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A COMMON FIXED POINT THEOREM ON ORDERED PARTIAL S-METRIC SPACES AND APPLICATIONS

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Abstract. A common fixed point result for weakly increasing mappings satisfying generalized contractive type in ordered partial S-metric spaces are derived. Also as an application of our results we consider a couple integral equations.to guarantee the existence of a common solution.

1. Introduction

Metrical fixed point theory became one of the most interesting area of research in the last fifty years. A lot of fixed and common fixed point results have been obtained by several authors in various types of spaces, such as metric spaces, fuzzy metric spaces, uniform spaces and others. One of the most interesting are partial metric spaces, which were defined by Matthews [5] in the following way.

Definition 1.1. [5] A partial metric on a nonempty set X is a function $p : X \times X \to [0, +\infty)$ such that, for all $x, y, z \in X$:

(p_1) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$,
(p_2) $p(x, x) \le p(x, y)$,
(p_3) $p(x, y) = p(y, x)$,
(p_4) $p(x, y) \le p(x, z) + p(z, y) - p(z, z)$.

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In this case, the pair (X, p) is called a partial metric space.

On the other hand, S-metric space were initiated by Sedghi, Shobe and Aliouche in [8] (see also [3] and references cited therein).

Definition 1.2. [8] An S-metric on a nonempty set X is a function $S : X \times X \times X \to [0, +\infty)$ such that for all $x, y, z, a \in X$, the following conditions are satisfied:

(s_1) $S(x, y, z) = 0$ if and only if $x = y = z$,

(s_2) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$.

In this case, the pair (X, S) is called an S-metric space.

It is easy to see that in an S-metric space (X, S) we always have $S(x, x, y) = S(y, y, x)$ for all $x, y \in X$.

In this paper, combining these two concepts, we introduce the notion of partial S-metric space and prove a common fixed point theorem for weakly increasing mappings in ordered spaces of this kind.

We recall some notions and properties in S-metric spaces.

Definition 1.3. [9] Let (X, S) be an S-metric space and $\{x_n\}$ be a sequence in X.

(a) The sequence $\{x_n\}$ converges to $x \in X$ if $S(x_n, x_n, x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.

(b) $\{x_n\}$ is said to be a Cauchy sequence if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for $S(x_n, x_n, x_m) < \epsilon$ for all $n, m \geq n_0$.

(c) The space (X, S) is said to be complete if every Cauchy sequence in X is convergent.

Lemma 1.4. [9] *Let* (X, S) *be an* S*-metric space. If* $\{x_n\}$ *and* $\{y_n\}$ *are sequences such that* $\lim_{n\to\infty} x_n = x$ *and* $\lim_{n\to\infty} y_n = y$*, then*

$$\lim_{n\to\infty} S(x_n, x_n, y_n) = S(x, x, y).$$

2. Partial S-metric spaces

In this section, we introduce partial S-metric spaces and investigate some of their simple properties.

Definition 2.1. A partial S-metric on a nonempty set X is a function $S^* : X \times X \times X \to [0, +\infty)$ such that for all $x, y, z, a \in X$:

(s_{p1}) $x = y = z$ if and only if $S^*(x,y,z) = S^*(x,x,x) = S^*(y,y,y) = S^*(z,z,z)$,
(s_{p2}) $S^*(x,x,x) \le S^*(x,x,y)$,
(s_{p3}) $S^*(x,y,z) \le S^*(x,x,a) + S^*(y,y,a) + S^*(z,z,a) - 2S^*(a,a,a)$.

The pair (X, S^*) is then called a partial S-metric space.

Each S-metric space is also a partial S-metric space. The converse is not true, as shown by the following examples.

Example 2.2. Let $X = [0,+\infty)$ and let $S^* : X \times X \times X \to [0,+\infty)$ be defined by $S^*(x,y,z) = \max\{x,y,z\}$. Then, it is easy to check that (X, S^*) is a partial S-metric space. Obviously, (X, S^*) is not an S-metric space.

Example 2.3. Let $X = [0,+\infty)$ and let $S^* : X \times X \times X \to [0,+\infty)$ be defined by $S^*(x,y,z) = \max\{x,y,z\} + |x-z| + a$ for every $x,y,z \in X$, where $a \in X$ is a constant. Then, it is easy to check that (X, S^*) is a partial S-metric space. Obviously, (X, S^*) is not an S-metric space.

Lemma 2.4. *For a partial S-metric S^* on X, we have, for all $x, y \in X$:*
(a) $S^*(x,x,y) = S^*(y,y,x)$,
(b) *if* $S^*(x,x,y) = 0$ *then* $x = y$.

Proof. (a) By the condition (s_{p3}), we have

$$\begin{aligned} S^*(x,x,y) &\le S^*(x,x,x) + S^*(x,x,x) + S^*(y,y,x) - 2S^*(x,x,x) \\ &= S^*(y,y,x) \qquad (1) \end{aligned}$$

and similarly

$$\begin{aligned} S^*(y,y,x) &\le S^*(y,y,y) + S^*(y,y,y) + S^*(x,x,y) - 2S^*(y,y,y) \\ &= S^*(x,x,y). \qquad (2) \end{aligned}$$

By (1) and (2), we get $S^*(x,x,y) = S^*(y,y,x)$.

(b) By the condition (s_{p2}), we have:

$$S^*(x,x,x) \le S^*(x,x,y) = 0 \qquad (3)$$

and similarly by relation (a), we also have:

$$S^*(y,y,y) \le S^*(y,y,x) = S^*(x,x,y) = 0. \qquad (4)$$

By (3), (4), we get $S^*(x,x,y) = S^*(x,x,x) = S^*(y,y,y) = 0$, which, by the condition (s_{p1}) implies that $x = y$. □

Remark 2.5. Dung, Hieu and Radojević noted in [4, Examples 2.1 and 2.2] that the class of S-metric spaces is incomparable with the the class of G-metric spaces, in the sense of Mustafa and Sims [6]. The same examples show that the class of partial S-metric spaces is incomparable with the class of GP-metric spaces, in the sense of Zand and Nezhad [12].

Definition 2.6. Let (X, S^*) be a partial S-metric space and $\{x_n\}$ be a sequence in X.

(a) The sequence $\{x_n\}$ converges to $x \in X$ (denoted as $x_n \to x$ as $n \to \infty$) if

$$\lim_{n\to\infty} S^*(x_n, x_n, x) = \lim_{n\to\infty} S^*(x_n, x_n, x_n) = S^*(x, x, x).$$

(b) The sequence $\{x_n\}$ is said to be a Cauchy sequence if there exists (finite) $\lim_{n,m\to\infty} S^*(x_n, x_n, x_m)$.

