ON THE STABILITY OF THE QUADRATIC-ADDITIVE TYPE FUNCTIONAL EQUATION IN RANDOM NORMED SPACES VIA FIXED POINT METHOD

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Abstract. In this paper, we prove the stability in random normed spaces via fixed point method for the functional equation

\[ 2f(x + y) + f(x - y) + f(y - x) - f(2x) - f(2y) = 0. \]

1. Introduction

In 1940, S. M. Ulam [23] raised a question concerning the stability of homomorphisms: Given a group \( G_1 \), a metric group \( G_2 \) with the metric \( d(\cdot, \cdot) \), and a positive number \( \varepsilon \), does there exist a \( \delta > 0 \) such that if a mapping \( f : G_1 \rightarrow G_2 \) satisfies the inequality

\[ d(f(xy), f(x)f(y)) < \delta \]

for all \( x, y \in G_1 \) then there exists a homomorphism \( F : G_1 \rightarrow G_2 \) with

\[ d(f(x), F(x)) < \varepsilon \]

for all \( x \in G_1 \)? As mentioned above, when this problem has a solution, we say that the homomorphisms from \( G_1 \) to \( G_2 \) are stable. In 1941, D. H. Hyers [5] gave a partial solution of Ulam’s problem for the case of approximate additive mappings under the assumption that \( G_1 \) and \( G_2 \) are Banach spaces. Hyers’ result was generalized by T. Aoki [1] for additive mappings and Th. M. Rassias [19] for linear mappings.


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by considering the stability problem with unbounded Cauchy differences. The paper of Th. M. Rassias has provided a lot of influence in the development of stability problems. The terminology Hyers-Ulam-Rassias stability originated from these historical background. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians, see [2]-[4], [6]-[15].

Recall, almost all subsequent proofs in this very active area have used Hyers’ method, called a direct method. Namely, the function $F$, which is the solution of a functional equation, is explicitly constructed, starting from the given function $f$, by the formulae $F(x) = \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)$ or $F(x) = \lim_{n \to \infty} 2^n f(\frac{x}{2^n})$. In 2003, V. Radu [18] observed that the existence of the solution $F$ of a functional equation and the estimation of the difference with the given function $f$ can be obtained from the fixed point alternative. In 2008, D. Miheţ and V. Radu [17] applied this method to prove the stability theorems of the Cauchy functional equation:

\[(1.1) \quad f(x + y) - f(x) - f(y) = 0\]

in random normed spaces. We call solutions of (1.1) by additive mappings.

In this paper, using the fixed point method, we will prove the stability for the quadratic-additive type functional equation:

\[(1.2) \quad 2f(x + y) + f(x - y) + f(y - x) - f(2x) - f(2y) = 0\]

in random normed spaces. It is easy to see that the mappings $f(x) = ax^2 + bx$ is a solution of the functional equation (1.2). The solution of the quadratic-additive type functional equation (1.2) is said to be a quadratic-additive mapping.

2. Preliminaries

In this section, we state the usual terminology, notations and conventions of the theory of random normed spaces, as in [21,22]. Firstly,
the space of all probability distribution functions is denoted by
\[ \Delta^+ := \{ F : \mathbb{R} \cup \{-\infty, \infty\} \to [0, 1] | F \text{ is left-continuous and nondecreasing on } \mathbb{R}, \text{ where } F(0) = 0 \text{ and } F(+\infty) = 1 \}. \]

And let the subset \( D^+ \subseteq \Delta^+ \) be the set \( D^+ := \{ F \in \Delta^+ | l^- F(+\infty) = 1 \} \), where \( l^- f(x) \) denotes the left limit of the function \( f \) at the point \( x \). The space \( \Delta^+ \) is partially ordered by the usual pointwise ordering of functions, that is, \( F \leq G \) if and only if \( F(t) \leq G(t) \) for all \( t \in \mathbb{R} \). The maximal element for \( \Delta^+ \) in this order is the distribution function \( \varepsilon_0 : \mathbb{R} \cup \{0\} \to [0, \infty) \) given by
\[
\varepsilon_0(t) = \begin{cases} 
0, & \text{if } t \leq 0, \\
1, & \text{if } t > 0.
\end{cases}
\]

