CHARACTERIZING FUNCTIONS FIXED BY A WEIGHTED BEREZIN TRANSFORM IN THE BIDISC

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Abstract. For $c > -1$, let $\nu_c$ denote a weighted radial measure on $\mathbb{C}$ normalized so that $\nu_c(D) = 1$. For $c_1, c_2 > -1$ and $f \in L^1(D^2, \nu_{c_1} \times \nu_{c_2})$, we define the weighted Berezin transform $B_{c_1, c_2}f$ on $D^2$ by

$$(B_{c_1, c_2}f)(z, w) = \int_D \int_D f(\varphi_z(x), \varphi_w(y)) \ d\nu_{c_1}(x) d\nu_{c_2}(y).$$

This paper is about the space $M_{c_1, c_2}^p$ of function $f \in L^p(D^2, \nu_{c_1} \times \nu_{c_2})$ satisfying $B_{c_1, c_2}f = f$ for $1 \leq p < \infty$. We find the identity operator on $M_{c_1, c_2}^p$ by using invariant Laplacians and we characterize some special type of functions in $M_{c_1, c_2}^p$.

1. Introduction

Let $D$ be the unit discs of $\mathbb{C}$ and $\nu$ be the Lebesgue measure on $\mathbb{C}$ normalized to $\nu(D) = 1$. For $c > -1$, we define a measure $\nu_c$ by $d\nu_c(z) = (\alpha + 1)(1 - |z|^2)^\alpha \ d\nu(z)$ so that $\nu_c(D) = 1$. If $u \in L^1(D, \nu_c)$ and $z \in D$, we define $T_c u$ the weighted Berezin transform of $u$ by

$$(T_c u)(z) = \int_D (u \circ \varphi_z) \ d\nu_c,$$

where $\varphi_a \in \text{Aut}(D)$ is defined by $\varphi_a(z) = \frac{a - z}{1 - \overline{a}z}$. 

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For $c_1, c_2 > -1$ and $f \in L^1(D^2, \nu_{c_1} \times \nu_{c_2})$, we define the weighted Berezin transform $B_{c_1, c_2} f$ on $D^2$ by

$$(B_{c_1, c_2}) f(z, w) = \int_D \int_D f(\varphi_z(x), \varphi_w(y)) \, d\nu_{c_1}(x) d\nu_{c_2}(y).$$

A function $f \in C^2(D)$ with $\Delta_1 f = \Delta_2 f = 0$ (i.e., harmonic in each variable) is called 2–harmonic. If $f \in L^1(D^2, \nu_{c_1} \times \nu_{c_2})$ is 2–harmonic, then we can easily see that $B_{c_1, c_2} f = f$ for every $c_1, c_2 > -1$. Conversely, Furstenberg ([3]) proved that a function $f \in L^\infty(D^2)$ satisfying $B_{c_1, c_2} f = f$ for some $c_1, c_2 > -1$ has to be 2–harmonic, whose complete analytic proof is given in [5]. The author([4]) proved that for every $1 \leq p < \infty$ and $c_1, c_2 > -1$, a function $f \in L^p(D^2, \nu_{c_1} \times \nu_{c_2})$ satisfying $B_{c_1, c_2} f = f$ needs not be 2–harmonic. Indeed, for every $1 \leq p < \infty$ and $c_1, c_2 > -1$, there exist uncountably many joint eigenfunctions $f \in L^p(D^2, \nu_{c_1} \times \nu_{c_2})$ of invariant Laplacians satisfying $B_{c_1, c_2} f = f$ (theorem 1.1 of [4]).

This paper is about the space $M^p_{c_1, c_2}$ of function $f \in L^p(D^2, \nu_{c_1} \times \nu_{c_2})$ satisfying $B_{c_1, c_2} f = f$ for $1 \leq p < \infty$ and $c_1, c_2 > -1$. We express the identity operator on $M^p_{c_1, c_2}$ as an entire function of invariant Laplacians. Then we find the joint spectrum of invariant Laplacians in an attempt to express $M^p_{c_1, c_2}$ by using spectral decompositions. Our original aim is to prove that the space $M^p_{c_1, c_2}$ is generated by the joint eigenfunctions of $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$. However, we are unable to provide the entire proof so that we leave it as a conjecture. In this paper, instead, we prove the conjecture for $f \in M^p_{c_1, c_2}$ of the form $f(z, w) = u(z) v(w)$.

In Section 2, we mention some preliminaries on eigenspaces and eigenvalues of invariant Laplacians, most of which have appeared in [4] and [7]. In Section 3, we mention some important properties of the operator the operator $B_{k_1 + c_1, k_2 + c_2}$ where $k_1, k_2$ are non-negative integers. In Section 4, we suggest a conjecture on $M^p_{c_1, c_2}$ and provide related propositions.

