SOLUTIONS OF VECTOR VARIATIONAL INEQUALITY PROBLEMS

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Abstract. In this paper, we prove the existence results of the solutions for vector variational inequality problems by using the \( \| \cdot \| \)-sequentially continuous mapping.

1. Introduction

Based on the research works originated by Hartmann and Stampacchia [12] in finite dimensional Euclidean spaces, Giannessi [11] studied the vector version of scalar variational inequalities. Vector variational inequalities have been developed and extended in several areas including vector equilibrium problems and vector optimization problems, see [1,4,6,9,10,15].

Inspired and motivated by recent works [2, 5, 8, 10, 13, 14, 17, 18], in this paper we prove the existence of solutions for vector variational inequality problems by using the \( \| \cdot \| \)-sequentially continuous mapping.

Suppose that \( X \) and \( Y \) are two Banach spaces. A nonempty subset \( P \) of \( X \) is called convex cone, if \( \lambda P \subseteq P \) for all \( \lambda \geq 0 \) and \( P + P \subseteq P \). A cone \( P \) is called pointed cone if \( P \) is a cone and \( P \cap (-P) = \{0\} \), where 0 denotes the zero vector. Also, a cone \( P \) is called proper if it is properly contained in \( X \). Let \( K \) be a nonempty closed convex subset of \( X \) and \( C : K \rightarrow 2^Y \) be a multivalued mapping such that for each \( x \in K, C(x) \)...
is a closed convex cone with \( \text{int} \mathcal{C}(x) \neq \emptyset \), where \( \text{int} \mathcal{C}(x) \) denotes the interior of \( \mathcal{C}(x) \). The partial order \( \leq_{\mathcal{C}(x)} \) on \( Y \) induced by \( \mathcal{C}(x) \) is defined by declaring \( y \leq_{\mathcal{C}(x)} z \) if and only if \( z - y \in \mathcal{C}(x) \) for all \( x, y, z \in K \). We will write \( y \leq_{\mathcal{C}(x)} z \) if \( z - y \in \text{int} \mathcal{C}(x) \) in the case \( \text{int} \mathcal{C}(x) \neq \emptyset \). Let \( \mathcal{A} : K \subseteq X \to L(X,Y) \) be a mapping where \( L(X,Y) \) be the family of all bounded linear mapping from \( X \) to \( Y \) and \( \zeta : K \to X \) be a given operator. The vector variational inequality problems for finding \( x \in K \) such that

\[
\langle \mathcal{A}(x), \zeta(y) - \zeta(x) \rangle \not\in -\text{int} \mathcal{C}(x), \forall y \in K.
\]

**Special Cases:**

(i) We note that \( \zeta \equiv id_K, id_K : K \to K, id_K(x) = x \). Then (1.1) reduces to the vector variational inequality problems for finding \( x \in K \) such that

\[
\langle \mathcal{A}(x), y - x \rangle \not\in -\text{int} \mathcal{C}(x), \forall y \in K.
\]

(ii) If \( \mathcal{C}(x) = \mathbb{R}_+ \) for all \( x \in X \), then (1.1) reduces to general variational inequality problems for finding \( x \in K \) such that

\[
\langle \mathcal{A}(x), \zeta(y) - \zeta(x) \rangle \geq 0, \forall y \in K.
\]

(iii) If \( \mathcal{C}(x) = \mathbb{R}_+ \) for all \( x \in X \), then (1.2) reduces to variational inequality problems for finding \( x \in K \) such that

\[
\langle \mathcal{A}(x), y - x \rangle \geq 0, \forall y \in K.
\]

studied by Hartmann and Stampacchia [12].

**DEFINITION 1.1.** Let \( \mathcal{C} : K \to 2^Y \) be a multifunction such that \( \mathcal{C}(x) \) is a proper closed convex cone with \( \text{int} \mathcal{C}(x) \neq \emptyset \), then a mapping \( g : K \to X \) is called \( \mathcal{C}_x \)-convex if for each \( x, y \in K \) and \( \lambda \in [0,1] \),

\[
(1 - \lambda)g(x) + \lambda g(y) - g((1 - \lambda)x + \lambda y) \in \mathcal{C}(x),
\]

and called affine if for each \( x, y \in K \) and \( \lambda \in R \),

\[
g((1 - \lambda)x + \lambda y) = \lambda g(x) + (1 - \lambda)g(y).
\]

