ROTA-BAXTER OPERATORS OF 3-DIMENSIONAL HEISENBERG LIE ALGEBRA

GUANGZHI JI† AND XIUYING HUA∗

ABSTRACT. In this paper, we consider the question of the Rota-Baxter operators of 3-dimensional Heisenberg Lie algebra on $F$, where $F$ is an algebraic closed field. By using the Lie product of the basis elements of Heisenberg Lie algebras, all Rota-Baxter operators of 3-dimensional Heisenberg Lie algebras are calculated and left symmetric algebras of 3-dimensional Heisenberg Lie algebra are determined by using the Yang-Baxter operators.

1. Introduction

Baxter proposed the concept of Rota-Baxter operator in 1960 (see [3]), while Rota further promoted the study of Baxter operator (see [8]). Rota-Baxter operator in various fields of mathematics has been widely used (see [2,4]). This year, many people have described the Rota-Baxter operator on low-dimensional algebra, for example, in [1,6] give the Rota-Baxter operators on low-dimensional pre-Lie algebras, in [7,9] give all Rota-Baxter operators on finite-dimensional Hamilton algebras and 3-, 4- and 5-dimensional Heisenberg Superalgebras. In [4] gives the Rota-Baxter operators on exterior algebras of two variables. By using the
Lie product of the basis elements of Heisenberg Lie algebras, all Rota-Baxter operators of 3-dimensional Heisenberg Lie algebras are calculated and left symmetric algebras of 3-dimensional Heisenberg Lie algebra are determined by using the Yang-Baxter operators.

2. Definition and basic properties

Definition 2.1. Let $G$ be Lie algebra on $F$ where $F$ is a field, we say that $R$ is a Rota-Baxter operator on $G$, if the following condition holds for any $x, y$ in $G$:

\[(1) \quad [R(x), R(y)] + \lambda R([x, y]) = R([R(x), y]) + R([x, R(y)]),\]

$\forall x, y \in G, \lambda \in F$.

In particular, we say that $R$ is a Yang-Baxter operator of $G$ it is the Rota-Baxter operator of the weight $\lambda = 0$. In this case the equation (1) becomes

\[(2) \quad [R(x), R(y)] = R([R(x), y]) + R([x, R(y)]), \quad \forall x, y \in G\]

which is called the classical Yang-Baxter equation of $G$ and the Rota-Baxter of weight $\lambda = 0$ will be a solution of the classical Yang-Baxter equation of $G$.

Obviously, $\lambda^{-1}R$ is the Rota-Baxter operator of the weight 1 when $\lambda \neq 0$, hence, We can get all Rota-Baxter operators of non-zero weight by applying the Rota-Baxter operator of weight 1. Hence, we only need to calculate Rota-Baxter operators of the weights 0 and 1.

One of the applications of the Yang-Baxter operators is to construct left symmetric algebras by using these operators and defining a new operation on $G$ as Lemma 2.2.

Lemma 2.2. Let $G$ be a Lie algebra and $R$ a solution of the classical Yang-Baxter equation of $G$. We define a new operation on $G$ as follows:

\[\ast : G \times G \rightarrow F\]

\[(x, y) \mapsto x \ast y := [R(x), y] \quad \forall x, y \in G\]

then $(G, \ast)$ will be a left symmetric algebra.

Now let us to consider the 3-dimensional Heisenberg Lie algebra $G$ with base elements $\{c, e, f\}$ satisfying in the relation
\[ \begin{align*}
\{ [e, f] = -[f, e] = c \\
[x, y] = 0 \quad \text{if } x, y \notin \{ c, e, f \} 
\end{align*} \]

Now let \( R \) be a linear operator on \( G \) such that

\[
\begin{align*}
R(c) &= a_{11}c + a_{21}e + a_{31}f \\
R(e) &= a_{12}c + a_{22}e + a_{32}f \\
R(f) &= a_{13}c + a_{23}e + a_{33}f
\end{align*}
\]

where \( a_{ij} \in \mathbb{F} \) for \( i, j \in \{1, 2, 3\} \).