2. Preliminaries

Here we mention some preliminaries on function theories in the bidisc, related with eigenspaces and eigenvalues of invariant Laplacians. For $u \in C^2(D)$, $\Delta u$ the invariant Laplacian of $u$ is defined by $\Delta u(z) = (1 - |z|^2)^2 \Delta u(z)$. Acting on $f \in C^2(D^2)$, $\tilde{\Delta}_1, \tilde{\Delta}_2$ are the invariant Laplacians with respect to the first and second variable respectively, such as
$(\tilde{\Delta}_1 f)(z, w) = (1 - |z|^2) \Delta_1 f(z, w)$. For $\lambda, \mu \in \mathbb{C}$, we define the joint eigenspace $X_{\lambda, \mu}$ by

$$X_{\lambda, \mu} = \{ f \in C^2(D^2) \mid \tilde{\Delta}_1 f = \lambda f \text{ and } \tilde{\Delta}_2 f = \mu f \}.$$ 

It is known that (section 3 of [4]) for $c_1, c_2 > -1$ and $1 \leq p < \infty$,

$$L^p(D^2, \nu_{c_1} \times \nu_{c_2}) \cap X_{\lambda, \mu} \neq \{0\} \text{ if and only if } \alpha \in \Sigma_{c_1, p} \text{ and } \beta \in \Sigma_{c_2, p} \text{ where } \alpha, \beta \in \mathbb{C} \text{ satisfy }$$

$$\lambda = -4\alpha(1 - \alpha), \mu = -4\beta(1 - \beta)$$

and there exist uncountably many pairs of $(\alpha, \beta) \in \Sigma_{c_1, p} \times \Sigma_{c_2, p}$ satisfying

$$\frac{\Gamma(c_1 + 1 + \alpha)\Gamma(c_1 + 2 - \alpha)}{\Gamma(c_1 + 1)\Gamma(c_1 + 2)} \cdot \frac{\Gamma(c_2 + 1 + \beta)\Gamma(c_2 + 2 - \beta)}{\Gamma(c_2 + 1)\Gamma(c_2 + 2)} = 1.$$

Moreover, if $\lambda = -4\alpha(1 - \alpha)$ then we get

$$\frac{\Gamma(c_1 + 1 + \alpha)\Gamma(c_1 + 2 - \alpha)}{\Gamma(c_1 + 1)\Gamma(c_1 + 2)} = \frac{1}{G_{c_1}(\lambda)},$$

where

$$G_{c}(z) = \prod_{j=1}^{\infty} \left(1 + \frac{z}{(j + c)(j + c + 1)}\right)$$

is an entire function.

3. the operator $B_{k+c_1, \ell+c_2}$

DEFINITION 3.1. For $f \in L^1(D^2, \nu_{c_1} \times \nu_{c_2})$ and $k, \ell = 0, 1, 2, \cdots$, we define the operator $B_{k+c_1, \ell+c_2}$ on $L^1(D^2, \nu_{c_1} \times \nu_{c_2})$ by the obvious way such as

$$(B_{k+c_1, \ell+c_2}f)(z, w) = (k + 1)(\ell + 1) \cdot$$

$$\int \int_{D^2} (1 - |x|^2)^k \ (1 - |y|^2)^\ell \ f(\varphi_z(x), \varphi_w(y)) \ d\nu_{c_1}(x)d\nu_{c_2}(y).$$
Just the same way as the proof of Proposition 3.2 of [4], it is easy to see that
\[(B_{k+c_1,\ell+c_2} f) \circ \psi = B_{k+c_1,\ell+c_2} (f \circ \psi)\]
for \(k, \ell \geq 0\), \(\psi \in \text{Aut}(D^2)\) and \(f \in L^1(D^2, \nu_{c_1} \times \nu_{c_2})\). Also, the operators \(B_{c_1,c_2}\) and \(B_{k+c_1,\ell+c_2}\) commute on \(L^1(D^2, \nu_{c_1} \times \nu_{c_2})\).

The following lemma comes directly from Proposition 2.1 and Proposition 2.2 of [6].

**Lemma 3.2.** For \(k, \ell \geq 0\), \(B_{k+c_1,\ell+c_2}\) is a bounded operator on \(L^p(D^2, \nu_{c_1} \times \nu_{c_2})\) when \(p > 1\). And for \(k, \ell > 0\), \(B_{k+c_1,\ell+c_2}\) is a bounded linear operator on \(L^1(D^2, \nu_{c_1} \times \nu_{c_2})\).

