IHARA ZETA FUNCTION OF DUMBBELL GRAPHS

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Abstract. We study the Ihara zeta function of the dumbbell graph $D_{1,1,n}$ of type $(1,1,n)$ and $D_{1,2,n}$ of type $(1,2,n)$. Explicit formulas of the zeta functions of the graphs, their radius of convergence, and the connection with the number of closed cycles are given.

1. Introduction

Let $G$ be a finite connected undirected graph with no degree 1 vertices. Let $V_G$ and $E_G$ be the set of vertices and the set of edges of $G$, respectively. In addition, we denote by $E^\pm_G$ the set of all oriented edges of $G$. Thus, we have $|E^\pm_G| = 2|E_G|$.

Let $P = (e_1, e_2, \ldots, e_{l(P)-1}, e_{l(P)})$ be a primitive closed cycle without backtracking. That is, $o(e_1) = t(e_{l(P)})$, $e_{i+1} \neq e_i^{-1} \pmod{l(P)}$ for all $i$ and $P \neq D^m$ for any integer $m \geq 2$ and a path $D$ in $A$. If a closed cycle $Q$ is obtained by changing the cyclic order of $P$, then we say $P$ and $Q$ are equivalent. A prime $[P]$ in $G$ is an equivalence class of primitive closed cycle without backtracking.

The Ihara zeta function of $G$ is defined at $u \in \mathbb{C}$, for which $|u|$ is sufficiently small, by

$$Z_G(u) = \prod_{[P]} (1 - u^{l(P)})^{-1}$$
where \([P]\) runs over the primes of \(G\).

The Ihara determinant formula \([1]\) gives that \(Z_{G}(u)\) is a rational function, given by

\[
Z_{G}(u) = \frac{1}{(1 - u^2)\chi(x) - 1 \det(I - A(u) + Bu^2)}
\]

where \(\chi = |EG| - |VG| + 1\), \(A\) is the vertex adjacency matrix of \(G\) and \(B\) is diagonal matrix whose \(j\)-th diagonal entry is \(\deg(v_j) - 1\). Let \(R_G\) be the radius of convergence of \(Z_{G}(u)\).

We denote by \(D_{a,b,n}\) the **dumbbell** graph of type \((a, b, n)\) defined as a graph consisting of two vertex-disjoint cycles \(C_a, C_b\) and a path \(P_n\) \((a, b \geq 1, n \geq 2)\) joining them having only its end-vertices in common with the two cycles. It has \(a + b + n - 2\) number of vertices and \(a + b + n - 1\) number of edges. Below is the figure of the graph \(D_{a,b,n}\).

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{dumbbell_graph.png}
\end{array}
\]

**Figure 1.** Dumbbell graph \(D_{a,b,n}\)

In this article, we study the Ihara-zeta function of the graph \(D_{1,1,n}\) and \(D_{1,2,n}\). If \(G = D_{1,1,n}\), then we have \(|VG| = n\) and \(|EG| = n + 1\). If \(G = D_{1,2,n}\), then we have \(|VG| = n + 1\) and \(|EG| = n + 2\).

The following two theorems are main results of the paper.

**Theorem 1.1.** Let \(G\) be the dumbbell graph \(D_{1,1,n}\) of type \((1, 1, n)\). If \(n\) is odd, then

\[
Z_{G}(u) = \frac{1}{(1 - u^2)(u - 1)(2u^{n-1} - 2u^{n-2} + \cdots + 2u - 1)(2u^n + u - 1)}
\]

and if \(n\) is even, then

\[
Z_{G}(u) = \frac{1}{(1 - u^2)(u - 1)(2u^{n-1} - 2u^{n-2} + \cdots + 2u - 1)(2u^n - u + 1)}.
\]
Theorem 1.2. Let $G$ be the dumbbell graph $D_{1,2,n}$ of type $(1,2,n)$. Then, we have

$$Z_G(u) = \frac{1}{(1-u^2)(u-1)(4u^{2n-1} - u^3 + u^2 + u - 1)}.$$ 

From the formula of the zeta function $Z_G(u)$ and the Perron-Frobenius theory, we get the irreducible polynomial of the radius of convergence of $Z_G(u)$.

Corollary 1.3. Let $G$ be the dumbbell graph $D_{1,1,n}$. The radius $R_G$ of convergence of the rational function $Z_G(u)$ the unique real root of $P(u)$ where $P(u)$ is given by

$$P(u) = \begin{cases} 
2u^n + u - 1, & n \text{ is odd} \\
2u^{n-1} - 2u^{n-2} + \cdots - 2u^2 + 2u - 1, & n \text{ is even}.
\end{cases}$$

Finally, the prime geodesic theorem gives the following.

Corollary 1.4. Let $G$ be the dumbbell graph $D_{1,1,n}$. If we denote by $\pi(m)$ the number of prime cycles of length $m$, then we have

$$\lim_{m \to \infty} \frac{m\pi(m)}{\lambda_G^m} = 1.$$
Here, $\lambda_G$ is the unique real root of $Q(u)$ where $Q(u)$ is given by
\[
Q(u) = \begin{cases} 
  u^n - u^{n-1} - 2, & n \text{ is odd} \\
  u^{n-1} - 2u^{n-2} + \cdots - 2u^2 + 2u - 2, & n \text{ is even}.
\end{cases}
\]
The analogous results for $D_{1,2,n}$ hold.