**Definition 2.1.** ([21]) A mapping \( \tau : [0, 1] \times [0, 1] \to [0, 1] \) is called a **continuous triangular norm** (briefly, a **continuous t-norm**) if \( \tau \) satisfies the following conditions:

(a) \( \tau \) is commutative and associative;
(b) \( \tau \) is continuous;
(c) \( \tau(a, 1) = a \) for all \( a \in [0, 1] \);
(d) \( \tau(a, b) \leq \tau(c, d) \) whenever \( a \leq c \) and \( b \leq d \) for all \( a, b, c, d \in [0, 1] \).

Typical examples of continuous t-norms are \( \tau_P(a, b) = ab \), \( \tau_M(a, b) = \min(a, b) \) and \( \tau_L(a, b) = \max(a + b - 1, 0) \).

**Definition 2.2.** ([22]) A random normed space (briefly, **RN-space**) is a triple \((X, \Lambda, \tau)\), where \( X \) is a vector space, \( \tau \) is a continuous t-norm, and \( \Lambda \) is a mapping from \( X \) into \( D^+ \) such that the following conditions hold:

(RN1) \( \Lambda_x(t) = \varepsilon_0(t) \) for all \( t > 0 \) if and only if \( x = 0 \),
(RN2) \( \Lambda_{\alpha x}(t) = \Lambda_x(t/|\alpha|) \) for all \( x \) in \( X \), \( \alpha \neq 0 \) and all \( t \geq 0 \),
(RN3) \( \Lambda_{x+y}(t+s) \geq \tau(\Lambda_x(t), \Lambda_y(s)) \) for all \( x, y \in X \) and all \( t, s \geq 0 \).

If \((X, \| \cdot \|)\) is a normed space, we can define a mapping \( \Lambda : X \to D^+ \) by
\[
\Lambda_x(t) = \frac{t}{t + \|x\|}
\]
for all \( x \in X \) and \( t > 0 \). Then \((X, \Lambda, \tau_M)\) is a random normed space, which is called the **induced random normed space**.
Definition 2.3. Let \((X, \Lambda, \tau)\) be an \(RN\)-space.

(i) A sequence \(\{x_n\}\) in \(X\) is said to be convergent to a point \(x \in X\) if, for every \(t > 0\) and \(\varepsilon > 0\), there exists a positive integer \(N\) such that \(\Lambda x_n - x(t) > 1 - \varepsilon\) whenever \(n \geq N\).

(ii) A sequence \(\{x_n\}\) in \(X\) is called a Cauchy sequence if, for every \(t > 0\) and \(\varepsilon > 0\), there exists a positive integer \(N\) such that \(\Lambda x_n - x_m(t) > 1 - \varepsilon\) whenever \(n \geq m \geq N\).

(iii) An \(RN\)-space \((X, \Lambda, \tau)\) is said to be complete if and only if every Cauchy sequence in \(X\) is convergent to a point in \(X\).

Theorem 2.4. ([21]) If \((X, \Lambda, \tau)\) is an \(RN\)-space and \(\{x_n\}\) is a sequence such that \(x_n \to x\), then \(\lim_{n \to \infty} \Lambda x_n(t) = \Lambda x(t)\).

3. Main results

We recall the fundamental result in the fixed point theory.

Theorem 3.1. ([16] or [20]) Suppose that a complete generalized metric space \((X, d)\), which means that the metric \(d\) may assume infinite values, and a strictly contractive mapping \(J : X \to X\) with the Lipschitz constant \(0 < L < 1\) are given. Then, for each given element \(x \in X\), either

\[d(J^n x, J^{n+1} x) = +\infty, \quad \forall n \in \mathbb{N} \cup \{0\},\]

or there exists a nonnegative integer \(k\) such that:

1. \(d(J^n x, J^{n+1} x) < +\infty\) for all \(n \geq k\);
2. the sequence \(\{J^n x\}\) is convergent to a fixed point \(y^*\) of \(J\);
3. \(y^*\) is the unique fixed point of \(J\) in \(Y := \{y \in X, d(J^k x, y) < +\infty\}\);
4. \(d(y, y^*) \leq (1/(1 - L))d(y, Jy)\) for all \(y \in Y\).

