Korean J. Math. **26** (2018), No. 2, pp. 271–283
https://doi.org/10.11568/kjm.2018.26.2.271

SOME RESULTS ABOUT THE REGULARITIES OF MULTIFRACTAL MEASURES

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Abstract. In this paper, we generelize the Olsen's density theorem to any measurable set, allowing us to extend the main results of H.K. Baek in (Proc. Indian Acad. Sci. (Math. Sci.) Vol. **118**, (2008), pp. 273-279.). In particular, we tried through these results to improve the decomposition theorem of Besicovitch's type for the regularities of multifractal Hausdorff measure and packing measure.

1. Introduction

The density theorems are used in geometric measure theory to derive geometric information from given metric information. Classically, they deal with the distribution of the s-dimensional Hausdorff measure, $\mathcal{H}^s$ and the t-dimensional packing measure, $\mathcal{P}^t$. Many researchers had formulated density theorems with respect to Hausdorff measure or packing measure in some spaces. See for example [1, 4–10, 14, 15, 17–21, 23–26]. Regular sets are defined by density with respect to the Hausdorff and the packing measure [2, 10–16, 21, 22, 25, 26]. More precisely, Tricot et al. [21, 25] showed that a subset of $\mathbb{R}^n$ has an integer Hausdorff and packing dimension if it is strongly regular. Moreover, the results of [21] are improved to a generalized ϕ-Hausdorff measure in a Polish space by Mattila and Mauldin in [15]. Later, Baek [3] used the multifractal density theorems [5, 17, 18] to prove the decomposition theorem for the

Received March 6, 2018. Revised April 8, 2018. Accepted April 9, 2018.
2010 Mathematics Subject Classification: 28A78, 28A80.
Key words and phrases: Multifractal measures, Hausdorff measure, packing measure, regularity, densities.

regularities of multifractal Hausdorff measure and packing measure in Euclidean space which enables him to split a set into regular and irregular parts. In addition, he extended the Olsen's density theorem to any measurable set.

The first aim of this paper is to establish a new version of Olsen's density theorem given in [17] under less restrictive hypotheses, which will enable us to give more generalized variant of the essential results of Baek in [3]. In particular, we tried through these results to improve the decomposition theorem for the regularities of multifractal Hausdorff measure and packing measure in $\mathbb{R}^n$.

Let us recall the multifractal formalism introduced by Olsen in [17]. This formalism was motivated by Olsen's wish to provide a general mathematical setting for the ideas present in the physics literature on multifractals.

Fix an integer $n \geq 1$ and denote by $\mathcal{P}(\mathbb{R}^n)$ the family of compactly supported Borel probability measures on $\mathbb{R}^n$. Let $\mu \in \mathcal{P}(\mathbb{R}^n)$, for $q, t \in \mathbb{R}$, $E \subseteq \mathbb{R}^n$ and $\delta > 0$, we define the generalized packing pre-measure,

$$\overline{\mathcal{P}}_{\mu}^{q,t}(E) = \inf_{\delta>0} \sup \left\{ \sum_i \mu(B(x_i, r_i))^q (2r_i)^t;\ \Big(B(x_i, r_i)\Big)_i \text{ is a centered } \delta\text{-packing of } E \right\}.$$

In a similar way we define the generalized Hausdorff pre-measure,

$$\overline{\mathcal{H}}_{\mu}^{q,t}(E) = \sup_{\delta>0} \inf \left\{ \sum_i \mu(B(x_i, r_i))^q (2r_i)^t;\ \Big(B(x_i, r_i)\Big)_i \text{ is a centered } \delta\text{-covering of } E \right\},$$

with the conventions $0^q = \infty$ for $q \leq 0$ and $0^q = 0$ for $q > 0$.

The function $\overline{\mathcal{H}}_{\mu}^{q,t}$ is δ-subadditive but not increasing and the function $\overline{\mathcal{P}}_{\mu}^{q,t}$ is increasing but not δ-subadditive. That is the reason why Olsen introduced the modifications of the generalized Hausdorff and packing measures $\mathcal{H}_{\mu}^{q,t}$ and $\mathcal{P}_{\mu}^{q,t}$:

$$\mathcal{H}_{\mu}^{q,t}(E) = \sup_{F \subseteq E} \overline{\mathcal{H}}_{\mu}^{q,t}(F) \quad \text{and} \quad \mathcal{P}_{\mu}^{q,t}(E) = \inf_{E \subseteq \bigcup_i E_i} \sum_i \overline{\mathcal{P}}_{\mu}^{q,t}(E_i)$$

The functions $\mathcal{H}_{\mu}^{q,t}$ and $\mathcal{P}_{\mu}^{q,t}$ are metric outer measures and thus measures on the Borel family of subsets of $\mathbb{R}^n$. An important feature of the Hausdorff and packing measures is that $\mathcal{P}_{\mu}^{q,t} \leq \overline{\mathcal{P}}_{\mu}^{q,t}$, and there exists an integer $\xi \in \mathbb{N}$, such that $\mathcal{H}_{\mu}^{q,t} \leq \xi \mathcal{P}_{\mu}^{q,t}$.

