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A NOTE ON SOME INEQUALITIES FOR THE b-NUMERICAL RADIUS AND b-NORM IN 2-HILBERT SPACE OPERATORS

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Abstract. In this paper, the definition b-numerical radius and b-norm is introduced and we present several b-numerical radius inequalities. Some applications of these inequalities are considered as well.

1. Introduction and preliminaries

Let $\mathcal{B}(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$. The numerical radius of $T \in \mathcal{B}(\mathcal{H})$, denoted by $\omega(T)$, is given by

$$\omega(T) = \sup_{\|x\|=1} |\langle Tx, x\rangle|.$$

It is well-known that $\omega(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H})$ which is equivalent to the usual operator norm $\|T\| = \sup_{\|x\|=1} \|Tx\|$. In fact for $T \in \mathcal{B}(\mathcal{H})$ we have

$$\frac{1}{2}\|T\| \leq \omega(T) \leq \|T\|.$$

Several numerical radius inequalities that provide alternative lower and upper bounds for $\omega(T)$ have received much attention from many authors. We refer the readers to [3] for the history and significance, and [4] for

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recent developments in this area. Kittaneh in [6] proved that for $T \in \mathcal{B}(\mathcal{H})$,

$$\frac{1}{4}\|T^*T + TT^*\| \leq \omega^2(T) \leq \frac{1}{2}\|T^*T + TT^*\|.$$

Let $\mathcal{X}$ be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $\langle \cdot, \cdot | \cdot \rangle$ is a $\mathbb{K}-$valued function defined on $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$ satisfying the following condition:

($2I_1$) $\langle x, x|z\rangle \geqslant 0$ and $\langle x, x|z\rangle = 0$ if and only if x, z are linearly dependent,
($2I_2$) $\langle x, x|z\rangle = \langle z, z|x\rangle$,
($2I_3$) $\langle x, y|z\rangle = \overline{\langle y, x|z\rangle}$,
($2I_4$) $\langle \alpha x, y|z\rangle = \alpha\langle x, y|z\rangle$ for any scaler $\alpha \in \mathbb{K}$,
($2I_5$) $\langle x + \acute{x}, y|z\rangle = \langle x, y|z\rangle + \langle \acute{x}, y|z\rangle$.

$\langle \cdot, \cdot | \cdot \rangle$ is called a 2-inner product on $\mathcal{X}$ and $(\mathcal{X}, \langle \cdot, \cdot | \cdot \rangle)$ is called a 2-inner product space (or 2-pre-Hilbert space). Some basic properties of 2-inner product spaces can be immediately obtained as follows [1]:
(i) If $\mathbb{K} = \mathbb{R}$, then ($2I_3$) reduces to

$$\langle y, x|z\rangle = \langle x, y|z\rangle,$$

(ii) From ($2I_3$) and ($2I_4$), we have

$$\langle 0, y|z\rangle = 0, \qquad \langle x, 0|z\rangle = 0$$

and also

$$\langle x, \alpha|z\rangle = \bar{\alpha}y\langle x, y|z\rangle. \tag{1.1}$$

(iii) Using ($2I_2$) $-$ ($2I_5$), we have

$$\langle z, z|x \pm y\rangle = \langle x \pm y, x \pm y|z\rangle = \langle x, x|z\rangle + \langle y, y|z\rangle \pm 2\mathrm{Re}\langle x, y|z\rangle,$$

and

$$\mathrm{Re}\langle x, y|z\rangle = \frac{1}{4}\Big[\langle z, z|x + y\rangle - \langle z, z|x - y\rangle\Big]. \tag{1.2}$$

