, there exists a graded ideal $I$ of $R$ such that $Rm = IM$. Let $N$ be a submodule of $M$. For each $x \in h(N)$ there exists a graded ideal $I_x$ such that $Rx = I_xM$. Let $I = \sum_{x \in h(N)} I_x$. Then $N = IM$. Therefore $\mathcal{M}$ is a gr-multiplication module. \hfill $\Box$

Let $\mathcal{M}$ be a graded $R$-module. If $P$ is a gr-maximal ideal of $R$, then we define $T_P(h(M)) = \{m \in h(M) \mid (1 - p)m = 0$ for some $p \in P\}$. 

**Lemma 2.4.** Let $\mathcal{M}$ be a gr-multiplication $R$-module and let $P$ be a gr-maximal ideal of $R$. Then $\mathcal{M} = PM$ if and only if $h(M) = T_P(h(M))$. 

**Proof.** Suppose that $\mathcal{M} = PM$. Let $m \in h(M)$. Then $Rm = IM$ for some graded ideal $I$ of $R$. Hence $Rm = IM = IPM = PM$ and $m = pm$ for some $p \in P$. Thus $(1 - p)m = 0$ and $m \in T_P(h(M))$. If follows that $h(M) = T_P(h(M))$. 

Conversely, suppose $h(M) = T_P(h(M))$. Let $m \in M$. Then $m = m_{\sigma_1} + \cdots + m_{\sigma_n}$ for some $m_{\sigma_i} \in M_{\sigma_i}$. Since $h(M) = T_P(h(M))$, $m_{\sigma_i} \in T_P(h(M))$ and hence $m = p_{\sigma_1}m_{\sigma_1} + \cdots + p_{\sigma_n}m_{\sigma_n}$ for some $p_{\sigma_i} \in P$. Thus $m \in PM$. It follows that $M = PM$. \hfill $\Box$

The following theorem can be found in [4]. For our purpose we modify the statement and provide the proof of the theorem for completeness of the paper.

**Theorem 2.5.** Let $R$ be a graded ring. Then a graded $R$-module $M$ is a gr-multiplication module if and only if for every gr-maximal ideal $P$ of $R$ either $h(M) = T_P(h(M))$ or there exist $p \in P$ and $m \in h(M)$ such that $(1 - p)M \subseteq Rm$.

**Proof.** Let $\mathcal{M}$ be a gr-multiplication module and let $P$ be a gr-maximal ideal of $R$. Suppose $\mathcal{M} = PM$. Then $h(M) = T_P(h(M))$ by Lemma 2.4. Now suppose $\mathcal{M} \neq PM$. Let $m \in h(M)$ with $m \notin PM$. Then there exists a graded ideal $I$ of $R$ such that $Rm = IM$. If $I \subseteq P$ then $Rm = IM \subseteq PM$ which gives a contradiction that $m \in PM$. Therefore $I \notin P$. Since $R = P + I$, $1 = p + i$ for some $p \in P$ and $i \in I$. Hence $1 - p \in I$. Thus $(1 - p)M \subseteq IM = Rm$. 

Conversely, let $N$ be a graded submodule of $M$ and let $I = (N : M)_p$. Then $IM \subseteq N$. Let $n \in h(N)$ and let $K = \{r \in R \mid rn \in IM\}$
be a graded ideal of $R$. Suppose $K \neq R$. Then there exists a gr-
maximal ideal $P$ of $R$ such that $K \subseteq P$. If $h(M) = T_P(h(M))$, then 
$(1 - p)n = 0$ for some $p \in P$. Hence $1 - p \in K \subseteq P$ which implies $1 \in P$. 
This is a contradiction. Thus by hypothesis, there exist $q \in P$ and $m \in h(M)$ such that $(1 - q)M \subseteq Rm$. It follows that $(1 - q)N$ is a graded 
submodule of $Rm$ and hence $(1 - q)N = JRm = Jm$ where $J = \{r \in R \mid rm \in (1 - q)N\}$ is a graded ideal of $R$. Note that 
$(1 - q)JM = J(1 - q)M \subseteq Jm \subseteq N$ and hence $(1 - q)J \subseteq I$. It follows 
that $(1 - q)^2n \in (1 - q)^2N = (1 - q)Jm \subseteq IM$. But this gives the 
contradiction $(1 - q)^2 \in K \subseteq P$. Thus $K = R$ and $n \in IM$. Hence 
h($N$) $\subseteq IM$. It follows that $N = IM$ and hence $M$ is a gr-
multiplication module.