2. Density Theorems

In this section, we consider μ, ν in $\mathcal{P}(\mathbb{R}^n)$ and q, t in $\mathbb{R}$. The first result of this section consists of a density theorem for the multifractal Hausdorff measure $\mathcal{H}_\mu^{q,t}$ and the multifractal packing measure $\mathcal{P}_\mu^{q,t}$ in $\mathbb{R}^n$. For $x \in \operatorname{supp}\mu$, we define the upper and lower (q,t)-densities of ν with respect to μ by

$$\overline{d}_\mu^{q,t}(x,\nu) = \limsup_{r\to 0} \frac{\nu\big(B(x,r)\big)}{\mu\big(B(x,r)\big)^q (2r)^t} \quad \text{and} \quad \underline{d}_\mu^{q,t}(x,\nu) = \liminf_{r\to 0} \frac{\nu\big(B(x,r)\big)}{\mu\big(B(x,r)\big)^q (2r)^t}.$$

We consider a Borel set E of $\mathbb{R}^n$ and we denote by $\mathcal{H}_\mu^{q,s}{}_{\llcorner E}$ (resp. $\mathcal{P}_\mu^{q,t}{}_{\llcorner E}$) the s-dimensional centered Hausdorff measure $\mathcal{H}_\mu^{q,s}$ (resp. t-dimensional centered packing measure $\mathcal{P}_\mu^{q,t}$) restricted to E. The density theorem was also proven with respect to multifractal Hausdorff measure and packing measure (see [18]). Let ξ be a constant that appears in Besicovitch's covering theorem (see [18]).

Theorem 1. [18] *Let E be a Borel subset of* $\operatorname{supp}\mu$.

1. Assume that $\mathcal{H}_\mu^{q,t}(E) < \infty$. We have,

$$\frac{1}{\xi}\mathcal{H}_\mu^{q,t}(E) \inf_{x\in E} \overline{d}_\mu^{q,t}(x,\nu) \le \nu(E) \le \mathcal{H}_\mu^{q,t}(E) \sup_{x\in E} \overline{d}_\mu^{q,t}(x,\nu), \tag{2.1}$$

and

$$1 \le \overline{d}_\mu^{q,t}(x, \mathcal{H}_\mu^{q,t}{}_{\llcorner E}) \le \xi, \quad \textit{for } \mathcal{H}_\mu^{q,t}{}_{\llcorner E}\text{-a.a. } x \in E. \tag{2.2}$$

2. If $\mathcal{P}_\mu^{q,t}(E) < \infty$, then

$$\mathcal{P}_\mu^{q,t}(E) \inf_{x\in E} \underline{d}_\mu^{q,t}(x,\nu) \le \nu(E) \le \mathcal{P}_\mu^{q,t}(E) \sup_{x\in E} \underline{d}_\mu^{q,t}(x,\nu), \tag{2.3}$$

and

$$\underline{d}_\mu^{q,t}(x, \mathcal{P}_\mu^{q,t}{}_{\llcorner E}) = 1, \quad \textit{for } \mathcal{P}_\mu^{q,t}{}_{\llcorner E}\text{-a.a. } x \in E. \tag{2.4}$$

Remark 1. Let $\mu, \nu \in \mathcal{P}(\mathbb{R}^n)$ and $q, t \in \mathbb{R}$. Assume either $q \le 0$, or $0 < q$ and μ is a doubling measure. Then for every set $E \subseteq \operatorname{supp}\mu$ such that $\mathcal{H}_\mu^{q,t}(E) < \infty$ we have

$$\mathcal{H}_\mu^{q,t}(E) \inf_{x\in E} \overline{d}_\mu^{q,t}(x,\nu) \le \nu(E) \le \mathcal{H}_\mu^{q,t}(E) \sup_{x\in E} \overline{d}_\mu^{q,t}(x,\nu),$$

and

$$\overline{d}_\mu^{q,t}(x, \mathcal{H}_\mu^{q,t}{}_{\llcorner E}) = 1, \quad \text{for } \mathcal{H}_\mu^{q,t}{}_{\llcorner E}\text{-a.a. } x \in E.$$

Definition 1. Let E be a Borel subset of $\operatorname{supp}\mu$.

1. We say that $\mathcal{H}_\mu^{q,t}$ and $\mathcal{P}_\mu^{q,t}$ are equivalent on E and we write $\mathcal{H}_\mu^{q,t}(E) \simeq \mathcal{P}_\mu^{q,t}(E)$ if
$$\mathcal{H}_\mu^{q,t}(E) = 0 \Leftrightarrow \mathcal{P}_\mu^{q,t}(E) = 0.$$
2. We write $\mathcal{H}_\mu^{q,t}(E) \sim \mathcal{P}_\mu^{q,t}(E)$ if for any $F \subseteq E$,
$$\frac{1}{\xi}\mathcal{H}_\mu^{q,t}(F) \le \mathcal{P}_\mu^{q,t}(F) \le \mathcal{H}_\mu^{q,t}(F).$$
3. If $\nu = \mathcal{H}_\mu^{q,s}{}_{\llcorner E}$, we put $\overline{D}_\mu^{q,t}(x,E) = \overline{d}_\mu^{q,t}(x,\nu)$ and $\underline{D}_\mu^{q,t}(x,E) = \underline{d}_\mu^{q,t}(x,\nu)$. When $\nu = \mathcal{P}_\mu^{q,t}{}_{\llcorner E}$, we also define $\overline{\Delta}_\mu^{q,t}(x,E) = \overline{d}_\mu^{q,t}(x,\nu)$ and $\underline{\Delta}_\mu^{q,t}(x,E) = \underline{d}_\mu^{q,t}(x,\nu)$. If $\overline{D}_\mu^{q,t}(x,E) = \underline{D}_\mu^{q,t}(x,E)$ $\big($resp. $\overline{\Delta}_\mu^{q,t}(x,E) = \underline{\Delta}_\mu^{q,t}(x,E)\big)$, we write $D_\mu^{q,t}(x,E)$ $\big($resp. $\Delta_\mu^{q,t}(x,E)\big)$ for the common value.

In the sequel, we prove our density theorems. In particular, we extend the Olsen's density theorem for the multifractal Hausdorff measure and packing measure in $\mathbb{R}^n$.

Proposition 1. *Let E be a Borel subset of* $\operatorname{supp}\mu$ *such that* $\mathcal{P}_\mu^{q,t}(E) < +\infty$.

