Korean J. Math. **20** (2012), No. 1, pp. 61–75

VARIOUS FRAMES AND CONNECTIONS

Yong Chan Kim* and Jung Mi Ko

Abstract. We investigate the properties of various frames and connections on partially ordered sets. In particular, we study the relations between various connections and various frames.

1. Introduction

Pawlak [6] introduced rough set theory to generalize the classical set theory. Rough approximations are defined by a partition of the universe which is corresponding to the equivalence relation about information. It is an important mathematical tool for data analysis and knowledge processing [1-8]. Järvinen et.al.[3] define rough approximations on partially order relations that are not necessarily equivalence relations. Wille [8] introduced the formal concept lattices by allowing some uncertainty in data as examples as Galois, dual Galois, residuated and dual residuated connections. Formal concept analysis is an important mathematical tool for data analysis and knowledge processing [1-7]. Orlowska and Rewitzky [5] investigated the algebraic structures of operators of Galois-style connections.

In this paper, we investigate the properties of various relations, frames and connections on partially ordered sets. In particular, we study the relations between various connections and various frames.

2. Preliminaries

Let X be a set. A pair (X, e_X) is called a partially order set (simply,

Received January 3, 2012. Revised February 27, 2012. Accepted March 5, 2012.

2010 Mathematics Subject Classification: 06A06, 06A15, 06B30, 54F05,68U35.

Key words and phrases: Preordered sets, lower and upper rough approximations, Galois, dual Galois, residuated and dual residuated connections and frames.

This work was supported by Research Institute of Natural Science of Gangneung-Wonju National University.

*Corresponding author.

poset)if $e_X \subset X \times X$ is reflexive, transitive and anti-symmetric. Let (X, e_X) be a poset. If we define a relation $(x, y) \in e_X^{-1}$ iff $(y, x) \in e_X$, then (X, e_X^{-1}) is a poset. We can define a poset $(P(X), e_{P(X)})$ where $e_{P(X)} \subset P(X) \times P(X)$ as $(A, B) \in e_{P(X)}$ iff $A \subset B$ for $A, B \in P(X)$.

Definition 2.1. [5] Let (X, e_X) and (Y, e_Y) be posets and $f : X \to Y$ and $g : Y \to X$ maps.

(1) (e_X, f, g, e_Y) is called a Galois connection if for all $x \in X, y \in Y$, $(y, f(x)) \in e_Y$ iff $(x, g(y)) \in e_X$.

(2)(e_X, f, g, e_Y) is called a dual Galois connection if for all $x \in X, y \in Y$, $(f(x), y) \in e_Y$ iff $(g(y), x) \in e_X$.

(3) (e_X, f, g, e_Y) is called a residuated connection if for all $x \in X, y \in Y$, $(f(x), y) \in e_Y$ iff $(x, g(y)) \in e_X$.

(4) (e_X, f, g, e_Y) is called a dual residuated connection if for all $x \in X, y \in Y$, $(y, f(x)) \in e_Y$ iff $(g(y), x) \in e_X$.

(5) f is an isotone map if $(f(x_1), f(x_2)) \in e_Y$ for all $(x_1, x_2) \in e_X$.

(6) f is an antitone map if $(f(x_2), f(x_1)) \in e_Y$ for all $(x_1, x_2) \in e_X$.

Definition 2.2. [4,5] Let $R \subset X \times Y$ be a relation. For each $B \in P(Y)$, we define operations $[R], [[R]], \langle R\rangle, [\langle R\rangle], [R]^c, \langle R\rangle^c : P(Y) \to P(X)$ as follows:

$$[R](B) = \{x \in X \mid (\forall y \in Y)((x, y) \in R \to y \in B)\},$$

$$[[R]](B) = \{x \in X \mid (\forall y \in Y)(y \in B \to (x, y) \in R)\}$$

$$\langle R\rangle(B) = \{x \in X \mid (\exists y \in Y)((x, y) \in R \,\&\, y \in B)\}$$

$$[\langle R\rangle](B) = \{x \in X \mid (\exists y \in Y)((x, y) \in R^c \,\&\, y \in B^c)\}.$$

$$[R]^c(B) = \{x \in X \mid (\forall y \in Y)((x, y) \in R \to y \in B^c)\}$$

$$\langle R\rangle^c(B) = \{x \in X \mid (\exists y \in Y)((x, y) \in R \,\&\, y \in B^c)\}.$$

For each $R^{-1} = \{(y, x) \in Y \times X \mid (x, y) \in R\}$, we similarly define operations $[R^{-1}], [[R^{-1}]], \langle R^{-1}\rangle, [\langle R^{-1}\rangle], [R^{-1}]^c, \langle R^{-1}\rangle^c : P(X) \to P(Y)$.

