MAPS PRESERVING JORDAN TRIPLE PRODUCT
$A^*B + BA^*$ ON $*$-ALGEBRAS

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ABSTRACT. Let $A$ and $B$ be two prime $*$-algebras. Let $\Phi : A \to B$ be a bijective and satisfies

$$\Phi(A \bullet B \bullet A) = \Phi(A) \bullet \Phi(B) \bullet \Phi(A),$$

for all $A, B \in A$ where $A \bullet B = A^*B + BA^*$. Then, $\Phi$ is additive. Moreover, if $\Phi(I)$ is idempotent then we show that $\Phi$ is $\mathbb{R}$-linear $*$-isomorphism.

1. Introduction

Let $\mathcal{R}$ and $\mathcal{R}'$ be rings. We say the map $\Phi : \mathcal{R} \to \mathcal{R}'$ preserves product or is multiplicative if $\Phi(AB) = \Phi(A)\Phi(B)$ for all $A, B \in \mathcal{R}$, see [9]. Motivated by this, many authors pay more attention to the map on rings (and algebras) preserving different kinds of products to establish characteristics of $\Phi$ on rings. A natural problem is to study whether the map $\Phi$ preserving the new product on ring or algebra $\mathcal{R}$ is a ring or algebraic isomorphism. (for example [1–4,6–8,10–12]).

Recently, Liu and Ji [5] proved that a bijective map $\Phi$ on factor von Neumann algebras preserves, $A^*B + BA^*$ if and only if $\Phi$ is a $*$-isomorphism. Also, the authors in [14] considered such a bijective map $\Phi : A \to B$ on prime $C^*$-algebras which preserves $A^*B + \eta BA^*$, where
$\eta$ is a non-zero scalar such that $\eta \neq \pm 1$. They proved that $\Phi$ is additive. Moreover, if $\Phi(I)$ is projection then $\Phi$ is $\ast$-isomorphism.

The authors of [13], proved that if the map $\Phi$ from a prime $\ast$-ring $\mathcal{A}$ onto a $\ast$-ring $\mathcal{B}$ is bijective and preserves Jordan triple product

$$\Phi(ABA) = \Phi(A)\Phi(B)\Phi(A)$$

or $\ast$-Jordan triple product

$$\Phi(AB^*A) = \Phi(A)\Phi(B^*)\Phi(A)$$

then it is additive. Also, we show that if $\Phi : \mathcal{A} \rightarrow \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are two prime rings, preserves Jordan triple product then it is multiplicative or anti-multiplicative. Also, we show that $\Psi(A) = \Phi(A)\Phi(I)^*$, for $A \in \mathcal{A}$, is a $\mathbb{C}$-linear or conjugate $\mathbb{C}$-linear $\ast$-isomorphism.

In this paper, motivated by the above results, we consider a map $\Phi$ on two prime $\ast$-algebras $\mathcal{A}$ and $\mathcal{B}$ with a nontrivial projection such that $\Phi$ is bijective and holds in the following condition

$$\Phi(A \bullet B \bullet A) = \Phi(A) \bullet \Phi(B) \bullet \Phi(A),$$

for all $A, B \in \mathcal{A}$ where $A \bullet B = A^*B + BA^*$. We show that $\Phi$ described in the above is additive. Also, if $\Phi(I)$ is idempotent then $\Phi$ is $\mathbb{R}$-linear $\ast$-isomorphism.

It is well known that $C^*$-algebra $\mathcal{A}$ is prime, in the sense that $AAB = 0$ for $A, B \in \mathcal{A}$ implies either $A = 0$ or $B = 0$.

2. Main Results

We need the following lemma for proving our theorems.

**Lemma 2.1.** Let $\mathcal{A}$ and $\mathcal{B}$ be two $\ast$-algebras and $\Phi : \mathcal{A} \rightarrow \mathcal{B}$ be a map which satisfies in the following case:

$$\Phi(A \bullet B \bullet A) = \Phi(A) \bullet \Phi(B) \bullet \Phi(A),$$

(2.1) If $\Phi(T) = \Phi(A) + \Phi(B)$ for $T, A, B \in \mathcal{A}$ then we have

$$\Phi(X \bullet T \bullet X) = \Phi(X \bullet A \bullet X) + \Phi(X \bullet B \bullet X)$$

for all $X, Y \in \mathcal{A}$.