1. *If $\underline{D}_\mu^{q,t}(x,E) = 1$ for $\mathcal{H}_\mu^{q,t}$-a.e. $x \in E$, then $\mathcal{P}_\mu^{q,t}(E) = \mathcal{H}_\mu^{q,t}(E)$.*
2. *If $\overline{\Delta}_\mu^{q,t}(x,E) = 1$ for $\mathcal{P}_\mu^{q,t}$-a.e. $x \in E$, then $\mathcal{H}_\mu^{q,t}(E) \simeq \mathcal{P}_\mu^{q,t}(E)$.*

Proof. Follows immediately from Theorem 1. □

Theorem 2. *Let E be a Borel subset of* $\operatorname{supp}\mu$ *such that* $\mathcal{P}_\mu^{q,t}(E) < \infty$.

1. *If $\mathcal{H}_\mu^{q,t}(E) \sim \mathcal{P}_\mu^{q,t}(E)$ then $1 \le \underline{D}_\mu^{q,t}(x,E) \le \overline{D}_\mu^{q,t}(x,E) \le \xi$ for $\mathcal{P}_\mu^{q,t}$-a.e. $x \in E$.*
2. *If $1 \le \underline{D}_\mu^{q,t}(x,E) \le \overline{D}_\mu^{q,t}(x,E) \le \xi$ for $\mathcal{P}_\mu^{q,t}$-a.e. $x \in E$, then $\mathcal{H}_\mu^{q,t}(E) \simeq \mathcal{P}_\mu^{q,t}(E)$.*

Proof. 1. From (2.2), if $\mathcal{H}_\mu^{q,t}(E) < \infty$, we have
$$1 \le \overline{D}_\mu^{q,t}(x,E) \le \xi \quad \text{for} \quad \mathcal{H}_\mu^{q,t}\text{-a.e. } x \in E. \tag{2.5}$$
The hypothesis $\mathcal{H}_\mu^{q,t}(E) \sim \mathcal{P}_\mu^{q,t}(E)$ implies that
$$\frac{1}{\xi}\mathcal{H}_\mu^{q,t}(F) \le \mathcal{P}_\mu^{q,t}(F) \le \mathcal{H}_\mu^{q,t}(F), \quad \text{for any } F \subset E. \tag{2.6}$$

Thanks to (2.5) and (2.6), we have

$$1 \le \overline{D}_{\mu}^{q,t}(x,E) \le \xi \quad \text{for} \quad \mathcal{P}_{\mu}^{q,t}\text{-a.e. } x \in E. \tag{2.7}$$

Now, we consider the set $F = \Big\{x \in E,\ \ \underline{D}_{\mu}^{q,t}(x,E) < 1\Big\}$, and for $m \in \mathbb{N}^*$

$$F_m = \left\{x \in E,\ \underline{D}_{\mu}^{q,t}(x,E) < 1 - \frac{1}{m}\right\}.$$

From (2.3) and (2.6), we have

$$\mathcal{P}_{\mu}^{q,t}(F_m) \le \mathcal{H}_{\mu}^{q,t}(F_m) \le \mathcal{P}_{\mu}^{q,t}(F_m)\left(1 - \frac{1}{m}\right).$$

This implies that $\mathcal{P}_{\mu}^{q,t}(F_m) = 0$. As $F = \bigcup_m F_m$, we obtain $\mathcal{P}_{\mu}^{q,t}(F) = 0$, i.e.

$$\underline{D}_{\mu}^{q,t}(x,E) \ge 1 \quad \text{for} \quad \mathcal{P}_{\mu}^{q,t}\text{-a.a. } x \in E. \tag{2.8}$$

Finally, (2.7) and (2.8) give the result.

2. Consider the set

$$F = \Big\{x \in E, 1 \le \underline{D}_{\mu}^{q,t}(x,E) \le \overline{D}_{\mu}^{q,t}(x,E) \le \xi\Big\}.$$

From Theorem 1 (assertion 2), we have

$$\mathcal{P}_{\mu}^{q,t}(E) = \mathcal{P}_{\mu}^{q,t}(F) \le \mathcal{H}_{\mu}^{q,t}(F) \le \mathcal{H}_{\mu}^{q,t}(E) \le \xi\mathcal{P}_{\mu}^{q,t}(E)$$

and then, $\mathcal{H}_{\mu}^{q,t}(E) \simeq \mathcal{P}_{\mu}^{q,t}(E)$.

□

THEOREM 3. *Let E be a Borel subset of* $\operatorname{supp}\mu$ *such that* $\mathcal{P}_{\mu}^{q,t}(E) < \infty$.

1. *If $\mathcal{H}_{\mu}^{q,t}(E) \sim \mathcal{P}_{\mu}^{q,t}(E)$ then $1 \le \underline{\Delta}_{\mu}^{q,t}(x,E) \le \overline{\Delta}_{\mu}^{q,t}(x,E) \le \xi$ for $\mathcal{P}_{\mu}^{q,t}$-a.e. $x \in E$.*
2. *If $1 \le \underline{\Delta}_{\mu}^{q,t}(x,E) \le \overline{\Delta}_{\mu}^{q,t}(x,E) \le \xi$ for $\mathcal{P}_{\mu}^{q,t}$-a.e. $x \in E$, then $\mathcal{H}_{\mu}^{q,t}(E) \simeq \mathcal{P}_{\mu}^{q,t}(E)$.*

Proof. 1. Thanks to (2.4), if $\mathcal{P}_{\mu}^{q,t}(E) < \infty$, we have

$$1 = \underline{\Delta}_{\mu}^{q,t}(x,E) \quad \text{for} \quad \mathcal{P}_{\mu}^{q,t}\text{-a.e. } x \in E. \tag{2.9}$$

Since $\mathcal{H}_{\mu}^{q,t}(E) \sim \mathcal{P}_{\mu}^{q,t}(E)$, then

$$\frac{1}{\xi}\mathcal{H}_{\mu}^{q,t}(F) \le \mathcal{P}_{\mu}^{q,t}(F) \le \mathcal{H}_{\mu}^{q,t}(F), \quad \text{for any } \ F \subset E. \tag{2.10}$$