Let \(X\) and \(Y\) be vector spaces. We use the following abbreviation for a given mapping \(f : X \to Y\)

\[Df(x, y) := 2f(x + y) + f(x - y) + f(y - x) - f(2x) - f(2y)\]

for all \(x, y \in X\). Now we will establish the stability for the functional equation (1.2) in random normed spaces via fixed point method.
THEOREM 3.2. Let $X$ be a linear space, $(Z, \Lambda', \tau_M)$ be an RN-space, $(Y, \Lambda, \tau_M)$ be a complete RN-space and $f : X \to Y$ be a mapping with $f(0) = 0$ for which there is $\varphi : X^2 \to Z$ such that

\[(3.1) \quad \Lambda_{Df(x,y)}(t) \geq \Lambda'_{\varphi(x,y)}(t)\]

for all $x, y \in X$ and $t > 0$. If for all $x, y \in X$ and $t > 0$ $\varphi$ satisfies one of the following conditions:

(i) $\Lambda'_{\alpha \varphi(x,y)}(t) \leq \Lambda'_{\varphi(2x,2y)}(t)$ for some $0 < \alpha < 2$,

(ii) $\Lambda'_{\varphi(2x,2y)}(t) \leq \Lambda'_{\alpha \varphi(x,y)}(t)$ for some $4 < \alpha$

then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

\[(3.2) \quad \Lambda_{f(x) - F(x)}(t) \geq \begin{cases} M(x, (2 - \alpha)t) & \text{if } \varphi \text{ satisfies (i)}, \\ M(x, (\alpha - 4)t) & \text{if } \varphi \text{ satisfies (ii)} \end{cases} \]

for all $x \in X$ and $t > 0$, where

$M(x, t) := \tau_M\{\Lambda'_{\varphi(x,0)}(t), \Lambda'_{\varphi(-x,0)}(t)\}$.

Moreover if $\alpha < 1$ and $\Lambda'_{\varphi(x,y)}$ is continuous in $x,y$ under the condition (i), then $f$ is a quadratic-additive mapping.

Proof. We will prove the theorem in two cases, $\varphi$ satisfies the condition (i) or (ii).

**Case 1.** Assume that $\varphi$ satisfies the condition (i). Let $S$ be the set of all functions $g : X \to Y$ with $g(0) = 0$ and introduce a generalized metric on $S$ by

$d(g, h) := \inf \{ u \in \mathbb{R}^+ : \Lambda_{g(x) - h(x)}(ut) \geq M(x, t) \text{ for all } x \in X \}$.

Consider the mapping $J : S \to S$ defined by

$Jf(x) := \frac{f(2x) - f(-2x)}{4} + \frac{f(2x) + f(-2x)}{8}$

then we have

$J^n f(x) = \frac{1}{2} \left( 4^{-n} (f(2^n x) + f(-2^n x)) + 2^{-n} (f(2^n x) - f(-2^n x)) \right)$
for all \( x \in X \) and \( n \in \mathbb{N} \). Let \( f, g \in S \) and let \( u \in [0, \infty) \) be an arbitrary constant with \( d(g, f) \leq u \). From the definition of \( d, (RN2) \), and \( (RN3) \), for the given \( 0 < \alpha < 2 \) we have

\[
\Lambda(Jg(x) - Jf(x)) \left( \frac{\alpha u}{2} \right) = \Lambda \frac{3(g(2x) - f(2x)) - g(-2x) - f(-2x)}{8} \left( \frac{\alpha u t}{2} \right)
\]

\[
\geq \tau M \left\{ \Lambda \frac{3(g(2x) - f(2x))}{8}, \frac{3\alpha u t}{8}, \Lambda \frac{g(-2x) - f(-2x)}{8} \left( \frac{\alpha u t}{8} \right) \right\}
\]

\[
\geq \tau M \left\{ \Lambda g(2x) - f(2x)(\alpha u t), \Lambda g(-2x) - f(-2x)(\alpha u t) \right\}
\]

\[
\geq \tau M \left\{ \Lambda' \varphi_{(2x,0)}(\alpha t), \Lambda' \varphi_{(-2x,0)}(\alpha t) \right\}
\]

\[
\geq M(x, t)
\]

for all \( x \in X \), which implies that

\[
d(Jf, Jg) \leq \frac{\alpha}{2} d(f, g).
\]