**Proof.** By assumption we have

$$\Phi(T)^* = \Phi(A)^* + \Phi(B)^*.$$
Maps preserving Jordan triple product $A^*B + BA^*$ on $*$-algebras

Multiplying the left and right sides of (2.2) by $\Phi(X)$, we obtain

\[(2.3) \quad 2\Phi(X)\Phi(T)^*\Phi(X) = 2\Phi(X)\Phi(A)^*\Phi(X) + 2\Phi(X)\Phi(B)^*\Phi(X).\]

Multiplying the left side of (2.2) by $\Phi(X)^2$, we obtain

\[(2.4) \quad \Phi(X)^2\Phi(T) = \Phi(X)^2\Phi(A)^* + \Phi(X)^2\Phi(B)^*.\]

Multiplying the right side of (2.2) by $\Phi(X)^2$, we obtain

\[(2.5) \quad \Phi(T)^*\Phi(X)^2 = \Phi(A)^*\Phi(X)^2 + \Phi(B)^*\Phi(X)^2.\]

Adding 2 times of (2.3), (2.4) and (2.5) together and making use of (2.1) we have

\[\Phi(X \cdot T \cdot X) = \Phi(X \cdot A \cdot X) + \Phi(X \cdot T \cdot X).\]

\[\square\]

Our first theorem is as follows:

**Theorem 2.2.** Let $\mathcal{A}$ and $\mathcal{B}$ be two prime $*$-algebras with unit $I_A$ and $I_B$ respectively, a nontrivial projection and $\Phi : \mathcal{A} \to \mathcal{B}$ be a bijective map which satisfies in the following condition

\[(2.6) \quad \Phi(A \cdot B \cdot A) = \Phi(A) \cdot \Phi(B) \cdot \Phi(A)\]

for all $A, B \in \mathcal{A}$. Then $\Phi$ is additive.

**Proof.** Let $P_1$ be a nontrivial projection in $\mathcal{A}$ and $P_2 = I_A - P_1$. Denote $\mathcal{A}_{ij} = P_i \mathcal{A} P_j$, $i, j = 1, 2$, then $\mathcal{A} = \sum_{i,j=1}^{2} \mathcal{A}_{ij}$. For every $A \in \mathcal{A}$ we may write $A = A_{11} + A_{12} + A_{21} + A_{22}$. In all that follow, when we write $A_{ij}$, it indicates that $A_{ij} \in \mathcal{A}_{ij}$. For showing additivity of $\Phi$ on $\mathcal{A}$, we use above partition of $\mathcal{A}$ and give some claims that prove $\Phi$ is additive on each $\mathcal{A}_{ij}$, $i, j = 1, 2$.

We prove the above theorem by several claims.

**Claim 1.** We show that $\Phi(0) = 0$.

We know that for all $A, B \in \mathcal{A}$, the following holds

\[\Phi(A \cdot B \cdot A) = \Phi(A) \cdot \Phi(B) \cdot \Phi(A)\]

Let $B = 0$ then

\[\Phi(0) = \Phi(A) \cdot \Phi(0) \cdot \Phi(A)\]

\[= \Phi(A)\Phi(0)^*\Phi(A) + \Phi(0)^*\Phi(A)\Phi(0)\]

\[+ \Phi(0)^2\Phi(0)^* + \Phi(A)\Phi(0)^*\Phi(A)\]

\[= \Phi(A)\Phi(0)^*\Phi(A) + \Phi(0)^*\Phi(A)\Phi(0) + \Phi(0)^2\Phi(0)^* + \Phi(A)\Phi(0)^*\Phi(A)\]

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for every $A \in \mathcal{A}$. Since $\Phi$ is surjective, we can find $A$ such that $\Phi(A) = 0$, then we have $\Phi(0) = 0$.

**Claim 2.** For each $A_{11} \in \mathcal{A}_{11}$ and $A_{22} \in \mathcal{A}_{22}$ we have

$$\Phi(A_{11} + A_{22}) = \Phi(A_{11}) + \Phi(A_{22}).$$

Since $\Phi$ is surjective, there exists $T = T_{11} + T_{12} + T_{21} + T_{22} \in \mathcal{A}$ such that

$$\Phi(T) = \Phi(A_{11}) + \Phi(A_{22}).$$

By applying Lemma (2.1) to (2.7) for $P_1$ and $P_2$, we have

$$\Phi(P_1 \cdot T \cdot P_1) = \Phi(P_1 \cdot A_{11} \cdot P_1) + \Phi(P_1 \cdot A_{22} \cdot P_1) = \Phi(4A_{11}^*)$$

and

$$\Phi(P_2 \cdot T \cdot P_2) = \Phi(P_2 \cdot A_{11} \cdot P_2) + \Phi(P_2 \cdot A_{22} \cdot P_2) = \Phi(4A_{22}^*).$$