In other words we can write

\[
(R(c), R(e), R(f)) = (c, e, f) \begin{bmatrix} a_{11} & a_{12} & a_{13} \\
               a_{21} & a_{22} & a_{23} \\
               a_{31} & a_{32} & a_{33} \end{bmatrix}
\]

3. Main Results

**THEOREM 3.1.** There is three types of the Rota-Baxter operators of weight 0 for the 3-dimensional Heisenberg Lie algebra \( G \), which are as follows:

\[
R_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\
                      0 & a_{22} & a_{23} \\
                      0 & a_{32} & a_{33} \end{bmatrix} \quad \text{where } a_{22} - a_{11} \neq 0
\]

\[
R_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\
                      0 & a_{22} & a_{23} \\
                      0 & -\frac{a_{11}}{a_{33}} & a_{33} \end{bmatrix} \quad \text{where } a_{22} = a_{11}, a_{23} \neq 0
\]

\[
R_3 = \begin{bmatrix} 0 & a_{12} & a_{13} \\
                      0 & 0 & 0 \\
                      0 & a_{32} & a_{33} \end{bmatrix} \quad \text{where } a_{ij} \in \mathbb{F}
\]

**Proof.** Since \( R \) is linear operator, so we only need to consider the base elements which are satisfying in the equation (2) which come from
the equation (1) by substituting 0 in stead of \( \lambda \) and also we have the equations:

\[
\begin{aligned}
\begin{cases}
a_{21} = 0 \\
a_{31} = 0 \\
(a_{22} - a_{11})a_{33} = a_{11}a_{22} + a_{23}a_{32}
\end{cases}
\end{aligned}
\]

(3)

where

\[
\begin{aligned}
[R(c), R(e)] &= R([R(c), e]) + R([c, R(e)]) \implies a_{31} = 0 \\
[R(c), R(f)] &= R([R(c), f]) + R([c, R(f)]) \implies a_{21} = 0 \\
[R(e), R(f)] &= R([R(e), f]) + R([e, R(f)]) \\
&\implies (a_{22} - a_{11})a_{33} = a_{11}a_{22} + a_{23}a_{32}
\end{aligned}
\]

Discuss the situation:

Situation 1: If \( a_{22} \neq a_{11} \), then (3) becomes

\[
\begin{aligned}
\begin{cases}
a_{21} = 0 \\
a_{31} = 0 \\
a_{33} = \frac{a_{11}a_{22} + a_{23}a_{32}}{a_{22} - a_{11}}
\end{cases}
\end{aligned}
\]

and this will yield us to the Rota-Baxter operator

\[
R_1 = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
0 & a_{22} & a_{23} \\
0 & \frac{a_{11}a_{22} + a_{23}a_{32}}{a_{22} - a_{11}} & a_{33}
\end{bmatrix}
\]

where \( a_{22} - a_{11} \neq 0 \).

Situation 2: If \( a_{22} = a_{11} \), \( a_{23} \neq 0 \), then (3) becomes

\[
\begin{aligned}
\begin{cases}
a_{21} = 0 \\
a_{31} = 0 \\
a_{32} = -\frac{a_{11}}{a_{23}}
\end{cases}
\end{aligned}
\]

and this will yield us to the Rota-Baxter operator

\[
R_2 = \begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
0 & a_{22} & a_{23} \\
0 & -\frac{a_{11}}{a_{23}} & a_{33}
\end{bmatrix}
\]

where \( a_{22} = a_{11}, a_{23} \neq 0 \).

Situation 3: If \( a_{11} = a_{22}, a_{23} = 0 \), then (3) becomes

\[
\begin{aligned}
\begin{cases}
a_{21} = 0 \\
a_{31} = 0 \\
a_{11} = a_{22} = a_{23}
\end{cases}
\end{aligned}
\]
and this will yield us to the Rota-Baxter operator

\[ R_3 = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix} \text{ where } a_{ij} \in \mathbb{F}. \]

\[ \square \]

**Theorem 3.2.** The Rota-Baxter operators of weight 1 of 3-dimensional Heisenberg Lie algebra \( G \) are the following:

\[ R_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} \end{bmatrix} \frac{a_{11}a_{22}-a_{11}a_{23}a_{32}}{a_{22}-a_{11}} \text{ where } a_{22} - a_{11} \neq 0 \]

\[ R_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} \end{bmatrix} \frac{a_{11}-a_{11}^2}{a_{23}} \text{ where } a_{22} = a_{11}, a_{23} \neq 0 \]

\[ R_3 = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix} \text{ where } a_{ij} \in \mathbb{F} \]

\[ R_4 = \begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix} \text{ where } a_{ij} \in \mathbb{F} \]