That is, \( J \) is a strictly contractive self-mapping of \( S \) with the Lipschitz constant \( \frac{\alpha}{2} \). Moreover, by (3.1), we see that

\[
\Lambda f(x) - Jf(x) \left( \frac{t}{2} \right) = \Lambda \frac{D_f(x, 0)}{8} - \frac{D_f(-x, 0)}{8} \left( \frac{t}{2} \right)
\]

\[
\geq \tau M \left\{ \Lambda \frac{D_f(x, 0)}{8}, \frac{3t}{8}, \Lambda \frac{D_f(-x, 0)}{8} \left( \frac{t}{8} \right) \right\}
\]

\[
\geq \tau M \left\{ \Lambda D_f(x, 0)(t), \Lambda D_f(-x, 0)(t) \right\}
\]

\[
\geq \tau M \left\{ \Lambda' \varphi_{(x,0)}(t), \Lambda' \varphi_{(-x,0)}(t) \right\}
\]

for all \( x \in X \). It means that \( d(f, Jf) \leq \frac{1}{2} < \infty \) by the definition of \( d \). Therefore according to Theorem 3.1, the sequence \( \{J^n f\} \) converges to the unique fixed point \( F : X \to Y \) of \( J \) in the set \( T = \{ g \in S | d(f, g) < \infty \} \), which is represented by

\[
F(x) := \lim_{n \to \infty} \left( \frac{f(2^n x) + f(-2^n x)}{2 \cdot 4^n} + \frac{f(2^n x) - f(-2^n x)}{2^{n+1}} \right)
\]

for all \( x \in X \). Since

\[
d(f, F) \leq \frac{1}{1 - \frac{\alpha}{2}} d(f, Jf) \leq \frac{1}{2 - \alpha}
\]
the inequality (3.2) holds. Next we will show that $F$ is a quadratic-
additive mapping. Let $x, y \in X$. Then by (RN3) we have

$$\Lambda_{DF(x,y)}(t) \geq \tau_M \left\{ \Lambda_{2(F-J^nf)(x+y)} \left( \frac{t}{10} \right), \Lambda_{(F-J^nf)(x-y)} \left( \frac{t}{10} \right), \right.$$  

$$\Lambda_{(F-J^nf)(y-x)} \left( \frac{t}{10} \right), \Lambda_{(J^nf-F)(2x)} \left( \frac{t}{10} \right), \Lambda_{(J^nf-F)(2y)} \left( \frac{t}{10} \right), \Lambda_{DJ^nf(x,y)} \left( \frac{t}{2} \right) \right\}$$  

(3.3)

for all $x, y \in X$ and $n \in \mathbb{N}$. The first five terms on the right hand side of the above inequality tend to 1 as $n \to \infty$ by the definition of $F$. Now consider that

$$\Lambda_{DJ^n f(x,y)} \left( \frac{t}{2} \right) \geq \tau_M \left\{ \Lambda_{\frac{DF(2x,2^n y)}{2^{2n}}} \left( \frac{t}{8} \right), \Lambda_{\frac{Df(-2^n x,-2^n y)}{2^{2n}}} \left( \frac{t}{8} \right), \right.$$  

$$\Lambda_{\frac{DF(2^n x,2^n y)}{2^{2n}}} \left( \frac{t}{8} \right), \Lambda_{\frac{Df(-2^n x,-2^n y)}{2^{2n}}} \left( \frac{t}{8} \right) \right\} \geq \tau_M \left\{ \Lambda_{\frac{DF(2^n x,2^n y)}{2^n}} \left( \frac{4^n t}{4} \right), \Lambda_{\frac{Df(-2^n x,-2^n y)}{2^n}} \left( \frac{4^n t}{4} \right), \right.$$  