Since $\Phi$ is injective, we obtain

$$T^* P_1 + 2P_1 T^* P_1 + P_1 T^* = 4A_{11}^* \quad \text{and} \quad T^* P_2 + 2P_2 T^* P_2 + P_2 T^* = 4A_{22}^*. $$

Hence, we have $A_{11} = T_{11}$, $A_{22} = T_{22}$ and $T_{12} = T_{21} = 0$. So,

$$\Phi(A_{11} + A_{22}) = \Phi(A_{11}) + \Phi(A_{22}).$$

**Claim 3.** For each $A_{12} \in \mathcal{A}_{12}$, $A_{21} \in \mathcal{A}_{21}$ we have

$$\Phi(A_{12} + A_{21}) = \Phi(A_{12}) + \Phi(A_{21}).$$

Since $\Phi$ is surjective, we can find $T = T_{11} + T_{12} + T_{21} + T_{22} \in \mathcal{A}$ such that

$$\Phi(T) = \Phi(A_{12}) + \Phi(A_{21}).$$

By applying Lemma (2.1) to (2.8) for $P_1 - P_2$, we have

$$\Phi((P_1 - P_2) \cdot T \cdot (P_1 - P_2))$$

$$= \Phi((P_1 - P_2) \cdot A_{12} \cdot (P_1 - P_2))$$

$$+ \Phi((P_1 - P_2) \cdot A_{21} \cdot (P_1 - P_2)) = 0.$$

Since $\Phi$ is injective, we have

$$(P_1 - P_2) \cdot T \cdot (P_1 - P_2) = 0.$$

So, we obtain

$$T_{11}^* + T_{22}^* = 0.$$
it follows that $T_{11} = T_{22} = 0$.

On the other hand, by applying Lemma (2.1) to (2.8) for $X_{12}$ and $X_{21}$ we have

$$
\Phi(X_{12} \bullet T \bullet X_{12}) = \Phi(X_{12} \bullet A_{12} \bullet X_{12}) + \Phi(X_{12} \bullet A_{21} \bullet X_{12}) = \Phi(2X_{12}A_{12}^*X_{12})
$$

and

$$
\Phi(X_{21} \bullet T \bullet X_{21}) = \Phi(X_{21} \bullet A_{12} \bullet X_{21}) + \Phi(X_{21} \bullet A_{21} \bullet X_{21}) = \Phi(2X_{21}A_{21}^*X_{21}).
$$

By injection, we have

$$
X_{12} \bullet T \bullet X_{12} = 2X_{12}A_{12}^*X_{12},
$$
for all $X_{12} \in A_{12}$ and

$$
X_{21} \bullet T \bullet X_{21} = 2X_{21}A_{21}^*X_{21},
$$
for all $X_{21} \in A_{21}$. Therefore, by primeness we have $T_{12} = A_{12} = A_{21}$.

**Claim 4.** For each $A_{11} \in A_{11}$, $A_{12} \in A_{12}$, $A_{21} \in A_{21}$ we have

$$
\Phi(A_{11} + A_{12} + A_{21}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})
$$

and

$$
\Phi(A_{22} + A_{12} + A_{21}) = \Phi(A_{22}) + \Phi(A_{12}) + \Phi(A_{21}).
$$

Since $\Phi$ is surjective, there exists $T = T_{11} + T_{12} + T_{21} + T_{22}$ such that

$$
(2.9) \quad \Phi(T) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}).
$$

By applying Lemma (2.1) to (2.9) for $P_2$ and implying Claim 3, we have

$$
\Phi(P_2 \bullet T \bullet P_2) = \Phi(P_2 \bullet A_{11} \bullet P_2) + \Phi(P_2 \bullet A_{12} \bullet P_2) + \Phi(P_2 \bullet A_{21} \bullet P_2) = \Phi(A_{12}^* + A_{21}^*).
$$