(c) The space (X, S^*) is complete if each Cauchy sequence in X is convergent.

Note that if $x_n \to x$ as $n \to \infty$, then for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|S^*(x_n, x_n, x) - S^*(x, x, x)| < \epsilon, \quad \text{for all } n \geq n_0, \tag{5}$$

and

$$|S^*(x_n, x_n, x_n) - S^*(x, x, x)| < \epsilon, \quad \text{for all } n \geq n_0. \tag{6}$$

Hence, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|S^*(x_n, x_n, x_n) - S^*(x_n, x_n, x)| < \epsilon, \quad \text{for all} n \geq n_0. \tag{7}$$

Lemma 2.7. *Let (X, S^*) be a partial S-metric space. If a sequence $\{x_n\}$ in X converges to $x \in X$, then x is unique.*

Proof. Let $\{x_n\}$ converges to x and y. Then we have

$$\lim_{n\to\infty} S^*(x_n, x_n, x_n) = \lim_{n\to\infty} S^*(x_n, x_n, x) = S^*(x, x, x), \tag{8}$$

and

$$\lim_{n\to\infty} S^*(x_n, x_n, x_n) = \lim_{n\to\infty} S^*(x_n, x_n, y) = S^*(y, y, y).$$

Then, by the condition (s_{p3}), relation (8) and Lemma 2.4, we have

$$\begin{aligned} S^*(x, x, y) &\leq 2S^*(x, x, x_n) + S^*(y, y, x_n) - 2S^*(x_n, x_n, x_n) \\ &= 2[S^*(x_n, x_n, x) - S^*(x_n, x_n, x_n)] \\ &\quad + S^*(x_n, x_n, y) - S^*(y, y, y) + S^*(y, y, y). \end{aligned}$$

By taking the limit as $n \to \infty$, we get $S^*(x,x,y) \le S^*(y,y,y)$. Also, by the condition (s_{p2}), we have $S^*(y,y,y) \le S^*(y,y,x) = S^*(x,x,y)$. Hence, we get $S^*(x,x,y) = S^*(y,y,y)$. Similarly, we have $S^*(x,x,y) = S^*(x,x,x)$. Hence, by the condition, (s_{p1}) it follows that $x = y$. □

Lemma 2.8. *Let (X, S^*) be a partial S-metric space. Then each convergent sequence $\{x_n\}$ in X is a Cauchy sequence.*

Proof. Let $\{x_n\}$ converges to x, that is for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that inequalities (5), (6) and (7) hold for all $n \ge n_0$. Then, by the condition (s_{p3}) and these inequalities, we have, for $m, n \ge n_0$,

$$\begin{aligned} S^*(x_n,x_n,x_m) &\le S^*(x_n,x_n,x) + S^*(x_n,x_n,x) + S^*(x_m,x_m,x) - 2S^*(x,x,x) \\ &\le 2[S^*(x_n,x_n,x) - S^*(x,x,x)] \\ &\quad + S^*(x_m,x_m,x) - S^*(x,x,x) + S^*(x,x,x) \\ &< 2\epsilon + \epsilon + S^*(x,x,x). \end{aligned} \tag{9}$$

Similarly, by the condition (s_{p3}) and Lemma 2.7,

$$\begin{aligned} S^*(x,x,x) &\le S^*(x,x,x_n) + S^*(x,x,x_n) + S^*(x,x,x_n) - 2S^*(x_n,x_n,x_n) \\ &= 2(S^*(x_n,x_n,x) - S^*(x_n,x_n,x_n)) + S^*(x,x,x_n) \\ &\le 2(S^*(x_n,x_n,x) - S^*(x_n,x_n,x_n)) + 2S^*(x,x,x_m) \\ &\quad + S^*(x_n,x_n,x_m) - 2S^*(x_m,x_m,x_m). \\ &< 2\epsilon + 2\epsilon + S^*(x_n,x_n,x_m). \end{aligned} \tag{10}$$

Hence, by (9) and (10), we have

$$|S^*(x_n,x_n,x_m) - S^*(x,x,x)| < 4\epsilon$$

for $m, n \ge n_0$. Thus, $\lim_{n,m\to\infty} S^*(x_n,x_n,x_m) = S^*(x,x,x)$, and the sequence $\{x_n\}$ is Cauchy. □

The notion of S_b-metric spaces was introduced independently in [10] and [11] (See also [7]).

Definition 2.9. Let X be a nonempty set and $b \ge 1$ a given real number. An S_b-metric on X, with parameter b, is a function $S_b : X \times X \times X \to [0, +\infty)$ such that for all $x, y, z, a \in X$, the following conditions are satisfied:

(s_{b1}) $S_b(x,y,z) = 0$ if and only if $x = y = z$,
(s_{b2}) $S_b(x,x,y) = S_b(y,y,x)$,
(s_{b3}) $S_b(x,y,z) \le b(S_b(x,x,a) + S_b(y,y,a) + S_b(z,z,a))$.

In this case, the pair (X, S_b) is called an S_b-metric space.

A connection between partial S-metric and S_b-metric spaces is given by the following lemma.

LEMMA 2.10. *If (X, S^*) is a partial S-metric space, then $S^s : X \times X \times X \to [0, +\infty)$, given by*

$$\begin{aligned} S^s(x,y,z) &= S^*(x,x,y) + S^*(y,y,z) + S^*(z,z,x) \\ &\quad - S^*(x,x,x) - S^*(y,y,y) - S^*(z,z,z), \end{aligned}$$

is an S_b-metric on X, with parameter $b = 2$.

Proof. First of all, by the condition (s_{p2}) and the definition of S^s, we have $S^s(x,y,z) \geq 0$. Further, we check that the conditions of Definition 2.9 are fulfilled.

(s_{b1}) If $S^s(x,y,z) = 0$ then it follows that $S^*(x,y,z) = S^*(x,x,x) = S^*(y,y,y) = S^*(z,z,z)$. That is, $x = y = z$. Conversely, if $x = y = z$, then we have $S^s(x,y,z) = 0$.