$$\Lambda_{\frac{DF(2^n x,2^n y)}{2^n}} \left( \frac{4^n t}{4} \right), \Lambda_{\frac{Df(-2^n x,-2^n y)}{2^n}} \left( \frac{4^n t}{4} \right) \right\} \geq \tau_M \left\{ \Lambda'_{\varphi(x,y)} \left( \frac{4^n t}{4^{\alpha n}} \right), \Lambda'_{\varphi(-x,-y)} \left( \frac{4^n t}{4^{\alpha n}} \right), \right.$$  

$$\Lambda'_{\varphi(x,y)} \left( \frac{2^n t}{4^{\alpha n}} \right), \Lambda'_{\varphi(-x,-y)} \left( \frac{2^n t}{4^{\alpha n}} \right) \right\}$$

which tends to 1 as $n \to \infty$ by (RN3) and $\frac{2}{\alpha} > 1$ for all $x, y \in X$. Therefore it follows from (3.3) that

$$\Lambda_{DF(x,y)}(t) = 1$$

for each $x, y \in X$ and $t > 0$. By (RN1), this means that $DF(x,y) = 0$ for all $x, y \in X$. Assume that $\alpha < 1$ and $\Lambda'_{\varphi(x,y)}$ is continuous in $x, y$. 


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If \( m, a, b, c, d \) are any fixed integers with \( a, c \neq 0 \), then we have

\[
\lim_{n \to \infty} \Lambda'(\varphi((2^n a + b)x, (2^n c + d)y) f(t)) \geq \lim_{n \to \infty} \Lambda'(\varphi((a + \frac{b}{2^n})x, (c + \frac{d}{2^n})y) (mt) = \Lambda'_{\varphi(ax,cy)}(mt)
\]

for all \( x, y \in X \) and \( t > 0 \). Since \( m \) is arbitrary, we have

\[
\lim_{n \to \infty} \Lambda'(\varphi((2^n a + b)x, (2^n c + d)y) f(t)) \geq \lim_{m \to \infty} \Lambda'_{\varphi(ax,cy)}(mt) = 1
\]

for all \( x, y \in X \) and \( t > 0 \). From these, we get the inequality

\[
\Lambda_2(f-F)(x)(5t) \geq \lim_{n \to \infty} \tau_M \left\{ \Lambda((DF-DF)((2^n+1)x, -2^n x), (2^n+1)x, -2^n x) f(t), \Lambda((DF-DF)((-2^n+1)x, (2-\alpha)t), -(2^n+1)x, (2-\alpha)t) \right\}
\]

\[
= \lim_{n \to \infty} \tau_M \left\{ \Lambda'_{\varphi((2^n+1)x, -2^n x)}(t), \Lambda_{\varphi((2^n+1)x, -2^n x)}(t), M((2^n+1+2)x, (2-\alpha)t), M(-2^n+1)x, (2-\alpha)t) \right\}
\]

\[
= 1
\]

for all \( x \in X \). From the above equality and the fact \( f(0) = 0 = F(0) \), we obtain \( f \equiv F \).

**Case 2.** We take \( \alpha > 4 \) and suppose that \( \varphi \) satisfies the condition (ii). Let the set \((S, d)\) be as in the proof of Case 1. Now we consider the mapping \( J : S \to S \) defined by

\[
Jg(x) := g\left(\frac{x}{2}\right) - g\left(-\frac{x}{2}\right) + 2\left(g\left(\frac{x}{2}\right) + g\left(-\frac{x}{2}\right)\right)
\]

for all \( g \in S \) and \( x \in X \). Notice that

\[
J^n g(x) = 2^{n-1} \left( g\left(\frac{x}{2^n}\right) - g\left(-\frac{x}{2^n}\right) \right) + \frac{4^n}{2} \left( g\left(\frac{x}{2^n}\right) + g\left(-\frac{x}{2^n}\right) \right)
\]