Using Lemma 3.2, we get the following proposition.

**Proposition 3.3.** If \(1 \leq p < \infty\), then for every \(f \in L^p(D^2, \nu_{c_1} \times \nu_{c_2})\) we have
\[
\lim_{n \to \infty} \|f - B_{n+c_1,n+c_2} f\|_p = 0.
\]

*Proof.* By Proposition 2.2 of [6], we get \(\lim_{n \to \infty} \|B_{n+c_1,n+c_2}\| = 1\) on \(L^1(D^2, \nu_{c_1} \times \nu_{c_2})\).

Since \(B_{n+c_1,n+c_2}\) is a contraction which fixes 2-harmonic functions on \(L^\infty(D^2)\), an interpolation theorem gives \(\lim_{n \to \infty} \|B_{n+c_1,n+c_2}\| = 1\) on \(L^p(D^2, \nu_{c_1} \times \nu_{c_2})\) for \(1 \leq p < \infty\).

If \(g \in C(\overline{D^2})\), then Definition 2.1 shows that \((B_{n+c_1,n+c_2}g)(z, w) \to g(z, w)\) for every \(z, w \in D\) as \(n \to \infty\). Hence by dominated convergence theorem, \(\lim_{n \to \infty} \|g - B_{n+c_1,n+c_2} g\|_p = 0\).

If \(1 \leq p < \infty\) and \(f \in L^p(D^2, \nu_{c_1} \times \nu_{c_2})\) then there is a sequence \(\{g_k\}\) in \(C(\overline{D^2})\) such that \(\|f - g_k\|_p \to 0\) as \(k \to \infty\). Hence, we get the proof from the inequality
\[
\|f - B_{n+c_1,n+c_2} f\|_p \leq \|f - g_k\|_p + \|g_k - B_{n+c_1,n+c_2} g_k\|_p + \|B_{n+c_1,n+c_2}(g_k - f)\|_p.
\]

The following lemma directly comes from Proposition 2.4 of [1].

**Lemma 3.4.** For \(k, \ell \geq 0\), \(\psi \in \text{Aut}(D^2)\), and \(f \in L^1(D^2, \nu_{c_1} \times \nu_{c_2})\) we get
\[
\hat{\Delta}_1 B_{k+c_1,\ell+c_2} f = 4(k+1+c_1)(k+2+c_1)(B_{k+c_1,\ell+c_2} f - B_{k+1+c_1,\ell+c_2} f)
\]
\[
\hat{\Delta}_2 B_{k+c_1,\ell+c_2} f = 4(\ell+1+c_2)(\ell+2+c_2)(B_{k+c_1,\ell+c_2} f - B_{k+c_1,\ell+1+c_2} f)
\]
and

\[ B_{k+c_1,\ell+c_2}f = G_{k,c_1}(\tilde{\Delta}_1)G_{\ell,c_2}(\tilde{\Delta}_2)B_{c_1,c_2}f \]

where

\[ G_{m,c}(z) = \prod_{i=1}^{m} \left( 1 - \frac{z}{4(i+c)(i+c+1)} \right) \]

is an entire function.

4. The space \( M_{c_1,c_2}^p \)

In this section, for \( c_1, c_2 > -1 \) and \( 1 \leq p < \infty \), we make an attempt to characterize

\[ M_{c_1,c_2}^p = \{ f \in L^p(D^2, \nu_{c_1} \times \nu_{c_2}) : B_{c_1,c_2}f = f \} \]

which is a Banach space of real analytic functions. The entire function

\[ G_c(z) = \prod_{j=1}^{\infty} \left( 1 + \frac{z}{(j+c)(j+c+1)} \right) \]

mentioned at the end of Section 2 plays an important role.

**Proposition 4.1.** \( G_{c_1}(\tilde{\Delta}_1)G_{c_2}(\tilde{\Delta}_2) \) is the identity operator on \( M_{c_1,c_2}^p \).