Using (2.9) and (2.10), we obtain

$$1 = \underline{\Delta}_\mu^{q,t}(x,E) \quad \text{for} \quad \mathcal{H}_\mu^{q,t}\text{-a.e. } x \in E. \tag{2.11}$$

Now, we consider the set $F = \Big\{x \in E, \overline{\Delta}_\mu^{q,t}(x,E) > \xi\Big\}$, and for $m \in \mathbb{N}^*$

$$F_m = \left\{x \in E, \overline{\Delta}_\mu^{q,t}(x,E) > \xi + \frac{1}{m}\right\}.$$

Using (2.1), (2.10), we get

$$\left(\xi + \frac{1}{m}\right)\frac{1}{\xi}\mathcal{H}_\mu^{q,t}(F_m) \le \mathcal{P}_\mu^{q,t}(F_m) \le \mathcal{H}_\mu^{q,t}(F_m).$$

This implies that $\mathcal{H}_\mu^{q,t}(F_m) = 0$. As $F = \bigcup_m F_m$, we obtain $\mathcal{H}_\mu^{q,t}(F) = 0$ and so, $\mathcal{P}_\mu^{q,t}(F) = 0$, i.e.

$$\overline{\Delta}_\mu^{q,t}(x,E) \le \xi \quad \text{for} \quad \mathcal{P}_\mu^{q,t}\text{-a.e. } x \in E. \tag{2.12}$$

Finally, (2.11) and (2.12) yields to the result.

2. We consider the set

$$F = \Big\{x \in E, 1 \le \underline{\Delta}_\mu^{q,t}(x,E) \le \overline{\Delta}_\mu^{q,t}(x,E) \le \xi\Big\}.$$

Using Theorem 1 (assertion 1) and since, $1 \le \underline{\Delta}_\mu^{q,t}(x,E) \le \overline{\Delta}_\mu^{q,t}(x,E) \le \xi$ for $\mathcal{P}_\mu^{q,t}$-a.e. $x \in E$, we get

$$\frac{1}{\xi}\mathcal{H}_\mu^{q,t}(E) \le \mathcal{P}_\mu^{q,t}(E) = \mathcal{P}_\mu^{q,t}(F) \le \xi\mathcal{H}_\mu^{q,t}(F) \le \xi\mathcal{H}_\mu^{q,t}(E).$$

□

Remark 2. The results in Theorems 2 and 3 are a generalization of the Olsen's density theorem in [17]. It is clear that if $\xi = 1$ and $E \subseteq \operatorname{supp}\mu$ such that $\mathcal{P}_\mu^{q,t}(E) < \infty$, then the following assertions are equivalent

1. $\mathcal{H}_\mu^{q,t}(E) = \mathcal{P}_\mu^{q,t}(E)$.
2. $\underline{D}_\mu^{q,t}(x,E) = 1 = \overline{D}_\mu^{q,t}(x,E)$ for $\mathcal{P}_\mu^{q,t}$-a.e. $x \in E$.
3. $\underline{\Delta}_\mu^{q,t}(x,E) = 1 = \overline{\Delta}_\mu^{q,t}(x,E)$ for $\mathcal{P}_\mu^{q,t}$-a.e. $x \in E$.

Question. If μ is not doubling and $q > 0$, then there exists a subset E of $\operatorname{supp}\mu$ such that $\mathcal{H}_\mu^{q,t}(E) \sim \mathcal{P}_\mu^{q,t}(E)$ if and only if $\mathcal{H}_\mu^{q,t}(E) \simeq \mathcal{P}_\mu^{q,t}(E)$?

3. Regularities of multifractal measures

In this section, we discuss the decomposition theorem for the regularities of multifractal measures. In [3], Baek proved the following statement:

THEOREM 4. [3] *Let μ be a doubling measure and E a Borel subset of* $\operatorname{supp}\mu$. *Consider the sets*

$$\mathcal{F}=\Big\{x\in E,\ \underline{D}_\mu^{q,t}(x,E)=\overline{D}_\mu^{q,t}(x,E)\Big\},$$
$$\mathcal{G}=\Big\{x\in E,\ \underline{\Delta}_\mu^{q,t}(x,E)=\overline{\Delta}_\mu^{q,t}(x,E)\Big\}.$$

1. *If $\mathcal{H}_\mu^{q,t}(E)<\infty$, then*
   (a) $\overline{D}_\mu^{q,t}(x,\mathcal{F})=\underline{D}_\mu^{q,t}(x,\mathcal{F})$, *for* $\mathcal{H}_\mu^{q,t}$-a.e. $x\in\mathcal{F}$.
   (b) $\mathcal{H}_\mu^{q,t}\Big(\Big\{x\in E\setminus\mathcal{F},\ \ \underline{D}_\mu^{q,t}(x,E\setminus\mathcal{F})=\overline{D}_\mu^{q,t}(x,E\setminus\mathcal{F})\Big\}\Big)=0.$
2. *If $\mathcal{P}_\mu^{q,t}(E)<\infty$, then*
   (a) $\overline{\Delta}_\mu^{q,t}(x,\mathcal{G})=\underline{\Delta}_\mu^{q,t}(x,\mathcal{G})$, *for* $\mathcal{P}_\mu^{q,t}$-a.e. $x\in\mathcal{G}$.
   (b) $\mathcal{P}_\mu^{q,t}\Big(\Big\{x\in E\setminus\mathcal{G},\ \ \underline{\Delta}_\mu^{q,t}(x,E\setminus\mathcal{G})=\overline{\Delta}_\mu^{q,t}(x,E\setminus\mathcal{G})\Big\}\Big)=0.$

Now, we extend the decomposition theorem of Baek for the regularities of the multifractal Hausdorff and packing measures in $\mathbb{R}^n$.