In the real case $\mathbb{K} = \mathbb{R}$, we have

$$\langle x, y|z\rangle = \frac{1}{4}\Big[\langle z, z|x + y\rangle - \langle z, z|x - y\rangle\Big] \tag{1.3}$$

and, using this formula, it is easy to see that, for any $\alpha \in \mathbb{R}$

$$\langle x, y|\alpha z\rangle = \alpha^2\langle x, y|z\rangle. \tag{1.4}$$

In the complex case, using (1.1) and (1.2), we have

$$\operatorname{Im}\langle x, y|z\rangle = \frac{1}{4}[\langle z, z|x + iy\rangle - \langle z, z|x - iy\rangle],$$

which, in combination with (1.2), yields

$$\langle x, y|z\rangle = \frac{1}{4}[\langle z, z|x + y\rangle - \langle z, z|x - y\rangle] + \frac{i}{4}[\langle z, z|x + iy\rangle - \langle z, z|x - iy\rangle]. \tag{1.5}$$

Using the above formula and (1.1), we have, for any $\alpha \in \mathbb{C}$,

$$\langle x, y|\alpha z\rangle = |\alpha|^2 \langle x, y|z\rangle. \tag{1.6}$$

However, for $\alpha \in \mathbb{R}$ (1.6) reduces to (1.4). Also, from (1.6) it follows that

$$\langle x, y|0\rangle = 0.$$

(iv) For any three given vectors $x, y, z \in \mathcal{X}$, consider the vector $u = \langle y, y|z\rangle x - \langle x, y|z\rangle y$. By $(2I_1)$, we know that $\langle u, u|z\rangle \geq 0$ with the equality if and only if u and z are linearly dependent. The inequality $\langle u, u|z\rangle \geq 0$ can be rewritten as,

$$\langle y, y|z\rangle \Big[\langle x, x|z\rangle\langle y, y|z\rangle - |\langle x, y|z\rangle|^2\Big] \geq 0. \tag{1.7}$$

For $x = z$, (1.7) becomes

$$-\langle y, y|z\rangle|\langle z, y|z\rangle|^2 \geq 0,$$

which implies that

$$\langle z, y|z\rangle = \langle y, z|z\rangle = 0 \tag{1.8}$$

provided y and z are linearly independent. Obviously, when y and z are linearly dependent, (1.8) holds too. Thus (1.8) is true for any two vectors $y, z \in X$. Now, if y and z are linearly independent, then $\langle y, y|z\rangle > 0$ and, from (1.7), it follows that

$$|\langle x, y|z\rangle|^2 \leq \langle x, x|z\rangle\langle y, y|z\rangle. \tag{1.9}$$

In any given 2-inner product space $(\mathcal{X}, \langle \cdot\,, \cdot|\cdot\rangle)$ we can define a function $\|\cdot|\cdot\|$ on $\mathcal{X} \times \mathcal{X}$

$$\|x|z\| = \sqrt{\langle x, x|z\rangle} \tag{1.10}$$

for all $x, z \in \mathcal{X}$. It is easy to see that this function satisfies the following condition:

$(2N_1)$ $\|x|z\| \geq 0$ and $\|x|z\| = 0$ if and only if x and z are linearly dependent,

$(2N_2)$ $\|x|z\| = \|z|x\|$,
$(2N_3)$ $\|\alpha x|z\| = |\alpha|\|z|x\|$, for any scaler $\alpha \in \mathbb{C}$,
$(2N_4)$ $\|x + \acute{x}|z\| \leq \|x|z\| + \|\acute{x}|z\|$.

Any function $\|\cdot|\cdot\|$ defined on $X \times \mathcal{X}$ and satisfying the conditions $(2N_1) - (2N_4)$ is called a 2-norm on $\mathcal{X}$ and $(\mathcal{X}, \|\cdot|\cdot\|)$ is called a linear 2-normed space [2]. Whenever a 2-inner product space $(\mathcal{X}, \langle\cdot, \cdot|\cdot\rangle)$ is given, we consider it as an inner 2-normed space $(\mathcal{X}, \|\cdot|\cdot\|)$ with the 2-norm defined by (1.10).