Definition 2.3. [4] Let (X, e_X) be a poset. A set $A \in P(X)$ is called an e_X-*upper set* if $(x \in A \,\&\, (x, y) \in e_X) \to y \in A$ for $x, y \in X$.

THEOREM 2.4. [4] *Let* (X, e_X) *and* (Y, e_Y) *be posets and* $R \subset X \times Y$ *be a relation. For* $A \in P(X)$ *and* $B \in P(Y)$, *we define operations as follows:*

$$[e_X](A) = \{x \in X \mid (\forall z \in X)((x, z) \in e_X \to z \in A)\},$$

$$\langle e_X \rangle(A) = \{x \in X \mid (\exists z \in X)((x, z) \in e_X \ \& \ z \in A)\},$$

$$I(X) = \{A \in P(X) \mid [e_X](A) = A\},$$

$$I(Y) = \{B \in P(Y) \mid [e_Y](B) = B\},$$

Then we have the following properties:

(1)*If* $(e_X)_x = \{z \in X \mid (x, z) \in e_X\}$ *and* $(e_X)_x^{-1} = \{z \in X \mid (z, x) \in e_X\}$, *then* $(e_X)_x$ *and* $((e_X)_x^{-1})^c$ *are* e_X*-upper sets.*

(2) $(e_X \circ R)^{-1} = R^{-1} \circ e_X^{-1}$ *and* $(R \circ e_Y)^{-1} = e_Y^{-1} \circ R^{-1}$.

(3) $e_X \circ R \subset R$ *iff* $e_X^{-1} \circ R^c \subset R^c$.

(4) $R \circ e_Y^{-1} \subset R$ *iff* $R^c \circ e_Y \subset R^c$.

(5) A *is an* e_X*-upper set iff* $[e_X](A) = A$ *iff* $[e_X^{-1}](A^c) = A^c$ *iff* $\langle e_X^{-1} \rangle(A) = A$.

(6) *If* $e_X \circ R \subset R$, *then* $(R_y^{-1})^*, [R](B), [R]^c(B), [\langle R \rangle](B), \langle R^c \rangle(B), \langle R^c \rangle^c(B), [[R^c]](B) \in I(X)$.

(7) *If* $e_X^{-1} \circ R \subset R$, *then* $R_y^{-1}, \langle R \rangle(B), \langle R \rangle^c(B), [[R]](B), [R^c](B), [R^c]^c(B), [\langle R^c \rangle](B) \in I(X)$.

(8) *If* $R \circ e_Y \subset R$, *then* $R_x, \langle R^{-1} \rangle(A), [[R^{-1}]](A), [(R^c)^{-1}](A),$
$[(R^c)^{-1}]^c(A), \langle R^{-1} \rangle^c(A) \in I(Y)$.

(9) *If* $R \circ e_Y^{-1} \subset R$, *then* $R_x^c, [R^{-1}](A), [R^{-1}]^c(A), [\langle R^{-1} \rangle](A),$
$\langle (R^c)^{-1} \rangle(A), \langle (R^c)^{-1} \rangle^c(A), [[(R^c)^{-1}]](A) \in I(Y)$.

DEFINITION 2.5 [3,6]. In above theorem, $[e_X](A)$ and $\langle e_X \rangle(A)$ are called *rough lower approximation* and *rough upper approximation*, respectively, for $A \in P(X)$ on a partially ordered set.

If e_X is an equivalence relation, $[e_X](A)$ and $\langle e_X \rangle(A)$ are rough lower approximation and rough upper approximation for $A \in P(X)$ in a Pawlak's sense [6].

3. The properties of rough approximations

DEFINITION 3.1 [3,5]. Let (X, e_X) and (Y, e_Y) be posets, $R \in P(X \times Y)$ and $S \in P(Y \times X)$. A structure (e_X, R, S, e_Y) is called:
(1) a *Galois frame* if $S = R^{-1}$ and $e_X \circ R \circ e_Y^{-1} \subset R$.
(2) a *dual Galois frame* if $S = R^{-1}$ and $e_X^{-1} \circ R \circ e_Y \subset R$.
(3) a *residuated frame* if $S = R^{-1}$ and $e_X \circ R \circ e_Y \subset R$.
(4) a *dual residuated frame* if $S = R^{-1}$ and $e_X^{-1} \circ R \circ e_Y^{-1} \subset R$.