**REMARK 1.2.** If \( g : K \to Y \) is a \( \mathcal{C}_x \)-convex vector valued function, then

\[
\sum_{i=1}^{n} \lambda_i g(y_i) - g(\sum_{i=1}^{n} \lambda_i y_i) \in \mathcal{C}(x), \forall y_i \in K, \lambda_i \in [0,1], i = 1,2,\ldots,n
\]

with \( \sum_{i=1}^{n} \lambda_i = 1 \).
**Definition 1.3.** Suppose $X$ and $Y$ are two Banach spaces and $T : D \subseteq X \to L(X, Y)$ is said to be weak to $\| \cdot \|$-sequentially continuous at $x \in D$ if for every sequence $\{x_n\} \subseteq D$ that converges weakly to $x \in D$, the sequence $\{T(x_n)\} \subseteq L(X, Y)$ converges to $T(x) \in L(X, Y)$ in the topology of the norm $L(X, Y)$. We say that $T$ is weak to $\| \cdot \|$-sequentially continuous on $D \subseteq X$ and it has the property at every point $x \in D$.

**Proposition 1.4.** [13] Let $A : K \subseteq X \to L(X, Y)$ be a given operator. If $A$ is weak to $\| \cdot \|$-sequentially continuous and $K$ is weakly compact and convex. Then variational inequality admits a solution.

Let $Z$ and $Y$ be two arbitrary sets. The inverse of a mapping $f : Z \to Y$ is defined as the set valued mapping $f^{-1} : Y \Rightarrow Z$,

$$f^{-1}(y) = \{ z \in Z : f(z) = y \}.$$  

A single valued selection of a multivalued mapping $F : Z \Rightarrow Y$ is the single valued mapping $f : Z \to Y$ satisfying

$$f(z) \in F(z), \ \forall z \in Z.$$  

**Theorem 1.5.** [7] Let $Y$ be a topological vector space with a pointed closed and convex cone $C$ such that $\text{int} C \neq \emptyset$, then for all $x, y, z \in Y$, we have

(i) $x - y \in -\text{int} C$ and $x \notin -\text{int} C \Rightarrow y \notin -\text{int} C$;

(ii) $x + y \in -C$ and $x + z \notin -\text{int} C \Rightarrow z - y \notin -\text{int} C$;

(iii) $x + z - y \notin -\text{int} C$ and $-y \in -C \Rightarrow x + z \notin -\text{int} C$;

(iv) $x + y \notin -\text{int} C$ and $y - z \in -C \Rightarrow x + z \notin -\text{int} C$.

**2. Main Results**

**Theorem 2.1.** Let $K$ be a nonempty subset of $X$. Let $A : K \subseteq X \to L(X, Y)$ and $\zeta : K \to X$ be the given operators. Assume that $\zeta(K)$ is weakly compact and convex. Assume further that for every sequence $\{x_n\} \subseteq K$ the following condition holds:

if the sequence $\{\zeta(x_n)\} \subseteq \zeta(K)$ converges weakly to $\zeta(x) \subseteq \zeta(K)$ then
the sequence \( \{A(x_n)\} \subseteq L(X,Y) \) is the norm convergent to \( A(x) \subseteq L(X,Y) \).

Then (1.1) admits a solution.

**Proof.** Consider \( \beta : \zeta(K) \rightarrow K \) is a single valued selection of \( \zeta^{-1} \). Let \( \{u_n\} \subseteq \zeta(K) \) be a weakly convergent sequence to \( u \in X \). From the weak compactness of \( \zeta(K) \), we have \( u \in \zeta(K) \). We show that

\[
(A \circ \beta)(u_n) \rightarrow (A \circ \beta)(u), \quad n \rightarrow \infty.
\]

Since \( \{u_n\} \subseteq \zeta(K) \), there exists a sequence \( \{x_n\} \subseteq K \) such that \( u_n = \zeta(x_n), n \in \mathbb{N} \).

Analogously, \( u = \zeta(x) \) for some \( x \in K \), then

\[
\zeta(\beta(u_n)) = u_n, n \in \mathbb{N} \quad \text{and} \quad \zeta(\beta(u)) = u.
\]

Hence the sequence \( \{\zeta(\beta(u_n))\} \) is converges weakly to \( \zeta(\beta(u)) \). From the hypothesis of the theorem

\[
(A \circ \beta)(u_n) \rightarrow (A \circ \beta)(u), n \rightarrow \infty.
\]

Hence the operator \( A \circ \beta : \zeta(K) \rightarrow L(X,Y) \) is weak to \( \|\cdot\|\)-sequentially continuous. From Proposition 1.4, there exists \( u \in \zeta(K) \) such that

\[
\langle (A \circ \beta)(u), v - u \rangle \notin -intC(x), \forall v \in \zeta(K).
\]