**Proof.** Since \( R \) is linear operator, hence we only need to consider the base elements which are satisfying in the equation

\[ [R(x), R(y)] + R([x, y]) = R([R(x), y]) + R([x, R(y)]) \]

which come from the equation (1) by substituting 1 in stead of \( \lambda \) and also we have the equations:

\[ \begin{cases} a_{21} = 0 \\ a_{31} = 0 \\ (a_{22} - a_{11})a_{33} = a_{11}a_{22} - a_{11} + a_{23}a_{32} \end{cases} \]

where

\[ [R(c), R(e)] = R([R(c), e]) + R([c, R(e)]) \implies a_{31} = 0 \]

\[ [R(c), R(f)] = R([R(c), f]) + R([c, R(f)]) \implies a_{21} = 0 \]

\[ [R(e), R(f)] = R([R(e), f]) + R([e, R(f)]) \]
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\[ (a_{22} - a_{11})a_{33} = a_{11}a_{22} - a_{11} + a_{23}a_{32} \]

Discuss the situation:

Situation 1: If \( a_{22} \neq a_{11} \), then (4) becomes

\[
\begin{align*}
& \exists a_{21} = 0 \\
& a_{31} = 0 \\
& a_{33} = \frac{a_{11}a_{22} - a_{11} + a_{23}a_{32}}{a_{22} - a_{11}}
\end{align*}
\]

and this will yield us to the Rota-Baxter operator

\[ R_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \text{ where } a_{22} - a_{11} \neq 0. \]

Situation 2: If \( a_{22} = a_{11}, a_{23} \neq 0 \), then (4) becomes

\[
\begin{align*}
& a_{21} = 0 \\
& a_{31} = 0 \\
& a_{32} = \frac{a_{11} - a_{12}^2}{a_{23}}
\end{align*}
\]

and this will yield us to the Rota-Baxter operator

\[ R_2 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix} \text{ where } a_{22} = a_{11}, a_{23} \neq 0. \]

Situation 3: If \( a_{11} = a_{22}, a_{23} = 0 \), then \( a_{11}^2 - a_{11} = 0 \)

(1) If \( a_{11} = 0 \), then (4) becomes

\[
\begin{align*}
& \exists a_{21} = 0 \\
& a_{31} = 0 \\
& a_{11} = a_{22} = a_{23} = 0
\end{align*}
\]

which will yield us to the Rota-Baxter operator

\[ R_3 = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix} \text{ where } a_{ij} \in \mathbb{F}. \]

(2) If \( a_{11} = 1 \), then (4) becomes

\[
\begin{align*}
& a_{21} = 0 \\
& a_{31} = 0 \\
& a_{23} = 0 \\
& a_{11} = a_{22} = 1
\end{align*}
\]
which will yield us to the Rota-Baxter operator

\[ R_4 = \begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix} \text{ where } a_{ij} \in \mathbb{F}. \]

**Theorem 3.3.** The structure of left symmetric algebra of 3-dimensional Heisenberg Lie algebra

1) \( e \ast e = -a_{32}c, \ f \ast f = a_{23}c, \ e \ast f = a_{22}c, \ f \ast e = \frac{a_{23}a_{32} - a_{11}a_{22}}{a_{22} - a_{11}}c. \)

2) \( e \ast e = \frac{a_{12}}{a_{23}}c, \ f \ast f = a_{23}c, \ e \ast f = a_{22}c, \ f \ast e = -a_{33}c. \)

3) \( e \ast e = -a_{32}c, \ f \ast e = -a_{33}c. \)

**Proof.** Considering the application of Yang-Baxter operators, we can calculate directly the structure of left symmetric algebra of Heisenberg Lie algebra by lemma 2.2 and theorem 3.1.

**Corollary 3.4.** The homomorphic operator of 3-dimensional Heisenberg Lie algebra of weight 0 is

\[ R_3 = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix} \text{ where } a_{ij} \in \mathbb{F}. \]

The homomorphic operator of 3-dimensional Heisenberg Lie algebra of weight 1 is

\[ R_3 = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & 0 \\ 0 & a_{32} & a_{33} \end{bmatrix} \text{ where } a_{ij} \in \mathbb{F}. \]

**Corollary 3.5.** Neither of the 3-dimensional Heisenberg Lie algebra of weight 0 and weight 1 have isomorphic operators.

**References**


Guangzhi Ji
No. 52 xuefu road, nangang district, Harbin city, heilongjiang province
College of Science
Harbin University of Science and Technology
Harbin 150080, China
E-mail: 245368609@qq.com

Xiuying Hua
No. 52 xuefu road, nangang district, Harbin city, heilongjiang province
College of Science
Harbin University of Science and Technology
Harbin 150080, China
E-mail: huaxiuyingnihao@163.com