THEOREM 5. *Let E be a Borel subset of* $\operatorname{supp}\mu$. *Consider the sets*

$$F=\Big\{x\in E,\ 1\le\underline{D}_\mu^{q,t}(x,E)\le\overline{D}_\mu^{q,t}(x,E)\le\xi\Big\},$$
$$G=\Big\{x\in E,\ 1\le\underline{\Delta}_\mu^{q,t}(x,E)\le\overline{\Delta}_\mu^{q,t}(x,E)\le\xi\Big\}.$$

1. *If $\mathcal{H}_\mu^{q,t}(E)<\infty$, then*
   (a) $1\le\overline{D}_\mu^{q,t}(x,F)\le\underline{D}_\mu^{q,t}(x,F)\le\xi$, *for* $\mathcal{H}_\mu^{q,t}$-a.e. $x\in F$.
   (b) $\mathcal{H}_\mu^{q,t}\Big(\Big\{x\in E\setminus F,\ \ 1\le\underline{D}_\mu^{q,t}(x,E\setminus F)\le\overline{D}_\mu^{q,t}(x,E\setminus F)\le\xi\Big\}\Big)=0.$
2. *If $\mathcal{P}_\mu^{q,t}(E)<\infty$, then*
   (a) $1\le\overline{\Delta}_\mu^{q,t}(x,G)\le\underline{\Delta}_\mu^{q,t}(x,G)\le\xi$, *for* $\mathcal{P}_\mu^{q,t}$-a.e. $x\in G$.
   (b) $\mathcal{P}_\mu^{q,t}\Big(\Big\{x\in E\setminus G,\ \ 1\le\underline{\Delta}_\mu^{q,t}(x,E\setminus G)\le\overline{\Delta}_\mu^{q,t}(x,E\setminus G)\le\xi\Big\}\Big)=0.$

To prove this theorem, we need the following lemmas

Lemma 1. *Let E be a Borel subset of* $\operatorname{supp}\mu$ *and A be a $\mathcal{H}_\mu^{q,t}$-measurable subset of E.*

1. Suppose that $\mathcal{H}_\mu^{q,t}(E)<\infty$, then

$$\overline{D}_\mu^{q,t}(x,E)=\overline{D}_\mu^{q,t}(x,A) \text{ and } \underline{D}_\mu^{q,t}(x,E)=\underline{D}_\mu^{q,t}(x,A), \text{ for } \mathcal{H}_\mu^{q,t}\text{-a.e. } x\in A.$$

2. Suppose that $\mathcal{P}_\mu^{q,t}(E)<\infty$, then

$$\overline{\Delta}_\mu^{q,t}(x,E)=\overline{\Delta}_\mu^{q,t}(x,A) \text{ and } \underline{\Delta}_\mu^{q,t}(x,E)=\underline{\Delta}_\mu^{q,t}(x,A), \text{ for } \mathcal{H}_\mu^{q,t}\text{-a.e. } x\in A.$$

Proof. Let $\theta\in\mathcal{P}(\mathbb{R}^n)$ and define the measure $\theta_{\llcorner E}$ by $\theta_{\llcorner E}(B)=\theta(E\cap B)$, for all Borel set B. Let $\nu=\theta_{\llcorner E}$. Then, we have

$$\overline{d}_\mu^{q,t}(x,\nu)=\overline{d}_\mu^{q,t}(x,\nu_{\llcorner A}) \text{ and } \underline{d}_\mu^{q,t}(x,\nu)=\underline{d}_\mu^{q,t}(x,\nu_{\llcorner A}), \text{ for } \mathcal{H}_\mu^{q,t}\text{-a.a. on } A \tag{3.1}$$

In fact, it is clear that

$$\underline{d}_\mu^{q,t}(x,\nu)\geq\underline{d}_\mu^{q,t}(x,\nu_{\llcorner A}) \qquad \text{and} \qquad \overline{d}_\mu^{q,t}(x,\nu)\geq\overline{d}_\mu^{q,t}(x,\nu_{\llcorner A}).$$

Let's set $\lambda(B)=\nu(B\setminus A)$, for all Borel set B. Then,

$$\nu(B)=\nu\big(B\cap(A^c\cup A)\big)=\nu(B\setminus A)+\nu(B\cap A)=\lambda(B)+\nu_{\llcorner A}(B).$$

A simple calculation shows that

$$\underline{d}_\mu^{q,t}(x,\nu)\leq\underline{d}_\mu^{q,t}(x,\nu_{\llcorner A})+\overline{d}_\mu^{q,t}(x,\lambda) \text{ and } \overline{d}_\mu^{q,t}(x,\nu)\leq\overline{d}_\mu^{q,t}(x,\nu_{\llcorner A})+\overline{d}_\mu^{q,t}(x,\lambda).$$

It becomes enough to prove that $\overline{d}_\mu^{q,t}(x,\lambda)=0$. For any integer $k\neq0$, let

$$A_k=\left\{x\in A, \quad \overline{d}_\mu^{q,t}(x,\lambda)\geq\frac{1}{k}\right\}.$$

Then $A_k\subset A$, for any $k\geq1$. So, by (2.1), we have

$$0\leq\frac{1}{\xi k}\mathcal{H}_\mu^{q,t}(A_k)\leq\lambda(A_k)=\nu(A_k\setminus A)=\nu(\emptyset)=0, \qquad \text{for all } k\geq1,$$

so, we get $\mathcal{H}_\mu^{q,t}(A_k)=0$ for all $k\geq1$ and $\overline{d}_\mu^{q,t}(x,\lambda)=0$, for $\mathcal{H}_\mu^{q,t}$-a.a. on A, which leads to (3.1). $\square$

Now, in (3.1), taking $\nu=\mathcal{H}_{\mu\ \llcorner E}^{q,t}$ (resp. $\nu=\mathcal{P}_{\mu\ \llcorner E}^{q,t}$) we obtain assertion (1) (resp. assertion (2)) of the Lemma.