So, we have

$$
T^*P_2 + 2P_2T^*P_2 + P_2T^* = A_{12}^* + A_{21}^*
$$

then

$$
4T_{22}^* + T_{12}^* + T_{21}^* = A_{12}^* + A_{21}^*.
$$
Therefore, we have
\[ T_{12} = A_{12}, \quad T_{21} = A_{21} \] and \( T_{22} = 0 \).
For showing that \( T_{11} = A_{11} \) we use the following trick. It is easy to check that
\[
\Phi(4T_{11}^*) = \Phi((P_1 - P_2) \bullet T \bullet (P_1 - P_2))
\]
\[
= \Phi((P_1 - P_2) \bullet A_{11} \bullet (P_1 - P_2)) + \Phi((P_1 - P_2) \bullet A_{12} \bullet (P_1 - P_2))
\]
\[
+ \Phi((P_1 - P_2) \bullet A_{21} \bullet (P_1 - P_2))
\]
\[
= \Phi(4A_{11}^*).
\]
By injection we have \( T_{11} = A_{11} \).
Similarly, one can prove
\[
\Phi(A_{22} + A_{12} + A_{21}) = \Phi(A_{22}) + \Phi(A_{12}) + \Phi(A_{21}).
\]

**Claim 5.** For each \( A_{11} \in A_{11}, \ A_{12} \in A_{12}, \ A_{21} \in A_{21}, \ A_{22} \in A_{22} \), we have
\[
\Phi(A_{11} + A_{12} + A_{21} + A_{22}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}).
\]

We assume \( T \) is an element in \( A \) such that
\[
(2.10) \quad \Phi(T) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}).
\]
By applying Lemma (2.1) to (2.10) for \( P_1 \) and Claim 4, we have
\[
\Phi(P_1 \bullet T \bullet P_1) = \Phi(P_1 \bullet A_{11} \bullet P_1) + \Phi(P_1 \bullet A_{12} \bullet P_1) + \Phi(P_1 \bullet A_{21} \bullet P_1)
\]
\[
+ \Phi(P_1 \bullet A_{22} \bullet P_1)
\]
\[
= \Phi(4A_{11}^*) + \Phi(A_{12}^*) + \Phi(A_{21}^*)
\]
\[
= \Phi(4A_{11}^* + A_{12}^* + A_{21}^*).
\]
Then, we have
\[
4T_{11}^* + T_{12}^* + T_{21}^* = 4A_{11}^* + A_{12}^* + A_{21}^*
\]
so, \( T_{11} = A_{11}, \ T_{12} = A_{12} \) and \( T_{21} = A_{21} \).
Similarly, by Lemma (2.1) to (2.10) for \( P_2 \) and Claim 4, we have
\[
\Phi(P_2 \bullet T \bullet P_2) = \Phi(4A_{22}^* + A_{12}^* + A_{21}^*).
\]
So, we obtain \( T_{22} = A_{22} \). Hence, we obtain
\[
\Phi(A_{11} + A_{12} + A_{21} + A_{22}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}).
\]

**Claim 6.** For each \( A_{ij}, B_{ij} \in A_{ij} \) such that \( 1 \leq i, j \leq 2 \) and \( i \neq j \), we have
\[
\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}).
\]
By a simple computation, we can show the following

\[(P_i + A_{ij}) \bullet (P_j + B_{ij}^*) \bullet (P_i + A_{ij}) = A_{ij} + B_{ij}.\]

By using Claim 5, we have

\[
\Phi(A_{ij} + B_{ij}) = \Phi((P_i + A_{ij}) \bullet (P_j + B_{ij}^*) \bullet (P_i + A_{ij})) = \Phi(P_i + A_{ij}) \bullet \Phi(P_j + B_{ij}) \bullet \Phi(P_i + A_{ij}) = (\Phi(P_i) + \Phi(A_{ij})) \bullet (\Phi(P_j) + \Phi(B_{ij}^*)) \bullet (\Phi(P_i) + \Phi(A_{ij})) = \Phi(P_i) \bullet \Phi(P_j) \bullet \Phi(P_i) + \Phi(P_i) \bullet \Phi(B_{ij}^*) \bullet \Phi(P_i) + \Phi(A_{ij}) \bullet \Phi(B_{ij}^*) \bullet \Phi(P_i) + \Phi(A_{ij}) \bullet \Phi(P_j) \bullet \Phi(A_{ij}) \bullet \Phi(B_{ij}^*) \bullet \Phi(A_{ij}) = \Phi(P_i \bullet P_j \bullet P_i) + \Phi(P_i \bullet B_{ij}^* \bullet P_i) + \Phi(A_{ij} \bullet P_j \bullet A_{ij}) + \Phi(A_{ij} \bullet B_{ij}^* \bullet A_{ij}) + \Phi(A_{ij} \bullet B_{ij}^* \bullet P_i) + \Phi(A_{ij} \bullet B_{ij}^* \bullet A_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}).
\]