**Corollary 2.6.** Let $M$ be a graded $R$-module such that $M = \bigoplus_{\lambda \in \Lambda} Rm_\lambda$ for some elements $m_\lambda \in h(M)$ ($\lambda \in \Lambda$). Then $M$ is a gr-
multiplication module if and only if there exist graded ideals $I_\lambda$ of $R$ 
such that $Rm_\lambda = I_\lambda M$ for all $\lambda \in \Lambda$.

**Proof.** The necessity is clear.

Conversely, suppose that there exist graded ideals $I_\lambda$ of $R$ such that 
$Rm_\lambda = I_\lambda M$ for all $\lambda \in \Lambda$. Let $P$ be a gr-maximal ideal of $R$. Suppose 
$I_\mu \nsubseteq P$ for some $\mu \in \Lambda$. Then there exist $p \in P$ such that $1 - p \in I_\mu$. 
Thus $(1 - p)M \subseteq I_\mu M = Rm_\mu$. Now suppose that $I_\lambda \subseteq P$ for all $\lambda \in \Lambda$. 
Then $Rm_\lambda \subseteq PM$ for all $\lambda \in \Lambda$ and hence $M = PM$. But for any 
$\lambda \in \Lambda$, this implies $Rm_\lambda = I_\lambda M = I_\lambda PM = PI_\lambda M = PRm_\lambda = Pm_\lambda$ 
and hence $m_\lambda \in T_P(h(M))$. It follows that $h(M) = T_P(h(M))$. By the 
Theorem 2.5, $M$ is a gr-multiplication module.

**3. Main Results**

**Definition 3.1.** An $R$-module $M$ is faithful if, whenever $r \in R$ is 
such that $rM = 0$, then $r = 0$.

The next proposition gives the conditions for a faithful graded module 
to be gr-multiplication module.

**Theorem 3.2.** Let $R$ be a graded ring and let $M$ be a faithful graded 
$R$-module. Then $M$ is a gr-multiplication module if and only if 
(i) $\bigcap_{\lambda \in \Lambda} I_\lambda M = (\bigcap_{\lambda \in \Lambda} I_\lambda)M$ for any non-empty collection of graded 
ideals $I_\lambda$ ($\lambda \in \Lambda$) of $R$, and
(ii) for any graded submodule $N$ of $M$ and graded ideal $A$ of $R$ such that $N \nsubseteq AM$ there exists an ideal $B$ with $B \nsubseteq A$ and $N \subseteq BM$.