2. Main results

Let $(\mathcal{X}, \langle\cdot, \cdot|\cdot\rangle)$ be a 2-inner product space and $b \in \mathcal{X}$, then the operator $T : \mathcal{X} \longrightarrow \mathcal{X}$ is said to be b-bounded if there exists $M \geq 0$ such that for all $x \in \mathcal{X}$

$$\|Tx|b\| \leq M\|x|b\|.$$

Definition 2.1. Let $b \in \mathcal{X}$. Then b, T are called linearly dependent if for all $x \in \mathcal{X}$, there exists $\lambda_x \in \mathbb{C}$ such that

$$Tx = \lambda_x b.$$

Definition 2.2. Let $\mathcal{B}_b(\mathcal{X})$ be the set of all b-bounded linear operators on space $\mathcal{X}$ and $b \in \mathcal{X}$, then the map $\|\cdot|b\| : \mathcal{B}_b(\mathcal{X}) \longrightarrow \mathbb{R}^+$ is called b-norm, if

(i) $\|T|b\| = 0$ if and only if T and b are linearly dependent,
(ii) $\|\lambda T|b\| = |\lambda|\|T|b\|$,
(iii) $\|T_1 + T_2|b\| \leq \|T_1|b\| + \|T_2|b\|$.

Remark 2.3. Let $b \in \mathcal{X}$, then the map

$$\|\cdot|b\| : \mathcal{B}_b(\mathcal{X}) \longrightarrow \mathbb{R}^+, \quad \|T|b\| = \sup_{\|x|b\|=1} \|Tx|b\|,$$

is a b-norm.

Theorem 2.4. *Let $T \in \mathcal{B}_b(\mathcal{X})$, then*

$$\|T|b\| = \sup_{\|x|b\|=\|y|b\|=1} |\langle Tx, y|b\rangle|.$$

Proof. For $x, y \in \mathcal{X}$, by (1.9), we have

$$|\langle Tx, y|b\rangle| \leq \|Tx|b\|\|y|b\|.$$

Thus

$$\sup_{\|x|b\|=\|y|b\|=1} |\langle Tx, y|b\rangle| \leq \|T|b\|.$$

On the other hand, we have

$$\sup_{\|x|b\|=\|y|b\|=1} |\langle Tx, y|b\rangle| \geq \sup_{\|x|b\|=1} |\langle Tx, \frac{Tx}{\|Tx|b\|}|b\rangle|,$$

therefore

$$\sup_{\|x|b\|=\|y|b\|=1} |\langle Tx, y|b\rangle| \geq \|T|b\|.$$

□

Let T be a b-bounded linear operator on the 2-inner product space $\mathcal{X}$. According to Riesz theorem in 2-inner product spaces which was proved in [5], for constant $y \in \mathcal{X}$, there exists a unique b-bounded operator T^* such that for all $x, y \in \mathcal{X}$ we have $\langle Tx, y|b\rangle = \langle x, T^*y|b\rangle$.

Definition 2.5. Let $T \in \mathcal{B}_b(\mathcal{X})$, the operator $T^* : \mathcal{X} \longrightarrow \mathcal{X}$ defined by

$$\langle Tx, y|b\rangle = \langle x, T^*y|b\rangle,$$

is called the adjoint operator of T. And T is called self-adjoint if

$$\langle Tx, y|b\rangle = \langle x, Ty|b\rangle.$$

Definition 2.6. An operator T in 2-inner product space is called positive if it is self-adjoint and $\langle Tx, x|b\rangle \geq 0$ for all $x \in \mathcal{X}$.

Theorem 2.7. *Let $T, S \in \mathcal{B}_b(\mathcal{X})$ and $b \in \mathcal{X}$, then*

(i) $\|T|b\| = \|T^*|b\|$,

(ii) $\|T^*T|b\| = \|T|b\|^2$,

(iii) If T is self-adjoint, then $\|T|b\|^n = \|T^n|b\|$,

(iv) $\|TS|b\| \leq \|T|b\|\|S|b\|$.