for all \( x \in X \) and \( n \in \mathbb{N} \). Let \( f, g \in S \) and let \( u \in [0, \infty) \) be an arbitrary constant with \( d(g, f) \leq u \). From the definition of \( d \), (RN2), and (RN3),
we have
\[
\Lambda J_g(x) - J_f(x) \left( \frac{4u}{\alpha} t \right) = \Lambda (g(\frac{x}{2}) - f(\frac{x}{2})) + g(-\frac{x}{2}) - f(-\frac{x}{2}) \left( \frac{4u}{\alpha} t \right)
\]
\[
\geq \tau \{ \Lambda (g(\frac{x}{2}) - f(\frac{x}{2})) \left( \frac{u}{\alpha} t \right), \Lambda g(-\frac{x}{2}) - f(-\frac{x}{2}) \left( \frac{u}{\alpha} t \right) \}
\]
\[
\geq \tau \{ \Lambda g(\frac{x}{2}) - f(\frac{x}{2}) \left( \frac{u}{\alpha} t \right), \Lambda g(-\frac{x}{2}) - f(-\frac{x}{2}) \left( \frac{u}{\alpha} t \right) \}
\]
\[
\geq \tau \{ \Lambda' \varphi(\frac{x}{2}, 0) \left( \frac{t}{\alpha} \right), \Lambda' \varphi(-\frac{x}{2}, 0) \left( \frac{t}{\alpha} \right) \}
\]
\[
\geq M(x, t)
\]
for all \( x \in X \), which implies that
\[
d(J f, J g) \leq \frac{4}{\alpha} d(f, g).
\]
That is, \( J \) is a strictly contractive self-mapping of \( S \) with the Lipschitz constant \( 0 < \frac{4}{\alpha} < 1 \). Moreover, by (3.1), we see that
\[
\Lambda f(x) - J_f(x) \left( \frac{t}{\alpha} \right) = \Lambda D_f(\frac{x}{2}, 0) \left( \frac{t}{\alpha} \right) \geq \Lambda' \varphi(\frac{x}{2}, 0) \left( \frac{t}{\alpha} \right) \geq \Lambda' \varphi(\frac{x}{2}, 0) (t)
\]
for all \( x \in X \). It means that \( d(f, J f) \leq \frac{1}{\alpha} < \infty \) by the definition of \( d \).
Therefore according to Theorem 3.1, the sequence \( \{J^n f\} \) converges to the unique fixed point \( F : X \to Y \) of \( J \) in the set \( T = \{g \in S | d(f, g) < \infty\} \), which is represented by
\[
F(x) := \lim_{n \to \infty} \left( 2^{n-1} \left( f \left( \frac{x}{2^n} \right) - f \left( -\frac{x}{2^n} \right) \right) + \frac{4^n}{2} \left( f \left( \frac{x}{2^n} \right) + f \left( -\frac{x}{2^n} \right) \right) \right)
\]
for all \( x \in X \). Since
\[
d(f, F) \leq \frac{1}{1 - \frac{2}{\alpha}} d(f, J f) \leq \frac{1}{\alpha - 4}
\]
the inequality (3.2) holds. Next we will show that \( F \) is quadratic-additive. Let \( x, y \in X \). Then by (RN3) we have the inequality (3.3)
for all \( x, y \in X \) and \( n \in \mathbb{N} \). The first five terms on the right hand side of the inequality (3.3) tend to 1 as \( n \to \infty \) by the definition of \( F \). Now consider that

\[
\Lambda_{Df(x,y)} \left( \frac{t}{2} \right) \geq \tau_M \left\{ \Lambda_{2^n-1Df(\frac{t}{2^n}, \frac{x}{2^n})} \left( t \right) , \Lambda_{2^n-1Df(\frac{t}{2^n}, \frac{-x}{2^n})} \left( \frac{t}{2^n} \right) \right\} \\
\geq \tau_M \left\{ \Lambda'_{\varphi(x,y)} \left( \frac{\alpha^n t}{4^n+1} \right) , \Lambda'_{\varphi(-x,-y)} \left( \frac{\alpha^n t}{4^n+1} \right) \right\} \\
\geq \tau_M \left\{ \Lambda'_{\varphi(x,y)} \left( \frac{\alpha^n t}{2n+2} \right) , \Lambda'_{\varphi(-x,-y)} \left( \frac{\alpha^n t}{2n+2} \right) \right\}
\]