Lemma 2. *Let E be a Borel subset of* $\operatorname{supp}\mu$, *such that $\mathcal{P}_\mu^{q,t}(E)<\infty$. For $K=\big\{x\in E, \ \overline{\Delta}_\mu^{q,t}(x,E)<+\infty\big\}$, we have for a Borel subset L of K such that $\mathcal{H}_\mu^{q,t}(L)=0$, $\mathcal{P}_\mu^{q,t}(L)=0$.*

Proof. It is sufficient to take $\nu = \mathcal{P}^{q,t}_{\mu}{}_{\llcorner E}$ in (2.1). □

Proof of Theorem 5.

1. Since $\mathcal{H}^{q,t}_{\mu}(E) < \infty$ and it is sufficient to take $A = F$ in assertion (1) of Lemma 1, we have $\overline{D}^{q,t}_{\mu}(x, F) = \overline{D}^{q,t}_{\mu}(x, E)$ and $\underline{D}^{q,t}_{\mu}(x, F) = \underline{D}^{q,t}_{\mu}(x, E)$, for $\mathcal{H}^{q,t}_{\mu}$-a.e. $x \in F$. So, $1 \le \underline{D}^{q,t}_{\mu}(x, F) \le \overline{D}^{q,t}_{\mu}(x, F) \le \xi$, for $\mathcal{H}^{q,t}_{\mu}$-a.e. $x \in F$.

   Taking $A = E \setminus F$ in assertion (1) of Lemma 1, we get $\overline{D}^{q,t}_{\mu}(x, E \setminus F) = \overline{D}^{q,t}_{\mu}(x, E)$ and $\underline{D}^{q,t}_{\mu}(x, E \setminus F) = \underline{D}^{q,t}_{\mu}(x, E)$ for $\mathcal{H}^{q,t}_{\mu}$-a.e. $x \in E \setminus F$. Hence,

$$\mathcal{H}^{q,t}_{\mu}\left(\left\{x \in E \setminus F,\ \ 1 \le \underline{D}^{q,t}_{\mu}(x, E \setminus F) \le \overline{D}^{q,t}_{\mu}(x, E \setminus F) \le \xi\right\}\right) = 0.$$

2. Since $\mathcal{P}^{q,t}_{\mu}(E) < \infty$, from (2.4), we obtain

$$\underline{\Delta}^{q,t}_{\mu}(x, G) = 1 \text{ for } \mathcal{P}^{q,t}_{\mu}\text{-a.e. } x \in G \text{ and } 1 \le \underline{\Delta}^{q,t}_{\mu}(x, G) \le \xi \text{ for } \mathcal{P}^{q,t}_{\mu}\text{-a.e. } x \in G.$$

   By Lemma 1, $\overline{\Delta}^{q,t}_{\mu}(x, G) = \overline{\Delta}^{q,t}_{\mu}(x, E)$, for $\mathcal{H}^{q,t}_{\mu}$-a.e. $x \in G$. So, $1 \le \overline{\Delta}^{q,t}_{\mu}(x, G) \le \xi$, for $\mathcal{H}^{q,t}_{\mu}$-a.e. $x \in G$. Using Lemma 2, we have $1 \le \overline{\Delta}^{q,t}_{\mu}(x, G) \le \xi$, for $\mathcal{P}^{q,t}_{\mu}$-a.e. $x \in G$ and $1 \le \underline{\Delta}^{q,t}_{\mu}(x, G) \le \overline{\Delta}^{q,t}_{\mu}(x, G) \le \xi$, for $\mathcal{P}^{q,t}_{\mu}$-a.e. $x \in G$. Due to Lemma 1, we have $\overline{\Delta}^{q,t}_{\mu}(x, E \setminus G) = \overline{\Delta}^{q,t}_{\mu}(x, E)$ and $\underline{\Delta}^{q,t}_{\mu}(x, E \setminus G) = \underline{\Delta}^{q,t}_{\mu}(x, E)$, for $\mathcal{H}^{q,t}_{\mu}$-a.e. $x \in E \setminus G$. Since $\overline{\Delta}^{q,t}_{\mu}(x, E) \le \xi < +\infty$, Lemma 2 implies that $\overline{\Delta}^{q,t}_{\mu}(x, E \setminus G) = \overline{\Delta}^{q,t}_{\mu}(x, E)$ and $\underline{\Delta}^{q,t}_{\mu}(x, E \setminus G) = \underline{\Delta}^{q,t}_{\mu}(x, E)$, for $\mathcal{P}^{q,t}_{\mu}$-a.e. $x \in E \setminus G$. Hence,

$$\mathcal{P}^{q,t}_{\mu}\left(\left\{x \in E \setminus G,\ \ 1 \le \underline{\Delta}^{q,t}_{\mu}(x, E \setminus G) \le \overline{\Delta}^{q,t}_{\mu}(x, E \setminus G) \le \xi\right\}\right) = 0.$$

Our purpose in the following theorem is to prove the result of Theorem 4 under less restrictive hypotheses.

THEOREM 6. *Let E be a Borel subset of* $\operatorname{supp}\mu$.

1. If $\mathcal{H}^{q,t}_{\mu}(E) < \infty$, then
   (a) $\overline{D}^{q,t}_{\mu}(x, \mathcal{F}) = \underline{D}^{q,t}_{\mu}(x, \mathcal{F})$, for $\mathcal{H}^{q,t}_{\mu}$-a.e. on $\mathcal{F}$.
   (b) $\mathcal{H}^{q,t}_{\mu}\left(\left\{x \in E \setminus \mathcal{F},\ \ \underline{D}^{q,t}_{\mu}(x, E \setminus \mathcal{F}) = \overline{D}^{q,t}_{\mu}(x, E \setminus \mathcal{F})\right\}\right) = 0.$

2. If $\mathcal{P}^{q,t}_{\mu}(E) < \infty$, then

(a) $\overline{\Delta}_\mu^{q,t}(x,\mathcal{G}) = \underline{\Delta}_\mu^{q,t}(x,\mathcal{G})$, *for* $\mathcal{P}_\mu^{q,t}$-a.e. *on* $\mathcal{G}$.
(b) $\mathcal{P}_\mu^{q,t}\left(\left\{x \in E \setminus \mathcal{G},\ \ \underline{\Delta}_\mu^{q,t}(x, E\setminus\mathcal{G}) = \overline{\Delta}_\mu^{q,t}(x, E\setminus\mathcal{G})\right\}\right) = 0.$

Proof. The proof is similar to the one of Theorem 5. □

Remark 3. We obtain the conclusion of Theorem 4 under less restrictive hypotheses on measure μ (we need not to assume that μ is a doubling measure).