(s_{b2}) By the definition of S^s and Lemma 2.4, we have

$$\begin{aligned} S^s(x,x,y) &= S^*(x,x,x) + S^*(x,x,y) + S^*(y,y,x) \\ &\quad - S^*(x,x,x) - S^*(x,x,x) - S^*(y,y,y) \\ &= S^*(x,x,x) + S^*(x,x,y) + S^*(x,x,y) \\ &\quad - S^*(x,x,x) - S^*(x,x,x) - S^*(y,y,y) \\ &= 2S^*(x,x,y) - S^*(x,x,x) - S^*(y,y,y). \end{aligned}$$

Similarly, we can show that

$$S^s(y,y,x) = 2S^*(x,x,y) - S^*(x,x,x) - S^*(y,y,y).$$

Therefore, $S^s(x,x,y) = S^s(y,y,x)$. Also, we have always that $S^*(x,x,y) - S^*(x,x,x) \leq S^s(x,x,y)$.

(s_{b3}) By the condition (s_{p3}) and Lemma 2.4, we have

$$\begin{aligned}
S^s(x,y,z) &= S^*(x,x,y) + S^*(y,y,z) + S^*(z,z,x) \\
&\quad - S^*(x,x,x) - S^*(y,y,y) - S^*(z,z,z) \\
&\leq 2S^*(x,x,a) - 2S^*(a,a,a) + S^*(y,y,a) \\
&\quad + 2S^*(y,y,a) - 2S^*(a,a,a) + S^*(z,z,a) \\
&\quad + 2S^*(z,z,a) - 2S^*(a,a,a) + S^*(x,x,a) \\
&\quad - S^*(x,x,x) - S^*(y,y,y) - S^*(z,z,z) \\
&\leq 3S^*(a,a,x) - 2S^*(a,a,a) - S^*(x,x,x) + S^*(a,a,x) - S^*(x,x,x) \\
&\quad + 3S^*(a,a,y) - 2S^*(a,a,a) - S^*(y,y,y) + S^*(a,a,y) - S^*(y,y,y) \\
&\quad + 3S^*(a,a,z) - 2S^*(a,a,a) - S^*(z,z,z) + S^*(a,a,z) - S^*(z,z,z) \\
&= 2[S^s(x,x,a) + S^s(y,y,a) + S^s(z,z,a)].
\end{aligned}$$

□

Lemma 2.11. *Let (X,S^*) be a partial S-metric space and S^s the respective S_b-metric introduced in Lemma 2.10. Then:*

(a) A sequence $\{x_n\}$ in X is a Cauchy sequence in (X,S^*) if and only if it is a Cauchy sequence in (X,S^s).

(b) The space (X,S^*) is complete if and only if the space (X,S^s) is complete. Furthermore, $\lim_{n\to\infty} S^s(x_n,x_n,x) = 0$ if and only if

$$S^*(x,x,x) = \lim_{n\to\infty} S^*(x_n,x_n,x) = \lim_{n,m\to\infty} S^*(x_n,x_n,x_m).$$

Proof. Let $\{x_n\}$ be a Cauchy sequence in (X,S^*). Then there exists (finite) $\lim_{n,m\to\infty} S^*(x_n,x_n,x_m) = \lim_{n\to\infty} S^*(x_n,x_n,x_n)$. Since

$$S^s(x_n,x_n,x_m) = 2S^*(x_n,x_n,x_m) - S^*(x_n,x_n,x_n) - S^*(x_m,x_m,x_m),$$

we have

$$\begin{aligned}
\lim_{n,m\to\infty} S^s(x_n,x_n,x_m) &= 2\lim_{n,m\to\infty} S^*(x_n,x_n,x_m) \\
&\quad - \lim_{n\to\infty} S^*(x_n,x_n,x_n) - \lim_{m\to\infty} S^*(x_m,x_m,x_m) \\
&= 0.
\end{aligned}$$

We conclude that $\{x_n\}$ is a Cauchy sequence in (X,S^s).

Next we prove that completeness of (X,S^s) implies completeness of (X,S^*). Indeed, if $\{x_n\}$ is a Cauchy sequence in (X,S^*) then it is also a Cauchy sequence in (X,S^s). Since the space (X,S^s) is complete, we deduce that there exists $y \in X$ such that $\lim_{n\to\infty} S^s(x_n,x_n,y) = 0$, since $S^s(x_n, x_n, y) = 2S^*(x_n, x_n, y) - S^*(y, y, y) - S^*(x_n, x_n, x_n)$. Also, we know that

$$0 \leq S^*(x_n, x_n, y) - S^*(y, y, y) < S^s(x_n, x_n, y),$$

and

$$0 \leq S^*(x_n, x_n, y) - S^*(x_n, x_n, x_n) < S^s(x_n, x_n, y).$$

Therefore, we have

$$\lim_{n\to\infty} S^*(x_n, x_n, y) = \lim_{n\to\infty} S^*(x_n, x_n, x_n) = S^*(y, y, y).$$

Hence, we deduce that $\{x_n\}$ is a convergent sequence in (X, S^*).

Now we prove that every Cauchy sequence $\{x_n\}$ in (X, S^s) is a Cauchy sequence in (X, S^*). Let $\epsilon = \frac{1}{2}$. Then there exists $n_0 \in \mathbb{N}$ such that $S^s(x_n, x_n, x_m) < \frac{1}{2}$ for all $n, m \geq n_0$. Since

$$\begin{aligned}
&S^*(x_n, x_n, x_n)\\
&\leq 4S^*(x_{n_0}, x_{n_0}, x_n) - 3S^*(x_n, x_n, x_n) - S^*(x_{n_0}, x_{n_0}, x_{n_0}) + S^*(x_n, x_n, x_n)\\
&\leq 2S^s(x_n, x_n, x_{n_0}) + S^*(x_{n_0}, x_{n_0}, x_{n_0}),
\end{aligned}$$

we have

$$\begin{aligned}
S^*(x_n, x_n, x_n) &\leq 2S^s(x_n, x_n, x_{n_0}) + S^*(x_{n_0}, x_{n_0}, x_{n_0})\\
&\leq 1 + S^*(x_{n_0}, x_{n_0}, x_{n_0}).
\end{aligned}$$

Consequently, the sequence $\{S^*(x_n, x_n, x_n)\}$ is bounded in $\mathbb{R}$, and so there exists an $\alpha \in \mathbb{R}$ such that a subsequence $\{S^*(x_{n_k}, x_{n_k}, x_{n_k})\}$ is convergent to α, i.e., $\lim_{k\to\infty} S^*(x_{n_k}, x_{n_k}, x_{n_k}) = \alpha$.