Proof. Suppose $M$ is a gr-multiplication module. Let $I_\lambda (\lambda \in \Lambda)$ be a non-empty collection of graded ideals of $R$. Let $I = \cap_{\lambda \in \Lambda} I_\lambda$. Then $IM \subseteq \cap_{\lambda \in \Lambda} (I_\lambda M)$. Let $x \in h(\cap_{\lambda \in \Lambda} (I_\lambda M))$ and let $K = \{ r \in R \mid rx \in IM \}$ be a graded ideal of $R$. Suppose $K \neq R$. Then there exists a gr-maximal ideal $P$ of $R$ such that $K \subseteq P$. Then $x \not\in TP(h(M))$ and hence there exist $p \in P$ and $m \in h(M)$ such that $(1-p)M \subseteq R_m$. Then $(1-p)x \in (1-p)I_\lambda M = I_\lambda (1-p)M \subseteq I_\lambda m$ for all $\lambda \in \Lambda$. Thus $(1-p)x \in \cap_{\lambda \in \Lambda} (I_\lambda M)$. For each $\lambda \in \Lambda$, there exists $a_\lambda \in I_\lambda$ such that $(1-p)x = a_\lambda m$. Choose $\alpha \in \Lambda$. For each $\lambda \in \Lambda$, $a_\alpha m = a_\lambda m$ so that $(a_\alpha - a_\lambda)m = 0$. Now $(1-p)(a_\alpha - a_\lambda)M = (a_\alpha - a_\lambda)(1-p)M \subseteq (a_\alpha - a_\lambda)R_m = 0$ implies $(1-p)(a_\alpha - a_\lambda) = 0$. Therefore $(1-p)a_\alpha = (1-p)a_\lambda \in I_\lambda (\lambda \in \Lambda)$ and hence $(1-p)a_\alpha \in I$. Thus $(1-p)^2x = (1-p)a_\alpha m \in IM$. It follows that $(1-p)^2 \in K \subseteq P$, which is a contradiction. Thus $K = R$ and $x \in IM$. Hence $h(\cap_{\lambda \in \Lambda} (I_\lambda M)) \subseteq IM$. This shows that $\cap_{\lambda \in \Lambda} (I_\lambda M) \subseteq IM$ and (i) is proved. Now let $N$ be a graded submodule of $M$ and $A$ a graded ideal of $R$ such that $N \nsubseteq AM$. There exists a graded ideal $C$ of $R$ such that $N = CM$. Let $B = A \cap C$. Clear $B \nsubseteq A$ and $N = AM \cap CM = (A \cap C)M = BM$ by (i). This proves (ii).

Conversely, suppose that (i) and (ii) hold. Let $N$ be a graded submodule of $M$. Let $S = \{ I \mid I$ is a graded ideal of $R$ and $N \subseteq IM \}$. Clearly $R \in S$. Let $I_\lambda (\lambda \in \Lambda)$ be any non-empty collection of graded ideals in $S$. By (i), $\cap_{\lambda \in \Lambda} I_\lambda \in S$. By Zorn’s Lemma, $S$ has a minimal member, say $A$. Then $N \subseteq AM$. Suppose that $N \neq AM$. By (ii), there exists a graded ideal $B$ of $R$ with $B \nsubseteq A$ and $N \subseteq BM$. In this case $B \in S$, contradicting the choice of $A$. Thus $N = AM$. If follows that $M$ is a gr-multiplication module. \qed

A graded $R$-module $M$ is called finitely gr-cogenerated provided for every non-empty collection of graded submodules $N_\lambda (\lambda \in \Lambda)$ of $M$ with $\cap_{\lambda \in \Lambda} N_\lambda = 0$ there exists a finite subset $\Lambda'$ of $\Lambda$ such that $\cap_{\lambda \in \Lambda'} N_\lambda = 0$. The graded ring $R$ is called finitely gr-cogenerated provided it is finitely gr-cogenerated as an $R$-module.

COROLLARY 3.3. Let $M$ be a faithful gr-multiplication $R$-module. Then $M$ is finitely gr-cogenerated if and only if $R$ is finitely gr-cogenerated.
Proof. Suppose that $M$ is a finitely gr-cogenerated. Let $I_{\lambda}$ ($\lambda \in \Lambda$) be a non-empty collection of graded ideals of $R$ such that $\cap_{\lambda \in \Lambda}I_{\lambda} = 0$. Then $\cap_{\lambda \in \Lambda}(I_{\lambda} \cap M) = 0$ by Theorem 3.2. Since $M$ is finitely gr-cogenerated, it follows that there exists a finite subset $N$ of $\Lambda$ such that $\cap_{\lambda \in N}(I_{\lambda} \cap M) = 0$. Thus $(\cap_{\lambda \in N}I_{\lambda})M = 0$ and, because $M$ is faithful, $\cap_{\lambda \in N}I_{\lambda} = 0$. It follows that $R$ is finitely gr-cogenerated.