Definition 2. Let $(X, \mathcal{B}, \mu)$ be a measure space and E, F in $\mathcal{B}$. We will say that E is a subset of F μ-almost everywhere and write $E \subseteq F$ for μ-a.e., if $\mu(F \setminus E) = 0$.

We have the following general results.

Proposition 2. *Let E be a Borel subset of* $\operatorname{supp}\mu$ *such that* $\mathcal{P}_\mu^{q,t}(E) < +\infty$.

1. *If* $B \subseteq \left\{x \in E;\ \underline{D}_\mu^{q,t}(x,E) = 1\right\}$ *for* $\mathcal{H}_\mu^{q,t}$-a.e., *then* $\mathcal{H}_\mu^{q,t}(B) = \mathcal{P}_\mu^{q,t}(B)$.
2. *If* $B \subseteq \left\{x \in E;\ \overline{\Delta}_\mu^{q,t}(x,E) = 1\right\}$ *for* $\mathcal{P}_\mu^{q,t}$-a.e., *then* $\mathcal{H}_\mu^{q,t}(B) \simeq \mathcal{P}_\mu^{q,t}(B)$.

Proof. Follows immediately from Lemmas 1 and 2. □

Theorem 7. *Let E be a $\mathcal{P}_\mu^{q,t}$-measurable set with $\mathcal{P}_\mu^{q,t}(E) < \infty$ and let B be a measurable subset of E.*

1. *If* $\mathcal{H}_\mu^{q,t}(B) \sim \mathcal{P}_\mu^{q,t}(B)$ *then* $B \subseteq G$ *for* $\mathcal{P}_\mu^{q,t}$-a.e..
2. *If* $B \subseteq G$ *for* $\mathcal{P}_\mu^{q,t}$-a.e., *then* $\mathcal{H}_\mu^{q,t}(B) \simeq \mathcal{P}_\mu^{q,t}(B)$.

Proof. Let B be a measurable subset of E. Without loss of generality, we may assume that $\mathcal{P}_\mu^{q,t}(B) > 0$.

1. We suppose that $\mathcal{H}_\mu^{q,t}(B) \sim \mathcal{P}_\mu^{q,t}(B)$. By Theorem 3, we have $1 \le \underline{\Delta}_\mu^{q,t}(x,B) \le \overline{\Delta}_\mu^{q,t}(x,B) \le \xi$, for $\mathcal{P}_\mu^{q,t}$-a.e. $x \in B$ and so, $1 \le \underline{\Delta}_\mu^{q,t}(x,B) \le \overline{\Delta}_\mu^{q,t}(x,B) \le \xi$, for $\mathcal{H}_\mu^{q,t}$-a.e. $x \in B$. Using the assertion (2) of Lemma 1, we obtain

$$\underline{\Delta}_\mu^{q,t}(x,B) = \underline{\Delta}_\mu^{q,t}(x,E) \text{ and } \overline{\Delta}_\mu^{q,t}(x,B) = \overline{\Delta}_\mu^{q,t}(x,E) \ \text{ for } \ \mathcal{H}_\mu^{q,t}\text{-a.e. } x \in B,$$

and so,

$$1 \le \underline{\Delta}_\mu^{q,t}(x,E) \le \overline{\Delta}_\mu^{q,t}(x,E) \le \xi, \quad \text{for } \mathcal{H}_\mu^{q,t}\text{-a.e. } x \in B.$$

Since $\overline{\Delta}_{\mu}^{q,t}(x, E) < +\infty$, by using Lemma 2, we get

$$1 \leq \underline{\Delta}_{\mu}^{q,t}(x, E) \leq \overline{\Delta}_{\mu}^{q,t}(x, E) \leq \xi, \quad \text{for } \mathcal{P}_{\mu}^{q,t}\text{-a.e. } x \in B.$$

Therefore, $B \subseteq G$ for $\mathcal{P}_{\mu}^{q,t}$-a.e..

2. Now, we suppose that $B \subseteq G$ for $\mathcal{P}_{\mu}^{q,t}$-a.e., which implies that $1 \leq \underline{\Delta}_{\mu}^{q,t}(x, E) \leq \overline{\Delta}_{\mu}^{q,t}(x, E) \leq \xi$, for $\mathcal{P}_{\mu}^{q,t}$-a.e. $x \in B$. Then, we easily see that $1 \leq \underline{\Delta}_{\mu}^{q,t}(x, E) \leq \overline{\Delta}_{\mu}^{q,t}(x, E) \leq \xi$, for $\mathcal{H}_{\mu}^{q,t}$-a.e. $x \in B$. By using Lemma 1, we obtain

$$\underline{\Delta}_{\mu}^{q,t}(x, B) = \underline{\Delta}_{\mu}^{q,t}(x, E) \text{ and } \overline{\Delta}_{\mu}^{q,t}(x, B) = \overline{\Delta}_{\mu}^{q,t}(x, E), \quad \text{for } \mathcal{H}_{\mu}^{q,t}\text{-a.e. } x \in B.$$