**Claim 7.** For each \(A_{ii}, B_{ii} \in A_{ii}\) such that \(1 \leq i \leq 2\), we have

\[\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii}).\]

Since \(\Phi\) is surjective, we can find \(T = T_{ii} + T_{ij} + T_{ji} + T_{jj} \in A\) such that

\[(2.11) \quad \Phi(T) = \Phi(A_{ii}) + \Phi(B_{ii}).\]

By applying Lemma (2.1) to (2.11) for \(P_j\), we have

\[\Phi(P_j \bullet T \bullet P_j) = \Phi(P_j \bullet A_{ii} \bullet P_j) + \Phi(I \bullet P_j \bullet B_{ii} \bullet P_j) = 0.\]

Since \(\Phi\) is injective, we obtain

\[P_j T^* + T^* P_j + 2P_j T^* P_j = 0.\]

So, \(T_{ij} = T_{ji} = T_{jj} = 0\). Hence, we have \(T = T_{ii}\).

On the other hand, for each \(C_{ij} \in A_{ij}\) from Claim 6 and Lemma (2.1) for \(P_j + C_{ij}\) we have

\[
\Phi(T_{ii}^* C_{ij}) = \Phi((P_j + C_{ij}) \bullet T \bullet (P_j + C_{ij})) = \Phi((P_j + C_{ij}) \bullet A_{ii} \bullet P_j + C_{ij}) + \Phi((P_j + C_{ij}) \bullet B_{ii} \bullet (P_j + C_{ij})) = \Phi(A_{ii}^* C_{ij}) + \Phi(B_{ii}^* C_{ij}) = \Phi(A_{ii}^* C_{ij} + B_{ii}^* C_{ij}).
\]
So, we have
\[(T_{ii}^* - A_{ii}^* - B_{ii}^*)C_{ij} = 0\]
for all \(C_{ij} \in A_{ij}\). By primeness, we have \(T_{ii} = A_{ii} + B_{ii}\).
Hence, the additivity of \(\Phi\) comes from the above claims.

In the rest of this paper, we show that \(\Phi\) is \(*\)-isomorphism.

**Theorem 2.3.** Let \(A\) and \(B\) be two prime \(*\)-algebras with unit \(I_A\) and \(I_B\) respectively, a nontrivial projection and \(\Phi : A \to B\) be a bijective map which satisfies in the following condition
\[(2.12) \quad \Phi(A \bullet B \bullet A) = \Phi(A) \bullet \Phi(B) \bullet \Phi(A)\]
for all \(A, B \in A\). If \(\Phi(I_A)\) is idempotent, then \(\Phi\) is \(\mathbb{R}\)-linear \(*\)-isomorphism.

**Proof.** We prove the above theorem by some claims.

**Claim 1.** \(\Phi\) is a \(\mathbb{Q}\)-linear map.

By additivity of \(\Phi\), it is easy to check that \(\Phi\) is \(\mathbb{Q}\)-linear.

**Claim 2.** We show that \(\Phi\) is unital.

For any \(A \in A\), by knowing that \(\Phi(I)\) is idempotent, we have
\[
\Phi \left( \frac{I}{2} \bullet A \bullet \frac{I}{2} \right) = \Phi \left( \frac{I}{2} \right) \bullet \Phi(A) \bullet \Phi \left( \frac{I}{2} \right) \\
= \Phi \left( \frac{I}{2} \right)^2 \Phi(A)^* + \Phi(A)^* \Phi \left( \frac{I}{2} \right)^2 + 2 \Phi \left( \frac{I}{2} \right) \Phi(A)^* \Phi \left( \frac{I}{2} \right).
\]
Since \(\Phi\) is surjective, we can find \(A\) such that \(\Phi(A) = I_B\), then
\[
\Phi \left( \frac{I}{2} \bullet A \bullet \frac{I}{2} \right) = \Phi(I)
\]
by injectivity of \(\Phi\) we get
\[
\frac{I}{2} \bullet A \bullet \frac{I}{2} = I.
\]
So, \(A = I\).