Proof. These properties can be easily deduced by using the definition of $\|T|b\|$. □

Definition 2.8. Let $T \in \mathcal{B}_b(\mathcal{X})$ and $b \in \mathcal{X}$, then b-numerical radius is defined by

$$\omega(T|b) = \sup_{\|x|b\|=1} |\langle Tx, x|b\rangle|.$$

The next results represent some of the basic properties and sharp lower bound for the b-numerical radius. The following general result for the product of two operators holds:

Theorem 2.9. *For any $T, S \in \mathcal{B}_b(\mathcal{X})$, the b-numerical radius $\omega(\cdot\,|b)$: $\mathcal{B}_b(\mathcal{X}) \longrightarrow \mathbb{R}^+$ satisfies the following properties:*

(i) If $\omega(T|b) = 0$, then T and b are linearly depended,
(ii) $\omega(\lambda T|b) = |\lambda|\omega(T|b)$,
(iii) $\frac{1}{2}\|T|b\| \leq \omega(T|b) \leq \|T|b\|$,
(iv) $\omega(TS|b) \leq 4\,\omega(T|b)\,\omega(S|b)$.

Proof. (i) If $\omega(T|b) = 0$ for all $x \in \mathcal{X}$, then $\langle Tx, x|b\rangle = 0$, and by choosing

$$\begin{cases} x = x + y \Rightarrow \langle Tx, x|b\rangle + \langle Tx, y|b\rangle + \langle Ty, x|b\rangle + \langle Ty, y|b\rangle = 0, \\ \\ x = x + iy \Rightarrow \langle Tx, x|b\rangle - i\langle Tx, y|b\rangle + i\langle Ty, x|b\rangle + \langle Ty, y|b\rangle = 0. \end{cases}$$

Therefore

$$\begin{cases} \langle Tx, y|b\rangle + \langle Ty, x|b\rangle = 0, \\ \\ \langle Tx, y|b\rangle - \langle Ty, x|b\rangle = 0. \end{cases}$$

Thus

$$\langle Tx, y|b\rangle = 0.$$

By choosing $y = Tx$, we have

$$\langle Tx, Tx|b\rangle = 0 \Longrightarrow Tx = \lambda_x b.$$

(ii) This property can be easily deduced using the definition of $\omega(T|b)$.
(iii) For the first inequality, for any $x \in \mathcal{X}$, we have

$$|\langle Tx, x|b\rangle| \leq \omega(T|b)\|x|b\|^2,$$

and by (1.5), we have

$$\begin{aligned} 4\langle Tx, y|b\rangle &= \langle T(x+y), (x+y)|b\rangle - \langle T(x-y), (x-y)|b\rangle \\ &\quad + i\langle T(x+iy), (x+iy)|b\rangle - i\langle T(x-iy), (x-iy)|b\rangle, \end{aligned}$$

for all $x, y \in \mathcal{X}$. Hence

$$\begin{aligned} 4\langle Tx, y|b\rangle \leq \omega(T|b)\big(&\|(x+y)|b\| + \|(x-y)|b\| \\ &+ \|(x+iy)|b\| + \|(x-iy)|b\|\big). \end{aligned}$$

Choosing $\|x|b\| = \|y|b\| = 1$, we have

$$4|\langle Tx, y|b\rangle| \leq 8\,\omega(T|b),$$

which implies

$$\|T|b\| \leq 2\,\omega(T|b).$$

The second inequality can be easily deduced by using the definition of $\omega(T|b)$ and the inequality (1.9).
(iv) It follows from Theorem 2.7 (iv) that

$$\omega(TS|b) \leq \|TS|b\| \leq \|T|b\|\|S|b\| \leq 4\omega(T|b)\omega(S|b).$$

□

THEOREM 2.10. *If $T \in \mathcal{B}_b(\mathcal{X})$, then*

$$\frac{1}{4}\|T^*T + TT^*|b\| \leq \omega^2(T|b) \leq \frac{1}{2}\|T^*T + TT^*|b\|. \tag{2.1}$$