We have $1 \leq \underline{\Delta}_{\mu}^{q,t}(x, B) \leq \overline{\Delta}_{\mu}^{q,t}(x, B) \leq \xi$, for $\mathcal{H}_{\mu}^{q,t}$-a.e. $x \in B$. Now, from Lemma 2, we get $1 \leq \underline{\Delta}_{\mu}^{q,t}(x, B) \leq \overline{\Delta}_{\mu}^{q,t}(x, B) \leq \xi$, for $\mathcal{P}_{\mu}^{q,t}$-a.e. $x \in B$. Finally, by Theorem 2, we have

$$\mathcal{H}_{\mu}^{q,t}(B) \simeq \mathcal{P}_{\mu}^{q,t}(B).$$

□

THEOREM 8. *Let E be a $\mathcal{P}_{\mu}^{q,t}$-measurable set with $\mathcal{P}_{\mu}^{q,t}(E) < \infty$ and let B be a measurable subset of E.*

1. *If $\mathcal{H}_{\mu}^{q,t}(B) \sim \mathcal{P}_{\mu}^{q,t}(B)$, then $B \subseteq F$ for $\mathcal{P}_{\mu}^{q,t}$-a.e..*
2. *If $B \subseteq F$ for $\mathcal{P}_{\mu}^{q,t}$-a.e., then $\mathcal{H}_{\mu}^{q,t}(B) \simeq \mathcal{P}_{\mu}^{q,t}(B)$.*

Proof. The proof is similar to the one of Theorem 7. □

REMARK 4. The results in Theorems 7 and 8 are generalizations of those of Baek [3]. Notice that assertions (1) and (2) of Theorem 7 (resp. Theorem 8) are equivalent in the case where $\xi = 1$. In particular, we obtain the conclusion of Proposition 3.2 and Theorem 3.5 in [3].

ACKNOWLEDGMENTS

The author would like to thank the referee for carefully reading of the manuscript and for various suggestions and improvements.

References

[1] N. Attia, B. Selmi and Ch. Souissi. *Some density results of relative multifractal analysis*. Chaos, Solitons and Fractals. **103** (2017), 1–11.

[2] H.K. Baek and H.H. Lee. *Regularity of d-measure*. Acta Math. Hungarica. **99** (2003), 25–32.

[3] H.K. Baek. *Regularities of multifractal measures*. Proc. Indian Acad. Sci. **118** (2008), 273–279.

[4] A. S. Besicovitch. *On the fundamental geometrical properties of linearly measurable plane sets of points II*. Math. Ann. **155** (1938), 296–329.

[5] J. Cole and L. Olsen. *Multifractal Variation Measures and Multifractal Density Theorems*. Real Analysis Exchange. **28** (2003), 501–514.

[6] C. Cutler. *The density theorem and Hausdorff inequality for packing measure in general metric spaces*. Illinois J. Math. **39** (1995), 676–694.

[7] Z. Douzi and B. Selmi. *Multifractal variation for projections of measures*. Chaos, Solitons & Fractals. **91** (2016), 414–420.

[8] G.A. Edgar. *Packing measures a gauge variation*. Proceedings of the american mathematical society. **122** (1994), 167–174.

[9] G. A. Edgar. *Centered densities and fractal measures*. New York J. Math. **13** (2007), 33–87.

[10] K. J. Falconer. *The Geometry of Fractal sets: Mathematical Foundations and Applications*. John Wiley & Sons Ltd., (1990).

[11] H.H. Lee and I.S. Baek. *On d-measure and d-dimension*. Real Analysis Exchange **17** (1992), 590–596.

[12] H.H. Lee and I.S. Baek. *The comparison of d-meuasure with packing and Hausdorff measures*. Kyungpook Mathematical Journal **32** (1992), 523–531.

[13] H.H. Lee and I.S. Baek. *The relations of Hausdorff, $*$-Hausdorff, and packing measures*. Real Analysis Exchange **16** (1991), 497–507.

[14] P. Mattila.*The Geometry of Sets and Measures in Euclidean Spaces*. Cambridge University Press, Cambrdige. (1995).

[15] P. Mattila and R.D. Mauldin. *Measure and dimension functions: measurablility and densities*. Math. Proc. Camb. Phil. Soc. **121** (1997), 81–100.

[16] A.P. Morse and J.F. Randolph. *The ϕ-rectifiable subsets of the plane*. Am. Math. Soc. Trans. **55** (1944), 236–305.

[17] L. Olsen. *A multifractal formalism*. Advances in Mathematics. **116** (1995), 82–196.

[18] L. Olsen. *Dimension Inequalities of Multifractal Hausdorff Measures and Multifractal Packing Measures*. Math. Scand. **86** (2000), 109–129.

[19] D. Preiss. *Geometry of measures in $\mathbb{R}^n$: distribution, rectifiablity and densities*. Ann. Math. **125** (1987), 537–643.

[20] T. Rajala. *Comparing the Hausdorff and packing measures of sets of small dimension in metric spaces*. Monatsh. Math. **164** (2011), 313–323.

[21] X.S. Raymond and C. Tricot. *Packing regularity of sets in n-space*, Math. Proc. Camb. Philos. Soc. **103** (1988), 133–145.

[22] B. Selmi. *On the strong regularity with the multifractal measures in a probability space*. Preprint, (2017).

[23] V. Suomala. *On the conical density properties of measures on* $\mathbb{R}^n$. Math. Proc. Cambridge Philos. Soc, **138** (2005), 493–512.

[24] V. Suomala. *One-sided density theorems for measures on the real line*. Atti Sem. Mat. Fis. Univ. Modena e Reggio Emilia **53** (2005), 409–416.

[25] S.J. Taylor and C. Tricot. *The packing measure of rectifiable subsets of the plane.* Math. Proc. Camb. Philos. Soc. **99** (1986), 285-296.

[26] S.J. Taylor and C. Tricot. *Packing measure and its evaluation for a brownian path*. Trans. Am. Math. Soc. **288** (1985), 679–699.

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