LEMMA 3.2. *Let (X, e_X) and (Y, e_Y) be posets and $f : X \to Y$ and $g : Y \to X$ maps.*
(1) If g is antitone and define $(x, y) \in R_1$ iff $(x, g(y)) \in e_X$ (resp. $(x, y) \in R_2$ iff $(g(y), x) \in e_X$), then $e_X \circ R_1 \circ e_Y^{-1} \subset R_1$ (resp. $e_X^{-1} \circ R_2 \circ e_Y \subset R_2$).
(2) If g is isotone and define $(x, y) \in R_1$ iff $(x, g(y)) \in e_X$ (resp. $(x, y) \in R_2$ iff $(g(y), x) \in e_X$), then $e_X \circ R_1 \circ e_Y \subset R_1$ (resp. $e_X^{-1} \circ R_2 \circ e_Y^{-1} \subset R_2$).
(3) If f is antitone and $(x, y) \in R_1$ iff $(y, f(x)) \in e_Y$ (resp. $(x, y) \in R_2$ iff $(f(x), y) \in e_Y$), then $e_X \circ R_1 \circ e_Y^{-1} \subset R_1$ (resp. $e_X^{-1} \circ R_2 \circ e_Y \subset R_2$).
(4) If f is isotone and $(x, y) \in R_1$ iff $(y, f(x)) \in e_Y$ (resp. $(x, y) \in R_2$ iff $(f(x), y) \in e_Y$), then $e_X^{-1} \circ R_1 \circ e_Y^{-1} \subset R_1$ (resp. $e_X \circ R_2 \circ e_Y \subset R_2$).

Proof. (1) Since g is antitone, $(y, y_1) \in e_Y$ implies $(g(y_1), g(y)) \in e_X$. Then

$$
\begin{aligned}
&(x, x_1) \in e_X \ \& \ (x_1, y_1) \in R_1 \ \& \ (y_1, y) \in e_Y^{-1} \\
&\text{iff } (x, x_1) \in e_X \ \& \ (x_1, g(y_1)) \in e_X \ \& \ (y, y_1) \in e_Y \\
&(\Rightarrow) \ (x, x_1) \in e_X \ \& \ (x_1, g(y_1)) \in e_X \ \& \ (g(y_1), g(y)) \in e_X \\
&(\Rightarrow) \ (x, g(y)) \in e_X \text{ iff } (x, y) \in R
\end{aligned}
$$

Hence $e_X \circ R_1 \circ e_Y^{-1} \subset R_1$.
Other cases are similarly proved.

□

THEOREM 3.3. *Let (X, e_X) and (Y, e_Y) be posets and $f : X \to Y$ and $g : Y \to X$ maps.*
(1) If (e_X, f, g, e_Y) is a Galois connection and define relations $R \in P(X \times Y)$ and $S \in P(Y \times X)$ as

$$(x, y) \in R \text{ iff } (x, g(y)) \in e_X, \quad (y, x) \in S \text{ iff } (x, y) \in R.$$

Then (e_X, R, S, e_Y) *is a Galois frame.*

(2) *If* (e_X, f, g, e_Y) *is a dual Galois connection and define maps relations* $R \in P(X \times Y)$ *and* $S \in P(Y \times X)$ *as*

$$(x,y) \in R \text{ iff } (g(y), x) \in e_X, \ \ (y,x) \in S \text{ iff } (x,y) \in R.$$

Then (e_X, R, S, e_Y) *is a dual Galois frame.*

(3) *If* (e_X, f, g, e_Y) *is a residuated connection and define relations* $R \in P(X \times Y)$ *and* $S \in P(Y \times X)$ *as*

$$(x,y) \in R \text{ iff } (x, g(y)) \in e_X, \ \ (y,x) \in S \text{ iff } (x,y) \in R.$$

Then (e_X, R, S, e_Y) *is a residuated frame.*

(4) *If* (e_X, f, g, e_Y) *is a dual residuated connection and define relations* $R \in P(X \times Y)$ *and* $S \in P(Y \times X)$ *as*

$$(x,y) \in R \text{ iff } (g(y), x) \in e_X, \ \ (y,x) \in S \text{ iff } (x,y) \in R.$$