**Proof.** By Lemma 3.4, for \( f \in M_{c_1,c_2}^p \) we get

\[ \tilde{\Delta}_1 f = \tilde{\Delta}_1 B_{c_1,c_2}f = 4(1 + c_1)(2 + c_1)(f - B_{1+c_1,c_2}f). \]

Since \( B_{1+c_1,c_2} \) is bounded on \( L^p(D^2) \) and commutes with \( B_{c_1,c_2} \),

\[ B_{c_1,c_2}(\tilde{\Delta}_1 f) = 4(1 + c_1)(2 + c_1)(B_{c_1,c_2}f - B_{c_1,c_2}B_{1+c_1,c_2}f) \]

\[ = 4(1 + c_1)(2 + c_1)(f - B_{1+c_1,c_2}B_{c_1,c_2}f) \]

\[ = 4(1 + c_2)(2 + c_2)(f - B_{1+c_1,c_2}f) = \tilde{\Delta}_1 f \]

Likewise we get \( B_{c_1,c_2}(\tilde{\Delta}_2 f) = \tilde{\Delta}_2 f \) for \( f \in M_{c_1,c_2}^p \). Hence \( \tilde{\Delta}_1, \tilde{\Delta}_2 \) are bounded operators on \( M_{c_1,c_2}^p \). From Lemma 3.4, for \( f \in M_{c_1,c_2}^p \) and \( n \in \mathbb{N} \),

\[ B_{n+c_1,n+c_2}f = G_{n,c_1}(\tilde{\Delta}_1)G_{n,c_2}(\tilde{\Delta}_2)B_{c_1,c_2}f. \]

For \( c > -1 \), the function

\[ G_c(z) = \prod_{j=1}^{\infty} \left( 1 + \frac{z}{(j+c)(j+c+1)} \right) \]
is entire, so that we have
\[ G_{n,c_1}(\tilde{\Delta}_1) \to G_{c_1}(\tilde{\Delta}_1) \quad \text{and} \quad G_{n,c_2}(\tilde{\Delta}_2) \to G_{c_2}(\tilde{\Delta}_2) \]
in the operator norm since \( G_{n,c_1} \to G_{c_1} \) and \( G_{n,c_2} \to G_{c_2} \) uniformly on compact set of \( \mathbb{C} \).
Now take \( n \to \infty \), by Proposition 3.3 we get
\[ f = G_{c_1}(\tilde{\Delta}_1)G_{c_2}(\tilde{\Delta}_2)f. \]
Therefore, \( G_{c_1}(\tilde{\Delta}_1)G_{c_2}(\tilde{\Delta}_2) = I \) on \( M^{p}_{c_1,c_2} \). \( \square \)
On the other hand, from Section 2 we get
\[ G_{c}(\lambda) = \frac{\Gamma(c+1)\Gamma(c+2)}{\Gamma(c+1+\alpha)\Gamma(c+2-\alpha)} \]
if \( c > -1 \) and \( \lambda = -4\alpha(1-\alpha) \). Hence, if we define
\[ \Omega_{c,p} = \{ \lambda = -4\alpha(1-\alpha) \mid -\frac{c+1}{p} < \Re \alpha < 1 + \frac{c+1}{p} \}, \]
then we have
\[ L^p(D^2,\nu_{c_1} \times \nu_{c_2}) \cap X_{\lambda,\mu} \neq \{0\} \quad \text{if and only if} \quad \lambda \in \Omega_{c_1,p} \quad \text{and} \quad \mu \in \Omega_{c_2,p} \]
Therefore, the set
\[ E = \{ (\lambda,\mu) \in \Omega_{c_1,p} \times \Omega_{c_2,p} \mid G_{c_1}(\lambda)G_{c_2}(\mu) = 1 \} \]
is the set of all joint eigenvalues of \( \tilde{\Delta}_1 \) and \( \tilde{\Delta}_2 \) on \( M^{p}_{c_1,c_2} \). Since \( G_{c_1}(\tilde{\Delta}_1)G_{c_2}(\tilde{\Delta}_2) = I \) on \( M^{p}_{c_1,c_2} \), by the holomorphic functional calculus (3.11 of [2]),
\[ 1 = \sigma(G_{c_1}(\tilde{\Delta}_1)G_{c_2}(\tilde{\Delta}_2)) = \{ G_{c_1}(\lambda)G_{c_2}(\mu) \mid (\lambda,\mu) \in \sigma(\tilde{\Delta}_1,\tilde{\Delta}_2) \}. \]
Therefore, we get the following proposition.