which tends to 1 as \( n \to \infty \) by (RN3) for all \( x, y \in X \). Therefore it follows from (3.3) that

\[
\Lambda_{Df(x,y)}(t) = 1
\]

for each \( x, y \in X \) and \( t > 0 \). By (RN1), this means that \( DF(x, y) = 0 \) for all \( x, y \in X \). It completes the proof of Theorem 3.2.

Now we have a generalized Hyers-Ulam stability of the quadratic-additive functional equation (1.2) in the framework of normed spaces. Let \( \Lambda_x(t) = \frac{t}{t+\|x\|} \). Then \( (X, \Lambda, \tau_M) \) is an induced random normed space, which leads us to get the following result.

**Corollary 3.3.** Let \( X \) be a linear space, \( Y \) be a complete normed space and \( f : X \to Y \) be a mapping with \( f(0) = 0 \) for which there is \( \varphi : X^2 \to [0, \infty) \) such that

\[
\|Df(x,y)\| \leq \varphi(x,y)
\]

for all \( x, y \in X \). If for all \( x, y \in X \) \( \varphi \) satisfies one of the following conditions:

(i) \( \alpha \varphi(x,y) \geq \varphi(2x,2y) \) for some \( 0 < \alpha < 2 \),
(ii) \( \varphi(2x,2y) \geq \alpha \varphi(x,y) \) for some \( 4 < \alpha \)

then there exists a unique quadratic-additive mapping \( F : X \to Y \) such that

\[
\|f(x) - F(x)\| \leq \begin{cases} 
\frac{\phi(x)}{2-\alpha} & \text{if } \varphi \text{ satisfies (i)}, \\
\frac{\phi(x)}{\alpha-4} & \text{if } \varphi \text{ satisfies (ii)}
\end{cases}
\]
for all $x \in X$, where $\Phi(x)$ is defined by

$$\Phi(x) = \max(\varphi(x,0), \varphi(-x,0)).$$

Moreover, if $0 < \alpha < 1$ under the condition (i), then $f$ is a quadratic-additive mapping.

Now we have Hyers-Ulam-Rassias stability results of the quadratic-additive type functional equation (1.2).

**Corollary 3.4.** Let $X$ be a normed space, $p \in \mathbb{R}^+ \setminus [1,2]$ and $Y$ a complete normed-space. If $f : X \to Y$ is a mapping such that

$$\|Df(x,y)\| \leq \|x\|^p + \|y\|^p$$

for all $x, y \in X$ with $f(0) = 0$, then there exists a unique quadratic-additive mapping $F : X \to Y$ such that

$$\|f(x) - F(x)\| \leq \begin{cases} \frac{\|x\|^p}{2 - 2p} & \text{if } 0 \leq p < 1, \\ \frac{\|x\|^p}{2^p - 4} & \text{if } p > 2 \end{cases}$$

for all $x \in X$.

**Proof.** If we denote by $\varphi(x,y) = \|x\|^p + \|y\|^p$, then the induced random normed space $(X, \Lambda_x, \tau_M)$ holds the conditions of Theorem 3.3 with $\alpha = 2^p$.

**Corollary 3.5.** Let $X$ be a normed space and $Y$ a Banach space. Suppose that the mapping $f : X \to Y$ satisfies the inequality

$$\|Df(x,y)\| \leq \theta \|x\|^p \|y\|^q$$

for all $x, y \in X$, where $\theta \geq 0$, $p, q > 0$ and $p + q \in (0,1) \cup (2, \infty)$. Then $f$ is itself a quadratic additive mapping.

**Proof.** It follows from Theorem 3.2, by putting

$$\varphi(x,y) := \theta \|x\|^p \|y\|^q$$

for all $x, y \in X$ and $\alpha = 2^{p+q}$.
References


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