Proof. Let $T = C + iD$ be the Cartesian decomposition of T. Then C and D are self-adjoint, and $T^*T + TT^* = 2(C^2 + D^2)$. Let x be any vector in $\mathcal{X}$. Then by the convexity of the function $f(t) = t^2$, we have

$$\begin{aligned} |\langle Tx, x|b\rangle|^2 &= \langle Cx, x|b\rangle^2 + \langle Dx, x|b\rangle^2 \\ &\geq \frac{1}{2}(|\langle Cx, x|b\rangle| + |\langle Dx, x|b\rangle|)^2 \\ &\geq \frac{1}{2}|\langle (C \pm D)x, x|b\rangle|^2, \end{aligned}$$

and so we have

$$\begin{aligned} \omega^2(T|b) &= \sup_{\|x|b\|=1} |\langle Tx, x|b\rangle|^2 \\ &\geq \frac{1}{2}\sup_{\|x|b\|=1} |\langle (C \pm D)x, x|b\rangle|^2 \\ &= \frac{1}{2}\|C \pm D|b\|^2 = \frac{1}{2}\|(C \pm D)^2|b\|. \end{aligned}$$

Thus

$$2\omega^2(T|b) \geq \frac{1}{2}\|T^*T + TT^*|b\|.$$

This proves the first inequality.
To prove the second inequality, note that for every unit vector $x \in \mathcal{X}$, by (1.9), we have

$$\begin{aligned} |\langle Tx, x|b\rangle|^2 &= \langle Cx, x|b\rangle^2 + \langle Dx, x|b\rangle^2 \\ &\leq \|Cx|b\|^2 + \|Dx|b\|^2 = \langle C^2x, x|b\rangle + \langle D^2x, x|b\rangle \\ &= \langle (C^2 + D^2)x, x|b\rangle. \end{aligned}$$

Thus

$$\begin{aligned}\omega^2(T|b) &= \sup_{\|x|b\|=1} |\langle Tx, x|b\rangle|^2 \\ &\leq \sup_{\|x|b\|=1} \langle (C^2 + D^2)x, x|b\rangle \\ &= \|C^2 + D^2|b\| = \frac{1}{2}\|T^*T + TT^*|b\|.\end{aligned}$$

This proves the second inequality, and completes the proof of the theorem.

□

THEOREM 2.11. *Let $T, S : \mathcal{X} \longrightarrow \mathcal{X}$ be two b-bounded linear operators on the 2-inner product space $(\mathcal{X}, \langle \cdot\,, \cdot\,|b\rangle)$, if $r \geq 0$ and*

$$\|T - S|b\| \leq r, \tag{2.2}$$

then

$$\left\|\frac{T^*T + S^*S}{2}|b\right\| \leq \omega(S^*T|b) + \frac{1}{2}r^2. \tag{2.3}$$

Proof. For any $x \in \mathcal{X}$, $\|x|b\| = 1$, we have from (2.2) that

$$\|Tx|b\|^2 + \|Sx|b\|^2 \leq 2\mathrm{Re}\langle Tx, Sx|b\rangle + r^2, \tag{2.4}$$

however

$$\|Tx|b\|^2 + \|Sx|b\|^2 = \langle (T^*T + S^*S)x, x|b\rangle,$$

and by (2.4) we obtain

$$\langle (T^*T + S^*S)x, x|b\rangle \leq 2|\langle S^*Tx, x|b\rangle| + r^2.$$

By taking the supremum we get

$$\omega(T^*T + S^*S|b) \leq 2\omega(S^*T|b) + r^2 \tag{2.5}$$

and since the operator $T^*T + S^*S$ is self-adjoint, hence $\omega(T^*T + S^*S|b) = \|T^*T + S^*S|b\|$ and by (2.5) we deduce the desired inequality (2.3). □

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