2. Proof of the results

In this section, we prove Theorem 1.1 and Theorem 1.2. First, we consider the case when the graph $G$ is $D_{1,1,n}$.

**Proof of Theorem 1.1.** By the Ihara determinant formula, we have

\[
Z_G(u)^{-1} = (1 - u^2) \det(B)
\]

where $B$ is an $n \times n$ tri-diagonal matrix given by

\[
B = \begin{pmatrix} 
  1 - 2u + 2u^2 & -u & \\
  -u & 1 + u^2 & -u \\
  & -u & 1 + u^2 & -u \\
  & & -u & 1 - 2u + 2u^2
\end{pmatrix}.
\]

Let us denote by $f(k)$ the determinant of the $k \times k$ matrix given by

\[
\begin{pmatrix} 
  1 + u^2 & -u & \\
  -u & 1 + u^2 & -u \\
  & -u & \cdots & -u \\
  & & -u & 1 + u^2
\end{pmatrix}.
\]

Then $f(k) = (1 + u^2)f(k-1) - u^2f(k-2)$. Since $f(1) = 1 + u^2$ and $f(2) = 1 + u^2 + u^4$, it follows that $f(k) = 1 + u^2 + \cdots + u^{2k}$. Hence, this yields

\[
\det(B) = (1 - 2u + 2u^2)[(1 - 2u + 2u^2)f(n - 2) - u^2f(n - 3)]
\]

\[
- u^2[(1 - 2u + 2u^2)f(n - 3) - u^2f(n - 4)]
\]

\[
= (1 - 2u + 2u^2)^2f(n - 2) - 2(1 - 2u + 2u^2)u^2f(n - 3) + u^4f(n - 4)
\]

\[
= 4u^{2n} - 8u^{2n-1} + 8u^{2n-2} - + \cdots - 8u^3 + 7u^2 - 4u + 1
\]

\[
= (u - 1)(2u^{n-1} - 2u^{n-2} + 2u^{n-3} + \cdots \pm 1)(2u^n \pm u \mp 1)
\]

Therefore, we have

\[
Z_G(u)^{-1} = (1 - u^2)(u - 1)(2u^{n-1} - 2u^{n-2} + \cdots \pm 1)(2u^n \pm u \mp 1)
\]
which completes the proof of the Theorem 1.1. □

Proof of Theorem 1.2. Similarly, if $G$ is the graph $D_{1,2,n}$, then the Ihara determinant formula implies that

$$Z_G(u)^{-1} = (1 - u^2) \det(C)$$

where

$$C = \begin{pmatrix}
1 - 2u + 2u^2 & -u & -u
-2u & 1 + u^2 & -u
-u & \cdots & -2u
1 + 2u^2 & -2u & -u
2u & 1 + u^2
\end{pmatrix}.$$  

Let us denote by $g(k)$ the determinant of the $k \times k$ matrix given by

$$\begin{pmatrix}
1 + u^2 & -u & -u & \cdots & -u \\
-u & 1 + u^2 & -u & \cdots & -u \\
-u & \cdots & -u & \cdots & -u \\
-u & \cdots & \cdots & \cdots & -u \\
-u & \cdots & \cdots & \cdots & 1 + 2u^2
\end{pmatrix}.$$  

Then $g(k) = (1 + u^2)g(k-1) - u^2g(k-2)$. Since $g(1) = 1 + 2u^2$ and $g(2) = 1 + 2u^2 + 2u^4$, it follows that $g(k) = 1 + 2u^2 + \cdots + 2u^{2k}$. This yields

$$\det(C) = (1 - 2u + 2u^2)[(1 + u^2)g(n-2) - 4u^2f(n-3)] - u^2[(1 + u^2)g(n-3) - 4u^2f(n-4)] = 4u^{2n} - 4u^{2n-1} - u^4 + 2u^3 - 2u + 1 = (u - 1)(4u^{2n-1} - u^3 + u^2 + u - 1).$$

Thus, we have

$$Z_G(u)^{-1} = (1 - u^2)(u - 1)(4u^{2n-1} - u^3 + u^2 + u - 1)$$

which completes the proof of the Theorem 1.2. □

Let us now prove Corollary 1.3 and 1.4. Let $L(G)$ be the vertex adjacency matrix of the oriented line graph of $G$ (see Section 3 of [4]). According to the determinant formula for the edge zeta function (Theorem 3.3 of [2]), we also have

$$Z_G(u) = \frac{1}{\det(I - uL(G))}.$$  

Since $G$ is connected, it follows that $L(G)$ is a non-negative irreducible matrix. The Perron-Frobenius theorem of the non-negative matrices...
(Section 4 of [4]) implies that the Perron-Frobenius eigenvalue $\lambda_G$ of $L(G)$ is simple and real. It follows that $0 < R_G < 1$ and $\lambda_G = R_G^{-1}$ is an algebraic integer. This gives us that $R_G$ is the unique real root of $P(u)$ where

$$P(u) = \begin{cases} 2u^n + u - 1, & n \text{ is odd} \\ 2u^{n-1} - 2u^{n-2} + \cdots - 2u^2 + 2u - 1, & n \text{ is even.} \end{cases}$$

Corollary 1.4 directly follows from the prime geodesic theorem in graphs (Theorem 2.10 of [2]) since $\Delta_G = 1$.

References


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