**Claim 3.** \(\Phi\) preserves projections on the both sides.
Suppose that $P \in \mathcal{A}$ is a projection. From the Claim 1 we have
\[
\Phi(P) = \Phi\left(\frac{I}{2} \bullet P \bullet \frac{I}{2}\right) = \left(\Phi\left(\frac{I}{2}\right)^* \Phi(P) + \Phi(P)\Phi\left(\frac{I}{2}\right)^*\right) \bullet \Phi\left(\frac{I}{2}\right) = \Phi(P) \bullet \Phi\left(\frac{I}{2}\right) = \Phi(P)^*.
\]
Then
\[
\Phi(P) = \Phi(P)^*.
\]
Also,
\[
\Phi(P) = \Phi\left(P \bullet \frac{I}{4} \bullet P\right) = \left(\Phi(P)^* \Phi\left(\frac{I}{4}\right) + \Phi\left(\frac{I}{4}\right) \Phi(P)^*\right) \bullet \Phi(P) = \frac{1}{2} \Phi(P) \bullet \Phi(P) = \frac{\Phi(P)^*\Phi(P) + \Phi(P)\Phi(P)^*}{2} = \Phi(P)^2.
\]
So,
\[
\Phi(P) = \Phi(P)^2.
\]
Since $\Phi^{-1}$ has the same characteristics of $\Phi$ then $\Phi$ is the preserver of the projections on the both sides.

**Remark 2.4.** We note here that if $P_i$ and $P_j = I - P_i$ are two orthogonal projections then $\Phi(P_i)$ and $\Phi(P_j)$ are so.
\[
\Phi(P_i)\Phi(P_j) = \Phi(P_i)(\Phi(I) - P_i) = \Phi(P_i)(\Phi(I) - \Phi(P_i)) = 0.
\]

**Remark 2.5.** From
\[
\Phi\left(\frac{I}{2} \bullet A \bullet \frac{I}{2}\right) = \Phi\left(\frac{I}{2}\right) \bullet \Phi(A) \bullet \Phi\left(\frac{I}{2}\right)
\]
we have $\Phi(A^*) = \Phi(A)^*$. It means that $\Phi$ preserves star.
Claim 4. $\Phi(A_{ii}) = B_{ii}$.

Let $X \in A_{ii}$ be an arbitrary element, then we obtain

$$
\Phi(4X) = \Phi(P_i \bullet X^* \bullet P_i)
= (\Phi(P_i)^*\Phi(X^*) + \Phi(X^*)\Phi(P_i)^*) \bullet \Phi(P_i)
= \Phi(P_i)\Phi(X) + \Phi(X)\Phi(P_i) + 2\Phi(P_i)\Phi(X)\Phi(P_i)
$$

since $\Phi$ is $\mathbb{Q}$-linear, so we show that

$$
4\Phi(X) = \Phi(P_i)\Phi(X) + 2\Phi(P_i)\Phi(X)\Phi(P_i) + \Phi(X)\Phi(P_i).
$$

From the above equation, we obtain the following relations

$$
\Phi(P_i)\Phi(X)\Phi(P_j) = 0,
\Phi(P_j)\Phi(X)\Phi(P_i) = 0
$$

and

$$
\Phi(P_j)\Phi(X)\Phi(P_j) = 0.
$$

So, we have

$$
\Phi(X) = \sum_{i,j=1}^{2} \Phi(P_i)\Phi(X)\Phi(P_j) = \Phi(P_i)\Phi(X)\Phi(P_i)
$$

it follows that

$$
\Phi(A_{ii}) \subseteq B_{ii}.
$$

Since $\Phi^{-1}$ has the properties as $\Phi$ then we have $\Phi(A_{ii}) = B_{ii}$.

Claim 5. $\Phi(A_{ij})\Phi(P_j) = \Phi(P_i)\Phi(A_{ij})\Phi(P_j)$ for $A_{ij} \in A_{ij}$ and $1 \leq i, j \leq 2$ such that $i \neq j$.

Since $\Phi$ is star preserving we have

$$
\Phi(A_{ij}) = \Phi(P_i \bullet A_{ij}^* \bullet P_i)
= \Phi(P_i) \bullet \Phi(A_{ij}^*) \bullet \Phi(P_i)
= (\Phi(P_i)^*\Phi(A_{ij}^*) + \Phi(A_{ij}^*)\Phi(P_i)) \bullet \Phi(P_i)
= \Phi(P_i)\Phi(A_{ij}) + \Phi(A_{ij})\Phi(P_i) + 2\Phi(P_i)\Phi(A_{ij})\Phi(P_i).
$$

So, we showed that

$$
\Phi(A_{ij}) = \Phi(P_i)\Phi(A_{ij}) + \Phi(A_{ij})\Phi(P_i) + 2\Phi(P_i)\Phi(A_{ij})\Phi(P_i).
$$

We multiply the right side of above equation by $\Phi(P_j)$, to obtain

(2.13) $\Phi(A_{ij})\Phi(P_j) = \Phi(P_i)\Phi(A_{ij})\Phi(P_j)$. 