It remains to prove that $\{S^*(x_n, x_n, x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\{x_n\}$ is a Cauchy sequence in (X, S^s), for given $\epsilon > 0$, there exists n_ϵ such that $S^s(x_n, x_n, x_m) < \frac{\epsilon}{2}$ for all $n, m \geq n_\epsilon$. Thus, for all $n, m \geq n_\epsilon$, if $S^*(x_n, x_n, x_n) \geq S^*(x_m, x_m, x_m)$ then

$$\begin{aligned}
S^*(x_n, x_n, x_n) - S^*(x_m, x_m, x_m) &\leq 4S^*(x_n, x_n, x_m) - 3S^*(x_n, x_n, x_n)\\
&\quad - S^*(x_m, x_m, x_m) + S^*(x_n, x_n, x_n)\\
&\quad - S^*(x_m, x_m, x_m)\\
&\leq 2S^s(x_n, x_n, x_m)\\
&< \epsilon.
\end{aligned}$$

Similarly, if $S^*(x_m, x_m, x_m) \geq S^*(x_n, x_n, x_n)$ then we can prove that

$$S^*(x_m, x_m, x_m) - S^*(x_n, x_n, x_n) < \epsilon.$$

Hence

$$|S^*(x_m, x_m, x_m) - S^*(x_n, x_n, x_n)| < \epsilon.$$

On the other hand,

$$\begin{aligned}
&|S^*(x_n, x_n, x_n) - \alpha| \\
&\quad \leq |S^*(x_n, x_n, x_n) - S^*(x_{n_k}, x_{n_k}, x_{n_k})| + |S^*(x_{n_k}, x_{n_k}, x_{n_k}) - \alpha| \\
&\quad < \epsilon + \epsilon \\
&\quad = 2\epsilon.
\end{aligned}$$

for all $n, n_k \geq n_\epsilon$. Hence $\lim_{n\to\infty} S^*(x_n, x_n, x_n) = \alpha$. Now,

$$\begin{aligned}
&|2S^*(x_n, x_n, x_m) - 2\alpha| \\
&\quad = |S^s(x_n, x_n, x_m) + S^*(x_n, x_n, x_n) - \alpha + S^*(x_m, x_m, x_m) - \alpha| \\
&\quad \leq S^s(x_m, x_m, x_m) + |S^*(x_n, x_n, x_n) - \alpha| + |S^*(x_m, x_m, x_m) - \alpha| \\
&\quad < \frac{\epsilon}{2} + 2\epsilon + 2\epsilon \\
&\quad = \frac{9}{2}\epsilon.
\end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence in (X, S^*).

In order to complete the proof, we have to prove that (X, S^s) is complete if such is (X, S^*). Let $\{x_n\}$ be a Cauchy sequence in (X, S^s). Then $\{x_n\}$ is a Cauchy sequence in (X, S^*), and so it is convergent to a point $y \in X$ with

$$\lim_{n,m\to\infty} S^*(x_n, x_n, x_m) = \lim_{n\to\infty} S^*(y, y, x_n) = S^*(y, y, y).$$

Thus, given $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that

$$|S^*(y, y, x_n) - S^*(y, y, y)| < \frac{\epsilon}{2}$$

and

$$|S^*(y, y, y) - S^*(x_n, x_n, x_n)| < \frac{\epsilon}{2},$$

whenever $n \geq n_\epsilon$. Hence, we have

$$\begin{aligned}
S^s(y, y, x_n) &= 2S^*(y, y, x_n) - S^*(x_n, x_n, x_n) - S^*(y, y, y) \\
&\leq |S^*(y, y, x_n) - S^*(y, y, y)| + |S^*(y, y, x_n) - S^*(x_n, x_n, x_n)| \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon,
\end{aligned}$$

whenever $n \geq n_\epsilon$. Therefore (X, S^s) is complete.

Finally, it is a simple matter to check that $\lim_{n\to\infty} S^s(a,a,x_n) = 0$ if and only if

$$S^*(a,a,a) = \lim_{n\to\infty} S^*(a,a,x_n) = \lim_{n,m\to\infty} S^*(x_n,x_n,x_m).$$

□

LEMMA 2.12. *Let $\{x_n\}$ and $\{y_n\}$ be two convergent sequences to $x \in X$ and $y \in X$, respectively, in a partial S-metric space (X,S^*). Then*

$$\lim_{n\to\infty} S^*(x_n,x_n,y_n) = S^*(x,x,y).$$

In particular, $\lim_{n\to\infty} S^*(x_n,x_n,z) = S^*(x,x,z)$ for every $z \in X$.

Proof. By the assumptions, for each $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|S^*(x_n,x_n,x) - S^*(x,x,x)| < \frac{\epsilon}{4},\ |S^*(y_n,y_n,y) - S^*(y,y,y)| < \frac{\epsilon}{4},$$
$$|S^*(x_n,x_n,x_n) - S^*(x,x,x)| < \frac{\epsilon}{4},\ |S^*(y_n,y_n,y_n) - S^*(y,y,y)| < \frac{\epsilon}{4},$$
$$|S^*(x_n,x_n,x_n) - S^*(x_n,x_n,x)| < \frac{\epsilon}{4},\ |S^*(y_n,y_n,y_n) - S^*(y_n,y_n,y)| < \frac{\epsilon}{4},$$

hold for all $n \geq n_0$. By the condition (s_{p3}), for $n \geq n_0$ we have

$$\begin{aligned}
S^*(x_n,x_n,y_n) &\leq S^*(x_n,x_n,x) + S^*(x_n,x_n,x) + S^*(y_n,y_n,x) - 2S^*(x,x,x)\\
&\leq S^*(x_n,x_n,x) + S^*(x_n,x_n,x) + S^*(y_n,y_n,y) + S^*(y_n,y_n,y)\\
&+ S^*(x,x,y) - 2S^*(y,y,y) - 2S^*(x,x,x)\\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + S^*(x,x,y),
\end{aligned}$$

and so we obtain

$$S^*(x_n,x_n,y_n) - S^*(x,x,y) < \epsilon.$$

Also,

$$\begin{aligned}
S^*(x,x,y) &\leq S^*(x,x,x_n) + S^*(x,x,x_n) + S^*(y,y,x_n) - 2S^*(x_n,x_n,x_n)\\
&\leq S^*(x,x,x_n) + S^*(x,x,x_n) + S^*(y,y,y_n) + S^*(y,y,y_n)\\
&+ S^*(x_n,x_n,y_n) - 2S^*(y_n,y_n,y_n) - 2S^*(x_n,x_n,x_n)\\
&< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + S^*(x_n,x_n,y_n).
\end{aligned}$$

Thus,

$$S^*(x,x,y) - S^*(x_n,x_n,y_n) < \epsilon.$$

Hence for all $n \geq n_0$, we have $|S^*(x_n, x_n, y_n) - S^*(x, x, y)| < \epsilon$ and the result follows. $\square$

3. Main Result

We begin this section giving the concept of weakly increasing mappings (see [2]).