Then (e_X, R, S, e_Y) *is a dual Galois frame.*

Proof. (1) Let (e_X, f, g, e_Y) be a Galois connection. If $(y_1, y_2) \in e_Y$, since $(g(y), g(y)) \in e_X$ iff $(y, f(g(y)) \in e_Y$, we have $(y_1, y_2) \in e_Y$ and $(y_2, f(g(y_2)) \in e_Y$ implies $(y_1, f(g(y_2)) \in e_Y$ iff $(g(y_2), g(y_1)) \in e_X$. Hence g is antitone. By Lemma 3.2(1), $e_X \circ R \circ e_Y^{-1} \subset R$. Thus (e_X, R, S, e_Y) is a Galois frame.

(2),(3) and (4) are similarly proved.

□

Theorem 3.4. *Let* (X, e_X) *and* (Y, e_Y) *be posets with* $I(X) = \{A \in P(X) \mid [e_X](A) = A\}$ *and* $I(Y) = \{B \in P(Y) \mid [e_Y](B) = B\}$.

(1) *If* (e_X, f, g, e_Y) *is a Galois connection and define relations* $R \in P(I(X) \times I(Y))$ *and* $S \in P(I(Y) \times I(X))$ *as*

$$(A, B) \in R \text{ iff } (\forall x \in X)(f(x) \in B \to x \in A^c),$$

$$(B, A) \in S \text{ iff } (\forall y \in Y)(g(y) \in A \to y \in B^c).$$

Then $(e_{I(X)}, R, S, e_{I(Y)})$ *is a Galois frame.*

(2) If (e_X, f, g, e_Y) is a dual Galois connection and define relations $R \in P(I(X) \times I(Y))$ and $S \in P(I(Y) \times I(X))$ as

$$(A, B) \in R \text{ iff } (\forall x \in X)(x \in A^c \to f(x) \in B),$$

$$(B, A) \in S \text{ iff } (\forall y \in Y)(y \in B^c \to g(y) \in A).$$

Then $(e_{I(X)}, R, S, e_{I(Y)})$ is a dual Galois frame.

(3) If (e_X, f, g, e_Y) is a residuated connection and define relations $R \in P(I(X) \times I(Y))$ and $S \in P(I(Y) \times I(X))$ as

$$(A, B) \in R \text{ iff } (\forall x \in X)(x \in A \to f(x) \in B),$$

$$(B, A) \in S \text{ iff } (\forall y \in Y)(y \in B \to g(y) \in A).$$

Then $(e_{I(X)}, R, S, e_{I(Y)})$ is a residuated frame.

(4) If (e_X, f, g, e_Y) is a dual residuated connection and define relations $R \in P(I(X) \times I(Y))$ and $S \in P(I(Y) \times I(X))$ as

$$(A, B) \in R \text{ iff } (\forall x \in X)(f(x) \in B \to x \in A),$$

$$(B, A) \in S \text{ iff } (\forall y \in Y)(y \in B \to g(y) \in A).$$

Then $(e_{I(X)}, R, S, e_{I(Y)})$ is a dual residuated frame.

Proof. (1) We have $(A, A') \in e_{P(X)}$ & $(A', B') \in R$ & $(B', B) \in e^{-1}_{P(Y)} \subset (A, B) \in R$ from:

$$(f(x) \in B \to f(x) \in B')\&(f(x) \in B' \to x \in A'^c)\&(x \in A'^c \to x \in A^c)$$
$$\to (f(x) \in B \to x \in A^c)$$

Hence $e_{P(X)} \circ R \circ e^{-1}_{P(Y)} \subset R$.

For $A \in I(X)$, $B \in I(Y)$ and a Galois connection (e_X, f, g, e_Y), since $x \in A$ iff $x \in A$ and $(x, g(f(x))) \in e_X$ implies $g(f(x)) \in A$ and $y \in B$ iff $y \in B$ and $(y, f(g(y))) \in e_Y$ implies $f(g(y)) \in B$, we have $R^{-1} = S$ from:

$$\begin{aligned}
(A, B) \in R \text{ iff } & (\forall x \in X)((f(x) \in B \to x \in A^c) \\
& (\Rightarrow)\ (\forall y \in Y)(f(g(y)) \in B \to g(y) \in A^c) \\
& (\Rightarrow)\ (\forall y \in Y)(y \in B \to g(y) \in A^c) \\
& (\Rightarrow)\ (\forall y \in Y)(g(y) \in A \to y \in B^c) \\
& (\Rightarrow)\ (B, A) \in S,
\end{aligned}$$