**Proposition 4.2.** The joint spectrum \( \sigma(\tilde{\Delta}_1,\tilde{\Delta}_2) \) of \( \tilde{\Delta}_1 \) and \( \tilde{\Delta}_2 \) on \( M^{p}_{c_1,c_2} \) is
\[ \sigma(\tilde{\Delta}_1,\tilde{\Delta}_2) = \{ (\lambda,\mu) \in \Omega_{c_1,p} \times \tilde{\Omega}_{c_2,p} \mid G_{c_1}(\lambda)G_{c_2}(\mu) = 1 \}. \]

In view of Proposition 4.2 we may conjecture that the space \( M^{p}_{c_1,c_2} \) is generated by the joint eigenfunctions of \( \tilde{\Delta}_1 \) and \( \tilde{\Delta}_2 \) in \( M^{p}_{c_1,c_2} \). But this conjecture is very hard for us to prove partly because the operators \( \tilde{\Delta}_1,\tilde{\Delta}_2 \) are not normal so that any type of spectral decomposition of \( M^{p}_{c_1,c_2} \) with respect to \( \tilde{\Delta}_1 \) and \( \tilde{\Delta}_2 \) is unavailable. The author hope to return to this problem in the future work.
Here instead, we will prove the conjecture for $f \in M_{c_1,c_2}^p$ of the form $f(z,w) = u(z)v(w)$.

**Proposition 4.3.** Given $1 \leq p < \infty$ and $c_1,c_2 > -1$, if $f(z,w) = u(z)v(w) \in M_{c_1,c_2}^p$, then $f$ can be written as a finite sum of joint eigenfunctions of $\tilde{\Delta}_1$ and $\tilde{\Delta}_2$.

**Proof.** If $f(z,w) = u(z)v(w)$ for some $u \in L^p(D,\nu_{c_1})$ and $v \in L^p(D,\nu_{c_2})$, then

\[
(B_{c_1,c_2}f)(z,w) = \int_D u(\varphi_z(x)) \, d\nu_{c_1}(x) \int_D v(\varphi_w(y)) \, d\nu_{c_2}(y)
= \left(T_{c_1}u(z)\right)\left(T_{c_2}v(w)\right).
\]

Fix $1 \leq p < \infty$ and $c_1,c_2 > -1$, then let $H$ be the set of all $r \in \mathbb{C} \setminus \{0\}$ such that both $\{u \in L^p(D,\nu_{c_1}) \mid T_{c_1}u = ru\}$ and $\{v \in L^p(D,\nu_{c_2}) \mid T_{c_2}v = \frac{1}{r}v\}$ are non-empty.

For $s,t \in H$ let us define the space

$K^1_s = \{u \in L^p(D,\nu_{c_1}) \mid T_{c_1}u = su\}$ and $K^2_t = \{u \in L^p(D,\nu_{c_2}) \mid T_{c_2}u = tu\}$.

Then, for $f = uv \in M_{c_1,c_2}^p$ there exists an $r \in H$, such that $u \in K^1_r$ and $v \in K^2_{1/r}$.

Just as in Lemma 3.4 and Proposition 4.1, for $r \in H$ we have the following:

1. $\tilde{\Delta}$ is a bounded operator on $K^1_r$.
2. $rG_{c_1}(\tilde{\Delta})$ is the identity operator on $K^1_r$.
3. The set $\{\lambda \in \Omega_{c_1,p} \mid G_{c_1}(\lambda) = \frac{1}{r}\}$ is the set of all eigenvalues of $\tilde{\Delta}$ on $K^1_r$.

Moreover, by Proposition 3.7 (d) of [1], $\{\lambda \in \Omega_{c_1,p} \mid G_{c_1}(\lambda) = \frac{1}{r}\}$ is a finite set.

Therefore, if $\{\lambda \in \Omega_{c_1,p} \mid G_{c_1}(\lambda) = \frac{1}{r}\} = \{\lambda_1, \ldots, \lambda_N\}$ then $Q(\tilde{\Delta}) = 0$ on $K^1_r$. Hence by Lemma 4.1 of [1], every $u \in K^1_r$ is a sum

$$u = u_{\lambda_1} + \cdots + u_{\lambda_N},$$
where $u_{\lambda_i} \in L^p(D, \nu_{c_1})$ satisfies $\tilde{\Delta} u_{\lambda_i} = \lambda_i u_{\lambda_i}$ for $1 \leq i \leq N$. By the same way every $v \in K_{1/r}^{2}$ is a sum
\[ v = v_{\mu_1} + \cdots + v_{\mu_m}, \]
where $\{\mu_1, \cdots, \mu_m\} = \{ \mu \in \Omega_{c_2,p} \mid G_{c_2}(\mu) = r \}$ and $v_{\mu_j} \in L^p(D, \nu_{c_2})$ satisfies
\[ \tilde{\Delta} v_{\mu_j} = \mu_j v_{\mu_j} \text{ for } 1 \leq j \leq m. \]
Hence we can write $f$ as
\[ f(z, w) = (u_{\lambda_1}(z) + \cdots + u_{\lambda_N}(z)) (v_{\mu_1}(w) + \cdots + v_{\mu_m}(w)) \]
which is a finite sum of joint eigenfunctions. □

References


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