Definition 3.1. Let $(X, \preceq)$ be a partially ordered set. Two mappings $S, T : X \longrightarrow X$ are said to be weakly increasing if $Sx \preceq TSx$ and $Tx \preceq STx$ for all $x \in X$.

Example 3.2. Let $X = [0, 1]$and let $S, T : X \longrightarrow X$ be defined by

$$Sx = \frac{x+1}{2},\ Tx = \frac{x+3}{4},$$

then

$$Sx = \frac{x+1}{2} < TSx = \frac{x+7}{8},$$

and

$$Tx = \frac{x+3}{4} < STx = \frac{x+7}{8}.$$

Thus S and T are weakly increasing mappings.

Note that, two weakly increasing mappings need not be nondecreasing. The following example illustrates this fact.

Example 3.3. [1] Let $X = \mathbb{R}^2$ be endowed with the lexicographical ordering, that is, $(x, y) \preceq (z, w) \Longleftrightarrow (x < z)$ or (if $x = z$, then $y \leq w$). Let $F, G : X \longrightarrow X$ be defined by

$$F(x, y) = (\max\{x, y\}, \min\{x, y\}),$$
$$G(x, y) = (\max\{x, y\}, \frac{x+y}{2}),$$

then

$$\begin{aligned} F(x, y) &= (\max\{x, y\}, \min\{x, y\}) \\ &\preceq GF(x, y) = G(\max\{x, y\}, \min\{x, y\}) \\ &= (\max\{\max\{x, y\}, \min\{x, y\}\}, \frac{\max\{x, y\} + \min\{x, y\}}{2}) \\ &= (\max\{x, y\}, \frac{x+y}{2}), \end{aligned}$$

$$\begin{aligned} G(x,y) &= (\max\{x,y\}, \frac{x+y}{2}) \\ &\preceq FG(x,y) = F(\max\{x,y\}, \frac{x+y}{2}) \\ &= (\max\{\max\{x,y\}, \frac{x+y}{2}\}, \min\{\max\{x,y\}, \frac{x+y}{2}\}) \\ &= (\max\{x,y\}, \frac{x+y}{2}). \end{aligned}$$

Thus F and G are weakly increasing mappings. Note that $(1,4) \preceq (2,3)$ but $F(1,4) = (4,1) \npreceq (3,2) = F(2,3)$, then F is not nondecreasing. Similarly, G is not nondecreasing.

In the sequel, we use the following notation : Let

$$\Delta = \{(x,y,z) \in X^3 | x \preceq y \preceq z \text{ or } z \preceq y \preceq x\}.$$

Our main result is as follows:

THEOREM 3.4. *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a partial S-metric S^* in X such that (X, S^*) is a complete partial S-metric space. Let $A, B : X \longrightarrow X$ are two weakly increasing mappings such that*

$$S^*(Ax, Ay, Bz) \leq \phi(x,y,z) \tag{11}$$

for all $(x,y,z) \in \Delta$, where

$$\phi(x,y,z) = q \max \left\{ \begin{array}{l} S^*(x,y,z), S^*(x,x,Ax), S^*(y,y,Ay), \\ S^*(z,z,Bz), \frac{S^*(x,x,Ay)+S^*(y,y,Bz)+S^*(z,z,Ax)}{4} \end{array} \right\},$$

and $0 < q < \frac{1}{2}$. If $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \longrightarrow x$ in X, then $x_n \preceq x$ hold for all $n \in \mathbb{N}$, then A and B have a common fixed point.

Proof. First of all we show that, if A or B has a fixed point, then it is a common fixed point of A and B. Indeed, let z be a fixed point of B. Now assume $S^*(Az, Az, z) > 0$. If we use the inequality (11), for $x = y = z$, we have

$$\begin{aligned} S^*(Az, Az, z) &= S^*(Az, Az, Bz) \leq \phi(z,z,z) = qS^*(Az, Az, z) \\ &< S^*(Az, Az, z) \end{aligned}$$

which is a contradiction. Thus $S^*(Az, Az, z) = 0$ and so z is a common fixed point of A and B. Similarly, if z is a fixed point of A, then it is also fixed point of B.

Now, let x_0 be an arbitrary point of X. We can define a sequence in X as follows:

$$x_{2n+1} = Ax_{2n} \text{ and } x_{2n+2} = Bx_{2n+1} \text{ for } n \in \{0, 1, \cdots\}.$$

Without lost of generality we can suppose that the successive term of $\{x_n\}$ are different. Otherwise we are finished. Note that, since A and B are weakly increasing, we have

$$x_1 = Ax_0 \preceq BAx_0 = Bx_1 = x_2 = Bx_1 \preceq ABx_1 = Ax_2 = x_3$$

and continuing this process we have

$$x_1 \preceq x_2 \cdots \preceq x_n \preceq x_{n+1} \preceq \cdots .$$

Now we claim that

$$S^*(x_{n+1}, x_{n+1}, x_n) < S^*(x_n, x_n, x_{n-1}).$$

Setting $x = y = x_{2n}$ and $z = x_{2n+1}$ in (11), we have

$$\begin{aligned}
&\phi(x_{2n}, x_{2n}, x_{2n+1}) \\
= \;& q \max \left\{ \begin{array}{l} S^*(x_{2n}, x_{2n}, x_{2n+1}), S^*(x_{2n}, x_{2n}, Ax_{2n}), \\ S^*(x_{2n}, x_{2n}, Ax_{2n}), S^*(x_{2n+1}, x_{2n+1}, Bx_{2n+1}), \\ \frac{S^*(x_{2n},x_{2n},Ax_{2n})+S^*(x_{2n},x_{2n},Bx_{2n+1})+S^*(x_{2n+1},x_{2n+1},Ax_{2n})}{4} \end{array} \right\} \\
\leq \;& q \max \left\{ \begin{array}{l} S^*(x_{2n}, x_{2n}, x_{2n+1}), S^*(x_{2n+1}, x_{2n+1}, x_{2n+2}), \\ \frac{S^*(x_{2n},x_{2n},x_{2n+1})+S^*(x_{2n},x_{2n},x_{2n+2})+S^*(x_{2n+1},x_{2n+1},x_{2n+1})}{4} \end{array} \right\}.
\end{aligned}$$