Since for every \( y \in K \), there exists \( v \in \zeta(K) \) such that

\[
\zeta(y) = v,
\]

and

\[
\langle (A \circ \beta)(u), \zeta(y) - u \rangle \notin -intC(x), \forall y \in K.
\]

Since \( \zeta(\beta(u)) = u \). Thus

\[
\langle A(\beta(u)), \zeta(y) - \zeta(\beta(u)) \rangle \notin -intC(x), \forall y \in K,
\]

or equivalently \( \beta(u) \in K \) is a solution of (1.1) \( \square \)

**Remark 2.2.** The condition \( \{\zeta(x_n)\} \subseteq \zeta(K) \) is converges weakly to \( \zeta(x) \subseteq \zeta(K) \), then the sequence \( \{A(x_n)\} \subseteq L(X,Y) \) is norm convergent to \( A(x) \subseteq L(X,Y) \). From Theorem 2.1 implies that

\[
\zeta^{-1}(\zeta(x)) \subseteq A^{-1}(A(x)) \quad \text{for} \quad x \in K.
\]

Let \( x \in K \) and \( \zeta(K) \) be the weakly sequentially closed, there exists a sequence \( \{\zeta(x_n)\} \subseteq \zeta(K) \) converges to \( \zeta(x) \) in the weak topology of \( X \). But the sequence \( \{A(x_n)\} \) converges strongly to \( A(x) \). Let \( y \in \zeta^{-1}(\zeta(x)) \), then

\[
\zeta(y) = \zeta(x),
\]
hence \( \{A(x_n)\} \) is converges strongly to \( A(y) \).
Therefore \( A(y) = A(x) \), hence \( y \in A^{-1}(A(x)) \).

**Corollary 2.3.** Let \( K \subseteq X \) be a weakly compact, \( A : K \subseteq X \to L(X,Y) \) and \( \zeta : K \to X \) be the given operators. Assume that \( \zeta(K) \) is convex, \( \zeta \) is weak to weak-sequentially continuous. Further assume that for every sequence \( \{x_n\} \subseteq K \) the following condition holds:

- If the sequence \( \{\zeta(x_n)\} \subseteq \zeta(K) \) is converges weakly to \( \zeta(x) \subseteq \zeta(K) \) then the sequence \( \{A(x_n)\} \subseteq L(X,Y) \) is norm convergent to \( A(x) \subseteq L(X,Y) \). Then (1.1) admits a solution.

**Proof.** We prove that \( \zeta(K) \) is weakly compact and conclusion follows from Theorem 2.1. From Eberlein Smulian Theorem [3], \( \zeta(K) \) is weakly compact if and only if, it is weakly sequentially compact. To prove that \( \zeta(K) \) is weakly sequentially compact, let \( \{u_n\} \) be an arbitrary sequence in \( \zeta(K) \). Then there exists a sequence \( \{x_n\} \subseteq K \) such that \( u_n = \zeta(x_n) \), \( n \in \mathbb{N} \).

We show that \( \{\zeta(x_n)\} \) has a weakly convergent subsequence in \( \zeta(K) \). Since \( \{x_n\} \) is a sequence in the weakly compact set \( K \) and \( \{x_n\} \) has a weakly convergent subsequence. Let \( \{x_{n_i}\} \) be a subsequence of \( \{x_n\} \), that is weakly converges to \( x \in K \). Since \( \zeta \) is weak to weak sequentially continuous, then \( \{\zeta(x_{n_i})\} \) is converges weakly to \( \zeta(x) \) and proof is completed. \( \square \)

**Definition 2.4.** [11] An operator \( T : D \subseteq X \to L(X,Y) \) is called monotone if for all \( x, y \in D \), we have

\[
\langle T(x) - T(y), y - x \rangle \geq 0.
\]

\( T \) is monotone relative to the operator \( \gamma : D \to X \) if for all \( x, y \in D \), we have

\[
\langle T(x) - T(y), \gamma(y) - \gamma(x) \rangle \geq 0.
\]

We note that \( \gamma = \text{id}_D \), then \( T \) is called continuous on finite dimensional subspaces if for every finite dimensional subspace \( M \subseteq X \), the restriction of \( T \) to \( D \cap M \) is weak continuous, that is for every sequence \( \{x_n\} \subseteq D \cap M \) converges to \( x \in M \), the sequence \( \{A(x_n)\} \subseteq L(X,Y) \) is converges to \( A(x) \) in the weak topology of \( L(X,Y) \), see [16].