$$
\begin{aligned}
(B, A) \in S \text{ iff } & (\forall y \in Y)(g(y) \in A \to y \in B^c) \\
& (\Rightarrow) \ (\forall x \in X)(g(f(x)) \in A \to f(x) \in B^c) \\
& (\Rightarrow) \ (\forall x \in X)(x \in A \to f(x) \in B^c) \\
& (\Rightarrow) \ (\forall x \in X)(f(x) \in B \to x \in A^c) \\
& (\Rightarrow) \ (A, B) \in R.
\end{aligned}
$$

(2) We have $(A, A') \in e_{P(X)}^{-1}$ & $(A', B') \in R$ & $(B', B) \in e_{P(Y)} \subset (A, B) \in R$ implies $e_{P(X)}^{-1} \circ R \circ e_{P(Y)} \subset R$ from:

$$
\begin{aligned}
&(x \in A' \to x \in A) \& (x \in (A')^c \to f(x) \in B') \& (f(x) \in B' \to f(x) \in B) \\
&\to (x \in A^c \to f(x) \in B)
\end{aligned}
$$

Since $f(g(y)) \in B$ & $(f(g(y)), y) \in e_Y$ implies $y \in B$, we have $R \subset S^{-1}$ from:

$$
\begin{aligned}
(A, B) \in R \text{ iff } & (\forall x \in X)(x \in A^c \to f(x) \in B) \\
& (\Rightarrow) \ (\forall y \in Y)(g(y) \in A^c \to f(g(y)) \in B) \\
& (\Rightarrow) \ (\forall y \in Y)(g(y) \in A^c \to y \in B) \\
& (\Rightarrow) \ (\forall y \in Y)(y \in B^c \to g(y) \in A) \\
& (\Rightarrow) \ (B, A) \in S,
\end{aligned}
$$

Similarly, $S \subset R^{-1}$.

(3) For $A, B \in I(X)$, since $f(g(y)) \in B$ & $(f(g(y)), y) \in e_Y \to x \in B$ and $x \in A$ & $(x, g(f(x))) \in e_X \to (g(f(x))) \in A$, we have

$$
\begin{aligned}
(A, B) \in R \text{ iff } & (\forall x \in X)(x \in A \to f(x) \in B) \\
& (\Rightarrow) \ (\forall y \in Y)(g(y) \in A \to f(g(y)) \in B) \\
& (\Rightarrow) \ (\forall y \in Y)(g(y) \in A \to y \in B) \\
& (\Rightarrow) \ (B, A) \in S,
\end{aligned}
$$

$$
\begin{aligned}
(B, A) \in S \text{ iff } & (\forall y \in Y)(g(y) \in A \to y \in B) \\
& (\Rightarrow) \ (\forall x \in X)(g(f(x)) \in A \to f(x) \in B) \\
& (\Rightarrow) \ (\forall x \in X)(x \in A \to f(x) \in B) \\
& (\Rightarrow) \ (A, B) \in R.
\end{aligned}
$$

(4) For $A \in I(X)$, since $g(f(x)) \in A$ & $(g(f(x)), x) \in e_X \to x \in A$ and $y \in B$ & $(y, f(g(y))) \in e_Y \to f(g(y)) \in B$, we have

$$\begin{aligned}
(A,B) \in R \text{ iff } & (\forall x \in X)((f(x) \in B \to x \in A) \\
& (\Rightarrow)\ (\forall y \in Y)(f(g(y)) \in B \to g(y) \in A) \\
& (\Rightarrow)\ (\forall y \in Y)(y \in B \to g(y) \in A) \\
& (\Rightarrow)\ (B,A) \in S,
\end{aligned}$$

$$\begin{aligned}
(B,A) \in S \text{ iff } & (\forall y \in Y)(y \in B \to g(y) \in A) \\
& (\Rightarrow)\ (\forall x \in X)(f(x) \in B \to g(f(x)) \in A) \\
& (\Rightarrow)\ (\forall x \in X)(f(x) \in B \to x \in A) \\
& (\Rightarrow)\ (A,B) \in R.
\end{aligned}$$