Since,

$$\begin{aligned}
S^*(x_{2n}, x_{2n}, x_{2n+2}) \;\leq\;& 2S^*(x_{2n}, x_{2n}, x_{2n+1}) + S^*(x_{2n+1}, x_{2n+1}, x_{2n+2}) \\
&-2S^*(x_{2n+1}, x_{2n+1}, x_{2n+1}),
\end{aligned}$$

therefore

$$\begin{aligned}
&\frac{S^*(x_{2n}, x_{2n}, x_{2n+1}) + S^*(x_{2n}, x_{2n}, x_{2n+2}) + S^*(x_{2n+1}, x_{2n+1}, x_{2n+1})}{4} \\
\leq\;& \frac{S^*(x_{2n}, x_{2n}, x_{2n+1}) + 2S^*(x_{2n}, x_{2n}, x_{2n+1}) + S^*(x_{2n+1}, x_{2n+1}, x_{2n+2})}{4}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\phi(x_{2n}, x_{2n}, x_{2n+1}) \\
\leq \;& q \max \left\{ \begin{array}{l} S^*(x_{2n}, x_{2n}, x_{2n+1}), S^*(x_{2n+1}, x_{2n+1}, x_{2n+2}), \\ \frac{S^*(x_{2n},x_{2n},x_{2n+1})+2S^*(x_{2n},x_{2n},x_{2n+1})+S^*(x_{2n+1},x_{2n+1},x_{2n+2})}{4} \end{array} \right\} \\
= \;& q \max\{S^*(x_{2n}, x_{2n}, x_{2n+1}), S^*(x_{2n+1}, x_{2n+1}, x_{2n+2})\}.
\end{aligned}$$

We prove that $S^*(x_{2n}, x_{2n}, x_{2n+1}) > S^*(x_{2n+1}, x_{2n+1}, x_{2n+2})$, for every $n \in \mathbb{N}$. If $S^*(x_{2n}, x_{2n}, x_{2n+1}) \leq S^*(x_{2n+1}, x_{2n+1}, x_{2n+2})$ for some $n \in \mathbb{N}$, then we get

$$\begin{aligned} S^*(x_{2n+1}, x_{2n+1}, x_{2n+2}) &= S^*(Ax_{2n}, Ax_{2n}, Bx_{2n+1}) \\ &\leq \phi(x_{2n}, x_{2n}, x_{2n+1}) \leq qS^*(x_{2n+1}, x_{2n+1}, x_{2n+2}) \\ &< S^*(x_{2n+1}, x_{2n+1}, x_{2n+2}), \end{aligned}$$

is a contradiction. Therefore, we have

$$S^*(x_{2n+1}, x_{2n+1}, x_{2n+2}) \leq qS^*(x_{2n}, x_{2n}, x_{2n+1}) < S^*(x_{2n}, x_{2n}, x_{2n+1}).$$

Similarly, we have

$$\begin{aligned} & \phi(x_{2n}, x_{2n}, x_{2n-1}) \\ = {} & q \max \left\{ \begin{array}{l} S^*(x_{2n}, x_{2n}, x_{2n-1}), S^*(x_{2n}, x_{2n}, Ax_{2n}), \\ S^*(x_{2n}, x_{2n}, Ax_{2n}), S^*(x_{2n-1}, x_{2n-1}, Bx_{2n-1}), \\ \frac{S^*(x_{2n}, x_{2n}, Ax_{2n}) + S^*(x_{2n}, x_{2n}, Bx_{2n-1}) + S^*(x_{2n-1}, x_{2n-1}, Ax_{2n})}{4} \end{array} \right\} \\ \leq {} & q \max \left\{ \begin{array}{l} S^*(x_{2n}, x_{2n}, x_{2n-1}), S^*(x_{2n}, x_{2n}, x_{2n+1}), \\ \frac{S^*(x_{2n}, x_{2n}, x_{2n+1}) + S^*(x_{2n}, x_{2n}, x_{2n}) + S^*(x_{2n-1}, x_{2n-1}, x_{2n+1})}{4} \end{array} \right\}. \end{aligned}$$

Since,

$$\begin{aligned} S^*(x_{2n-1}, x_{2n-1}, x_{2n+1}) &\leq 2S^*(x_{2n-1}, x_{2n-1}, x_{2n}) + S^*(x_{2n}, x_{2n}, x_{2n+1}) \\ &- 2S^*(x_{2n}, x_{2n}, x_{2n}), \end{aligned}$$

therefore

$$\begin{aligned} & \frac{S^*(x_{2n}, x_{2n}, x_{2n+1}) + S^*(x_{2n}, x_{2n}, x_{2n}) + S^*(x_{2n-1}, x_{2n-1}, x_{2n+1})}{4} \\ \leq {} & \frac{S^*(x_{2n}, x_{2n}, x_{2n+1}) + 2S^*(x_{2n-1}, x_{2n-1}, x_{2n}) + S^*(x_{2n}, x_{2n}, x_{2n+1})}{4}. \end{aligned}$$

Hence

$$\begin{aligned} & \phi(x_{2n}, x_{2n}, x_{2n-1}) \\ \leq {} & q \max \left\{ \begin{array}{l} S^*(x_{2n-1}, x_{2n-1}, x_{2n}), S^*(x_{2n}, x_{2n}, x_{2n+1}), \\ \frac{S^*(x_{2n}, x_{2n}, x_{2n+1}) + 2S^*(x_{2n-1}, x_{2n-1}, x_{2n}) + S^*(x_{2n}, x_{2n}, x_{2n+1})}{4} \end{array} \right\} \\ = {} & q \max\{S^*(x_{2n}, x_{2n}, x_{2n-1}), S^*(x_{2n}, x_{2n}, x_{2n+1})\}. \end{aligned}$$

We prove that $S^*(x_{2n-1}, x_{2n-1}, x_{2n}) > S^*(x_{2n}, x_{2n}, x_{2n+1})$, for every $n \in \mathbb{N}$. If $S^*(x_{2n-1}, x_{2n-1}, x_{2n}) \leq S^*(x_{2n}, x_{2n}, x_{2n+1})$ for some $n \in \mathbb{N}$, then we get

$$\begin{aligned} S^*(x_{2n}, x_{2n}, x_{2n-1}) &= S^*(x_{2n-1}, x_{2n-1}, x_{2n}) = S^*(Ax_{2n}, Ax_{2n}, Bx_{2n-1}) \\ &\leq \phi(x_{2n}, x_{2n}, x_{2n-1}) \leq qS^*(x_{2n}, x_{2n}, x_{2n-1}) \\ &< S^*(x_{2n-1}, x_{2n-1}, x_{2n}), \end{aligned}$$

is a contradiction. Therefore, we have

$$S^*(x_{2n}, x_{2n}, x_{2n+1}) \leq qS^*(x_{2n-1}, x_{2n-1}, x_{2n}).$$

Thus , we get

$$S^*(x_{n+1}, x_{n+1}, x_n) \leq qS^*(x_n, x_n, x_{n-1})$$

for all $n \in \mathbb{N}$.