**Theorem 2.5.** Let \( X \) and \( Y \) be the two reflexive Banach spaces. Let \( A : K \subseteq X \to L(X,Y) \) be the monotone relative to \( \zeta : K \to X \), where \( \zeta(K) \) is weakly compact and convex. Assume that for every
finite dimensional subset $L \subseteq \zeta(K)$ and for every sequence $\{x_n\} \subseteq X$ such that $\zeta(x_n) \subseteq L$, and if the sequence $\{\zeta(x_n)\} \subseteq L$ is converges to $\zeta(x) \subseteq \zeta(K)$, then the sequence $\{A(x_n)\} \subseteq L(X,Y)$ is weakly converges to $A(x) \subseteq L(X,Y)$.

Then (1.1) admits a solution.

Proof. Suppose $\beta : \zeta(K) \to K$ is a single valued selection of $\zeta^{-1}$ and $u, v \in \zeta(K)$. Then

$$
\langle (A \circ \beta)(u) - (A \circ \beta)(v), u - v \rangle = \langle A(x) - A(y), \zeta(x) - \zeta(y) \rangle
$$

where $x = \beta(u), y = \beta(v)$. Since $A$ is monotone relative to $\zeta$, we have

$$
\langle A(x) - A(y), \zeta(x) - \zeta(y) \rangle \notin -int C(x).
$$

Hence the operator $A \circ \beta : \zeta(K) \to L(X,Y)$ is monotone. Let $M$ be a finite dimensional subspace of $X$ and $L = M \cap \zeta(K)$. Let $\{u_n\} \subseteq L$ be a sequence converges to $u \in \zeta(K)$. Since $M$ is finite dimensional subspace then it is closed. Hence from weak compactness of $\zeta(K)$, we get that $u \in L$.

Now, we have to show that the sequence $\{(A \circ \beta)(u_n)\} \subseteq L(X,Y)$ converges to $\{\zeta(x_n)\} \subseteq \zeta(K)$ in the weak topology of $L(X,Y)$. Since $\{u_n\} \subseteq \zeta(K)$, there exists $\{x_n\} \subseteq K$ such that $u_n = \zeta(x_n)$. Analogously $u = \zeta(x)$ for some $x \in K$, since $\beta : \zeta(K) \to K$ is a single valued selection of $\zeta^{-1}$. Observe that $\zeta(\beta(u)) = u \in L$ and $\zeta(\beta(u)) = u \in L$. Hence $\{\zeta(\beta(u_n))\}$ is converges to $\zeta(\beta(u))$. From the hypothesis of the theorem, the sequence $\{A(\beta(u_n))\} \subseteq L(X,Y)$ converges weakly to $A(\beta(u)) \subseteq L(X,Y)$ as $n \to \infty$, which show that $A \circ \beta$ is continuous on finite dimensional subspace, there exists $u \in \zeta(K)$ such that

$$
\langle (A \circ \beta)(u), v - u \rangle \notin -int C(x), \forall v \in \zeta(K).
$$

Since for every $y \in K$, there exists $v \in \zeta(K)$ such that

$$
\zeta(y) = v,
$$

we have

$$
\langle (A \circ \beta)(u), \zeta(y) - u \rangle \notin -int C(x), \forall y \in K.
$$

Observe that $\zeta(\beta(u)) = u$. Thus

$$
\langle A(\beta(u)), \zeta(y) - \zeta(\beta(u)) \rangle \notin -int C(x), \forall v \in K,
$$

or equivalently $\beta(u) \in K$ is a solution of (1.1).
Corollary 2.6. Let $X$ and $Y$ be the two reflexive Banach spaces. Assume that $K$ is weakly compact, $\zeta(K)$ is convex and weak to weak sequentially continuous. Let $\mathcal{A}$ be a monotone relation to $\zeta$. Further, assume that for every finite dimensional subset $L \subseteq \zeta(K)$ and for every sequence $\{x_n\} \subseteq K$ such that $\zeta(x_n) \subseteq L$, and if the sequence $\{\zeta(x_n)\} \subseteq L$ converges to $\zeta(x) \subseteq \zeta(K)$, then the sequence $\{\mathcal{A}(x_n)\} \subseteq L(X,Y)$ is weakly converges to $\mathcal{A}(x) \subseteq L(X,Y)$. Then (1.1) admits a solution.

References


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