Similarly, one can show that
\[ \Phi(P_j)\Phi(A_{ji}) = \Phi(P_j)\Phi(A_{ji})\Phi(P_i). \]

**Claim 6.** Suppose that \( A_{ii}, B_{ii} \in A_{ii} \) for \( 1 \leq i \leq 2 \). Then
\[ \Phi(A_{ii}B_{ii}) = \Phi(A_{ii})\Phi(B_{ii}). \]

Let \( C_{ij} \in A_{ij} \) be an arbitrary element. Therefore, we have
\[
\Phi(A_{ii}B_{ii}C_{ij}) = \Phi((P_j + C_{ij}) \bullet (B_{ii}^*A_{ii}^*) \bullet (P_j + C_{ij}))
\]
\[
= \Phi((P_j + C_{ij}) \bullet \Phi(B_{ii}^*A_{ii}^*) \bullet \Phi(P_j + C_{ij})
\]
\[
= ((\Phi(P_j + C_{ij})^* \Phi(B_{ii}^*A_{ii}^*)
\]
\[
+ \Phi(B_{ii}^*A_{ii}^*) \Phi(P_j + C_{ij})^*) \bullet \Phi(P_j + C_{ij})
\]
\[
= (\Phi(C_{ij})^* \Phi(B_{ii}^*A_{ii}^*)) \bullet \Phi(P_j)
\]
\[
= \Phi(A_{ii}B_{ii})\Phi(C_{ij}).
\]

So, by the above equation, we obtain
\[
\Phi(A_{ii}B_{ii})\Phi(C_{ij}) = \Phi(A_{ii}B_{ii}C_{ij})
\]
\[
= \Phi(A_{ii})\Phi(B_{ii}C_{ij})
\]
\[
= \Phi(A_{ii})\Phi(B_{ii})\Phi(C_{ij}).
\]

We have
\[
(\Phi(A_{ii}B_{ii}) - \Phi(A_{ii})\Phi(B_{ii}))\Phi(C_{ij}) = 0.
\]

We multiply the above equation by \( \Phi(P_j) \) from the left side, then we have
\[
(\Phi(A_{ii}B_{ii}) - \Phi(A_{ii})\Phi(B_{ii}))\Phi(C_{ij})\Phi(P_j) = 0.
\]

By Claim 5, we have
\[
(\Phi(A_{ii}B_{ii}) - \Phi(A_{ii})\Phi(B_{ii}))\Phi(P_j)\Phi(C_{ij})\Phi(P_j) = 0.
\]

By primeness, we obtain
\[
\Phi(A_{ii}B_{ii}) = \Phi(A_{ii})\Phi(B_{ii}).
\]

**Claim 7.** Suppose that \( A_{ij} \in A_{ii} \) and \( B_{ji} \in B_{ji} \). Then
\[
\Phi(A_{ij}B_{ji}) = \Phi(A_{ij})\Phi(B_{ji}).
\]
Since $\Phi$ preserves star, we have
\[ \Phi(A_{ij}B_{ji} + B_{ji}A_{ij}) = \Phi \left( (A_{ij} + B_{ji}) \cdot \frac{I}{4} \cdot (A_{ij} + B_{ji}) \right) \]
\[ = \Phi(A_{ij} + B_{ji}) \cdot \Phi \left( \frac{I}{4} \right) \cdot \Phi(A_{ij} + B_{ji}) \]
\[ = \left( \Phi(A_{ij} + B_{ij})^* \cdot \Phi \left( \frac{I}{4} \right) \cdot \Phi(A_{ij} + B_{ji})^* \right) \cdot \Phi(A_{ij} + B_{ij}) \]
\[ = \frac{1}{2}(\Phi(A_{ij})^* + \Phi(B_{ji})^*) \cdot \Phi(A_{ij} + B_{ji}) \]
\[ = \Phi(A_{ij}) \Phi(B_{ji}) + \Phi(B_{ji}) \Phi(A_{ij}). \]

It follows that
\[ \Phi(A_{ij}B_{ji}) + \Phi(B_{ji}A_{ij}) = \Phi(A_{ij}) \Phi(B_{ji}) + \Phi(B_{ji}) \Phi(A_{ij}). \]