Next, we claim that $\{x_n\}$ is a Cauchy sequence in (X, S^*). Since

$$S^*(x_n, x_n, x_{n+1}) \leq qS^*(x_{n-1}, x_{n-1}, x_n) \leq \cdots \leq q^n S^*(x_0, x_0, x_1)$$

we have

$$\lim_{n\to\infty} S^*(x_n, x_n, x_{n+1}) = 0.$$

Hence

$$\begin{aligned} \lim_{n\to\infty} S^s(x_n, x_n, x_{n+1}) &= 2\lim_{n\to\infty} S^*(x_n, x_n, x_{n+1}) - \lim_{n\to\infty} S^*(x_n, x_n, x_n) \\ &\quad - \lim_{n\to\infty} S^*(x_{n+1}, x_{n+1}, x_{n+1}), \end{aligned}$$

which shows that $\lim_{n\to\infty} S^s(x_n, x_n, x_{n+1}) = 0$.

Since $S^s(x_n, x_n, x_{n+1}) \leq 2S^*(x_n, x_n, x_{n+1})$ we have

$$S^s(x_n, x_n, x_{n+1}) \leq 2S^*(x_n, x_n, x_{n+1}) \leq 2q^n S^*(x_0, x_0, x_1).$$

By the triangle inequality, for $m > n$ we have

$$\begin{aligned} & S^s(x_n, x_n, x_m) \\ \leq\ & 2.2S^s(x_n, x_n, x_{n+1}) + 2.2^2 S^s(x_{n+1}, x_{n+1}, x_{n+2}) \\ & + \cdots + 2.2^{m-n} S^s(x_{m-1}, x_{m-1}, x_m), \end{aligned}$$

hence we get

$$\begin{aligned} S^s(x_n, x_n, x_m) &\leq 2^3 q^n S^*(x_0, x_0, x_1) + 2^4 q^{n+1} S^*(x_0, x_0, x_1) \\ &\quad + \cdots + 2^{m-n} q^m S^*(x_0, x_0, x_1) \\ &\leq 2^3 q^n [1 + 2q + 2^2 q^2 + \cdots] S^*(x_0, x_0, x_1) \\ &\leq \frac{2^3 q^n}{1 - 2q} S^*(x_0, x_0, x_1) \longrightarrow 0. \end{aligned}$$

It follows that $\{x_n\}$ is a Cauchy sequence in the S_b-metric space (X, S^s). Since (X, S^*) is complete, then from Lemma 2.11 follows that the sequence $\{x_n\}$ converges to some x in the S_b-metric space (X, S^s). Hence

$$\lim_{n\to\infty} S^s(x_n, x_n, x) = 0.$$

Again, from Lemma 2.11 we have

$$S^*(x, x, x) = \lim_{n\to\infty} S^*(x_n, x_n, x) = \lim_{n,m\to\infty} S^*(x_n, x_n, x_m).$$

Since $\{x_n\}$ is a Cauchy sequence in the S_b-metric space (X, S^s) and

$$S^s(x_n, x_n, x_m) = 2S^*(x_n, x_n, x_m) - S^*(x_n, x_n, x_n) - S^*(x_m, x_m, x_m),$$

we have

$$\lim_{n,m\to\infty} S^s(x_n, x_n, x_m) = 0,$$

and

$$\lim_{n\to\infty} S^*(x_n, x_n, x_n) = 0.$$

Thus

$$\lim_{n,m\to\infty} S^*(x_n, x_n, x_m) = 0.$$

Therefore, we have

$$S^*(x, x, x) = \lim_{n\to\infty} S^*(x_n, x_n, x) = \lim_{n,m\to\infty} S^*(x_n, x_n, x_m) = 0.$$

That is,

$$\begin{aligned} \lim_{n\to\infty} x_n &= \lim_{n\to\infty} x_{2n+1} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Bx_{2n+1} \\ &= \lim_{n\to\infty} x_{2n+2} = x. \end{aligned}$$

Now, we show that $x = Ax = Bx$. Setting $x = y = x_{2n+2}$ and $z = x$ in (11), we have

$$S^*(x_{2n+2}, x_{2n+2}, Ax) = S^*(Ax, Ax, Bx_{2n+1}) \leq \phi(x, x, x_{2n+1}),$$

where

$$
\begin{aligned}
&\phi(x,x,x_{2n+1})\\
= \;& q\max\left\{\begin{array}{l} S^*(x,x,x_{2n+1}), S^*(x,x,Ax), S^*(x,x,Ax),\\ S^*(x_{2n+1},x_{2n+1},Bx_{2n+1}),\\ \frac{S^*(x,x,Ax)+S^*(x,x,Bx_{2n+1})+S^*(x_{2n+1},x_{2n+1},Ax)}{4}\end{array}\right\}\\
= \;& q\max\left\{\begin{array}{l} S^*(x,x,x_{2n+1}), S^*(x,x,Ax),\\ S^*(x,x,Bx_{2n+1}), \frac{S^*(x,x,Ax)+S^*(x,x,Bx_{2n+2})+S^*(x_{2n+1},x_{2n+1},Ax)}{4}\end{array}\right\}
\end{aligned}
$$

and so letting as $n \longrightarrow \infty$, by Lemma 2.12 we have

$$\lim_{n\to\infty} S^*(x_{2n+2},x_{2n+2},Ax) = S^*(x,x,Ax) \le qS^*(x,x,Ax) = \lim_{n\to\infty}\phi(x,x,x_{2n+1}).$$

Therefore we have

$$
\begin{aligned}
\lim_{n\to\infty} S^*(x_{2n+1},x_{2n+1},Ax) &= S^*(x,x,Ax)\\
&\le qS^*(x,x,Ax) < S^*(x,x,Ax),
\end{aligned}
$$

a contradiction. Therefore, $S^*(Ax,Ax,x) = 0$ and hence $Ax = x$. □

Corollary 3.5. *Let $(X,\preceq)$ be a partially ordered set and (X,S) is a complete S-metric space. Let $A,B : X \longrightarrow X$ are two weakly increasing mappings such that*

$$
\begin{aligned}
&S(Ax,Ay,Bz)\\
\le \;& q\max\left\{\begin{array}{l} S(x,y,z), S(x,x,Ax), S(y,y,Ay),\\ S(z,z,Bz), \frac{S(x,x,Ay)+S(y,y,Bz)+S(z,z,Ax)}{4}\end{array}\right\},
\end{aligned}
$$

for all $(x,y,z) \in \Delta$, where $0 < q < \frac{1}{2}$. If $\{x_n\} \subset X$ is a nondecreasing sequence with $x_n \longrightarrow x$ in X, then $x_n \preceq x$ hold for all $n \in \mathbb{N}$, then A and B have a common fixed point.