Conversely, let $N_\gamma$ ($\gamma \in \Gamma$) be a non-empty collection of graded submodules of $M$ such that $\cap_{\gamma \in \Gamma}N_\gamma = 0$. For each $\gamma \in \Gamma$, there exists a graded ideal $I_\gamma$ of $R$ such that $N_\gamma = I_\gamma \cap M$. Then $0 = \cap_{\gamma \in \Gamma}N_\gamma = \cap_{\gamma \in \Gamma}(I_\gamma \cap M) = (\cap_{\gamma \in \Gamma}I_\gamma) \cap M$. Thus $\cap_{\gamma \in \Gamma}I_\gamma = 0$ and by hypothesis, there exists a finite subset $\Gamma'$ of $\Gamma$ such that $\cap_{\gamma \in \Gamma'}I_\gamma = 0$. By Theorem 3.2, $\cap_{\gamma \in \Gamma'}N_\gamma = \cap_{\gamma \in \Gamma'}(I_\gamma \cap M) = (\cap_{\gamma \in \Gamma'}I_\gamma) \cap M = 0$. Hence $M$ is finitely gr-cogenerated.

A graded ideal $P$ of $R$ (i.e., a graded $R$-submodule of $R$) is called gr-prime if $P \neq R$ and whenever $rs \in P$ ($r, s \in h(R)$) then $r \in P$ or $s \in P$.

**Proposition 3.4.** Let $P$ be a gr-prime ideal of $R$ and $M$ a faithful gr-multiplication $R$-module. Let $a \in h(R)$ and $x \in h(M)$ satisfy $ax \in PM$. Then $a \in P$ or $x \in PM$.

**Proof.** Suppose $a \not\in P$. Let $K = \{ r \in R \mid rx \in PM \}$. Suppose $K \neq R$. Then there exists a gr-maximal ideal $Q$ of $R$ such that $K \subseteq Q$. Clearly $x \not\in T_Q(h(M))$. By Theorem 2.5, there exist $q \in Q$ and $m \in h(M)$ such that $(1 - q)M \subseteq Rm$. In particular, $(1 - q)x = sm$ for some $s \in R$ and $(1 - q)ax = pm$ for some $p \in P$. Thus $(as - p)m = 0$. Now $[(1 - q) \text{ann}(m)]M = 0$ implies $(1 - q) \text{ann}(m) = 0$, because $M$ is faithful, and hence $(1 - q)(as - p) = 0$. Then $(1 - q)as = (1 - q)p \in P$. But $P \subseteq K \subseteq Q$ so that $(1 - q) \not\in P$. Thus $s \in P$ and $(1 - q)x = sm \in PM$. Thus $1 - q \in K \subseteq Q$, which is a contradiction. It follows that $K = R$ and $x \in PM$, as required.

**Definition 3.5.** A graded submodule $N$ of a graded $R$-module $M$ is called gr-essential provided $N \cap K \neq 0$ for every nonzero graded submodule $K$ of $M$. A gr-essential ideal of $R$ is just a gr-essential submodule of the graded $R$-module $R$.

**Theorem 3.6.** Let $R$ be a graded ring and $M$ a faithful gr-multiplication $R$-module. A graded submodule $N$ of $M$ is gr-essential if and only if there exists a gr-essential ideal $E$ of $R$ such that $N = EM$. 
Proof. Suppose that \( N \) is a gr-essential submodule of \( M \). There exists a graded ideal \( A \) of \( R \) such that \( N = AM \). Suppose \( A \cap B = 0 \) for some graded ideal \( B \) of \( R \). By Theorem 3.2, we have \( N \cap (BM) = (AM) \cap (BM) = (A \cap B)M = 0 \), and hence \( BM = 0 \). Since \( M \) is faithful, \( B = 0 \). Hence \( A \) is a gr-essential ideal of \( R \).

Conversely, suppose that \( E \) is gr-essential ideal of \( R \). Let \( K \) be a graded submodule of \( M \) such that \((EM) \cap K = 0\). There exists a graded ideal \( C \) of \( R \) with \( K = CM \) and hence \((E \cap C)M = (EM) \cap K = 0\). Since \( M \) is faithful, it follows that \( E \cap C = 0 \) and hence \( C = 0 \). Therefore \( K = 0 \) and thus \( EM \) is a gr-essential submodule of \( M \).

References


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