Multiplying the left side of the above equation by $\Phi(P_i)$ and applying Claims 4 and 5 to obtain
\[ \Phi(P_i) \Phi(A_{ij}B_{ji}) + \Phi(P_i) \Phi(B_{ji}A_{ij}) = \Phi(P_i) \Phi(A_{ij}) \Phi(B_{ji}) + \Phi(P_i) \Phi(B_{ji}) \Phi(A_{ij}). \]

So,
\[ \Phi(A_{ij}B_{ji}) = \Phi(A_{ij}) \Phi(B_{ji}). \]

**Claim 8.** For $A_{ii} \in A_{ii}$ and $B_{ij} \in A_{ij}$ we have
\[ \Phi(A_{ii}B_{ij}) = \Phi(A_{ii}) \Phi(B_{ij}). \]

Let $T_{ji}$ in $A_{ji}$ such that $i \neq j$, Claims 6 and 7 imply that
\[ \Phi(A_{ii}B_{ij}) \Phi(T_{ji}) = \Phi(A_{ii}B_{ij}T_{ji}) \]
\[ = \Phi(A_{ii}) \Phi(B_{ij}T_{ji}) \]
\[ = \Phi(A_{ii}) \Phi(B_{ij}) \Phi(T_{ji}). \]

Since $B$ is prime and by Claim 5, we have
\[ \Phi(A_{ii}B_{ij}) = \Phi(A_{ii}) \Phi(B_{ij}). \]

**Claim 9.** For $A_{ij} \in A_{ij}$ and $B_{jj} \in A_{jj}$ we have
\[ \Phi(A_{ij}B_{jj}) = \Phi(A_{ij}) \Phi(B_{jj}). \]
For each $T_{ji} \in A_{ji}$ such that $i \neq j$, we have
$$\Phi(T_{ji})\Phi(A_{ij}B_{jj}) = \Phi(T_{ji}A_{ij}B_{jj})$$
$$= \Phi(T_{ji}A_{ij})\Phi(B_{jj})$$
$$= \Phi(T_{ji})\Phi(A_{ij})\Phi(B_{jj}).$$
So,
$$\Phi(A_{ij}B_{jj}) = \Phi(A_{ij})\Phi(B_{jj}).$$

It should be clear that
$$\Phi(AB) = \Phi((A_{ii} + A_{ij} + A_{ji} + A_{jj})(B_{ii} + B_{ij} + B_{ji} + B_{jj}))$$
$$= \Phi(A_{ii}B_{ii} + A_{ii}B_{ij} + A_{ij}B_{ji} + A_{ij}B_{jj} + A_{ji}B_{ii}$$
$$+ A_{ji}B_{ij} + A_{jj}B_{ji} + A_{jj}B_{jj})$$
$$= \Phi(A_{ii})\Phi(B_{ii}) + \Phi(A_{ij})\Phi(B_{ij}) + \Phi(A_{ji})\Phi(B_{ji}) + \Phi(A_{jj})\Phi(B_{jj})$$
$$+ \Phi(A_{ji})\Phi(B_{ii}) + \Phi(A_{ij})\Phi(B_{ji}) + \Phi(A_{jj})\Phi(B_{ji}) + \Phi(A_{jj})\Phi(B_{jj})$$
$$= (\Phi(A_{ii}) + \Phi(A_{ij}) + \Phi(A_{ji}) + \Phi(A_{jj}))(\Phi(B_{ii}) + \Phi(B_{ij}) + \Phi(B_{ji})$$
$$+ \Phi(B_{jj}))$$
$$= \Phi(A)\Phi(B).$$

**Claim 10.** $\Phi$ is $\mathbb{R}$-linear.

For every $\lambda \in \mathbb{R}$, there exists two rational number sequences $\{r_n\}, \{s_n\}$ such that $r_n \leq \lambda \leq s_n$ and $\lim r_n = \lim s_n = \lambda$ when $n \to \infty$. It is clear that $\Psi$ preserves positive elements, then $\Psi$ preserves order. So, by the additivity of $\Phi$ we have
$$r_nI = \Phi(r_nI) \leq \Phi(\lambda I) \leq \Phi(s_nI) = s_nI.$$
Hence,
$$\Phi(\lambda I) = \lambda I$$
for $\lambda \in \mathbb{R}$. It means that $\Phi$ is $\mathbb{R}$-linear.

References


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