Proof. If we take $S^* = S$, then from Theorem 3.4 follows that A and B have a common fixed point. □

Corollary 3.6. *Let $(X,\preceq)$ be a partially ordered set and (X,S) is a complete S-metric space. Let $T : X \longrightarrow X$ be a mapping such that*

$$
\begin{aligned}
&S(Tx,Ty,Tz)\\
\le \;& q\max\left\{\begin{array}{l} S(x,y,z), S(x,x,Tx), S(y,y,Ty),\\ S(z,z,Tz), \frac{S(x,x,Ty)+S(y,y,Tz)+S(z,z,Tx)}{4}\end{array}\right\},
\end{aligned}
$$

for all $(x, y, z) \in \Delta$, *where* $0 < q < \frac{1}{2}$. *If* $\{x_n\} \subset X$ *is a nondecreasing sequence with* $x_n \longrightarrow x$ *in* X, *then* $x_n \preceq x$ *hold for all* $n \in \mathbb{N}$ *and* $Tx \preceq T^2x$, *then* T *has a fixed point.*

Proof. If we take $A = B = T$, then from Corollary 3.6 follows that T have a fixed point. $\square$

4. Application

Consider the integral equations

$$\left\{ \begin{array}{l} x(t) = \int_a^b K_1(t, s, x(s))ds + g(t), \ t \in [a, b], \\ x(t) = \int_a^b K_2(t, s, x(s))ds + g(t), \ t \in [a, b]. \end{array} \right. \tag{12}$$

The purpose of this section is to give an existence theorem for solution of integral equations (12) using Corollary 3.5. Let $\ll$ be a partial order relation on $\mathbb{R}^n$.

Example 4.1. Consider the integral equations (12). Let $X = C([a, b], \mathbb{R}^n)$ with the usual supremum norm, that is, $||x|| = \max_{t\in[a,b]} |x(t)|$, for $x \in C([a, b], \mathbb{R}^n)$. and $S : X^3 \longrightarrow [0, \infty)$ defined by $S(u, v, w) = ||u - w|| + ||v - w||$ for every $u, v, w \in X$.

(i)

$K_1, K_2 : [a, b] \times [a, b] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are continuous,

(ii) for each $t, s \in [a, b]$,

$$K_1(t, s, x(s)) \ll K_2(t, s, \int_a^b K_1(s, \tau, x(\tau))d\tau + g(s)),$$

$$K_2(t, s, x(s)) \ll K_1(t, s, \int_a^b K_2(s, \tau, x(\tau))d\tau + g(s)),$$

(iii) there exist a continuous function $p : [a, b] \times [a, b] \longrightarrow \mathbb{R}^+$, such that

$$||K_1(t, s, u) - K_2(t, s, v)|| \leq p(t, s)||u - v||$$

for each $t, s \in [a, b]$ and comparable $u, v \in \mathbb{R}^n$,

(iv) $\sup_{t\in[a,b]} \int_a^b p(t, s)ds \leq q < \frac{1}{2}$.

Then the integral equations (12) have a solution x^* in $C([a, b], \mathbb{R}^n)$.

Proof. Consider on X the partial order defined by

$$x, y \in C([a,b], \mathbb{R}^n),\ x \preceq y \text{ iff } x(t) \ll y(t) \text{ for any } t \in [a,b].$$

Then $(X, \preceq)$ is a partially ordered set. Also (X, S) is a complete S-metric space. Moreover, for any increasing sequence $\{x_n\}$ in X converging to $x^* \in X$, we have $x_n(t) \ll x^*(t)$ for any $t \in [a,b]$.

Define $F, G : X \longrightarrow X$, by

$$Fx(t) = \int_a^b K_1(t, s, x(s))ds + g(t),\ t \in [a,b],$$

$$Gz(t) = \int_a^b K_2(t, s, z(s))ds + g(t),\ t \in [a,b].$$

Now from (ii), we have, for all $t \in [a,b]$,

$$\begin{aligned} Fx(t) &= \int_a^b K_1(t, s, x(s))ds + g(t) \\ &\ll \int_a^b K_2(t, s, \int_a^b K_1(s, \tau, x(\tau))d\tau + g(s))ds + g(t) \\ &= \int_a^b K_2(t, s, Fx(s))ds + g(t) = GFx(t) \end{aligned}$$

$$\begin{aligned} Gz(t) &= \int_a^b K_2(t, s, z(s))ds + g(t) \\ &\ll \int_a^b K_1(t, s, \int_a^b K_2(s, \tau, z(\tau))d\tau + g(s))ds + g(t) \\ &= \int_a^b K_1(t, s, Gz(s))ds + g(t) = FGz(t). \end{aligned}$$

Thus, we have $Fx \preceq GFx$ and $Gx \preceq FGx$ for all $x, z \in X$. This shows that F and G are weakly increasing. Also for each comparable $x, z \in X$, we have

$$\begin{aligned}
|Fx(t)-Gz(t)| &= |\int_a^b K_1(t,s,x(s))ds - \int_a^b K_2(t,s,z(s))ds| \\
&\leq \int_a^b |K_1(t,s,x(s)) - K_2(t,s,z(s))|ds \\
&\leq \int_a^b p(t,s)|x(s)-z(s)|ds \\
&\leq ||x-z|| \int_a^b p(t,s)ds \\
&\leq q||x-z||,
\end{aligned}$$

for any $t \in [a,b]$. Hence $||Fx - Gz|| \leq q||x-z||$ for each comparable $x, z \in X$. Similarly, we can prove that $||Fy - Gz|| \leq q||y-z||$ for each comparable $y, z \in X$. Therefore, for every $(x,y,z) \in \Delta$ we have

$$\begin{aligned}
S(Fx,Fy,Gz) &\leq qS(x,y,z) \\
&\leq q \max \left\{ \begin{array}{c} S(x,y,z), S(x,x,Ax), S(y,y,Ay), \\ S(z,z,Bz), \frac{S(x,x,Ay)+S(y,y,Bz)+S(z,z,Ax)}{4} \end{array} \right\},
\end{aligned}$$

hence all conditions of Corollary 3.5 are satisfied. Thus the conclusion follows. □

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