□

THEOREM 3.5. *Let (e_X, R, S, e_Y) be a Galois frame.*
(1) *If we define $[R^{-1}]^c : I(X) \to I(Y)$ and $[S^{-1}]^c : I(Y) \to I(X)$ as*

$$y \in [R^{-1}]^c(A) \text{ iff } (\forall x \in X)((x,y) \in R \to x \in A^c)$$

$$x \in [S^{-1}]^c(B) \text{ iff } (\forall y \in Y)((y,x) \in S \to y \in B^c)$$

Then $(e_{I(X)}, [R^{-1}]^c, [S^{-1}]^c, e_{I(Y)})$ is a Galois connection.
(2) *If we define $[\langle R^{-1}\rangle] : I(X) \to I(Y)$ and $[\langle S^{-1}\rangle] : I(Y) \to I(X)$ as*

$$y \in [\langle R^{-1}\rangle](A) \text{ iff } (\exists x \in X)(x \in A^c \ \&\ (x,y) \in R^c)$$

$$x \in [\langle S^{-1}\rangle](B) \text{ iff } (\exists y \in Y)(y \in B^c \ \&\ (y,x) \in S^c).$$

Then $(e_{I(X)}, [\langle R^{-1}\rangle], [\langle S^{-1}\rangle], e_{I(Y)})$ is a dual Galois connection.

Proof. (1) Since $R \circ e_Y^{-1} \subset e_X \circ R \circ e_Y^{-1} \subset R$, by Theorem 2.4 (9), $[R^{-1}]^c(A) \in I(Y)$. Since $(e_X \circ R \circ e_Y^{-1})^{-1} = e_Y \circ S \circ e_X^{-1} \subset S$, by Theorem 2.4 (9), $[S^{-1}]^c(B) \in I(X)$. Hence $[R^{-1}]^c$ and $[S^{-1}]^c$ is well defined. Moreover, $(e_{I(X)}, [R^{-1}]^c, [S^{-1}]^c, e_{I(Y)})$ is a Galois connection from

$$B \subset [R^{-1}]^c(A) \text{ iff } \vdash (\forall y \in Y)(y \in B \to y \in [R^{-1}]^c(A))$$
$$\text{iff } \vdash (\forall y \in Y)(y \in B \to (\forall x \in X)((x,y) \in R \to x \in A^c))$$
$$\text{iff } \vdash (\forall y \in Y)(\forall x \in X)(y \in B \to (x \in A \to (x,y) \in R^c))$$
$$\text{iff } \vdash (\forall y \in Y)(\forall x \in X)(x \in A \to (y \in B \to (x,y) \in R^c))$$
$$\text{iff } \vdash (\forall y \in Y)(\forall x \in X)(x \in A \to ((x,y) \in R \to y \in B^c))$$
$$\text{iff } \vdash (\forall x \in X)(x \in A \to (\forall y \in Y)((x,y) \in R \to y \in B^c))$$
$$\text{iff } \vdash (\forall x \in X)(x \in A \to x \in [S^{-1}]^c(B))$$
$$\text{iff } A \subset [S^{-1}]^c(B).$$

(2) Since $R \circ e_Y^{-1} \subset e_X \circ R \circ e_Y^{-1} \subset R$, by Theorem 2.4 (9), then $[\langle R^{-1}\rangle](A) \in I(Y)$. Since $(e_X \circ R \circ e_Y^{-1})^{-1} = e_Y \circ S \circ e_X^{-1} \subset S$, by Theorem 2.4 (9), $[\langle S^{-1}\rangle](B) \in I(X)$. Hence $[\langle R^{-1}\rangle]$ and $[\langle S^{-1}\rangle]$ is well defined. Moreover, $(e_{I(X)}, [\langle R^{-1}\rangle], [\langle S^{-1}\rangle], e_{I(Y)})$ is a dual Galois connection from

$$[\langle R^{-1}\rangle](A) \subset B \text{ iff } \vdash (\forall y \in Y)(y \in [\langle R^{-1}\rangle](A) \to y \in B)$$
$$\text{iff } \vdash (\forall y \in Y)((\exists x \in X)(x \in A^c \ \& \ (x,y) \in R^c) \to y \in B)$$
$$\text{iff } \vdash (\forall y \in Y)(\forall x \in X)((x,y) \in R^c) \to (x \in A^c \to y \in B)$$
$$\text{iff } \vdash (\forall y \in Y)(\forall x \in X)((x,y) \in R^c) \to (y \in B^c \to x \in A)$$
$$\text{iff } \vdash (\forall x \in X)((\exists y \in Y)(y \in B^c \ \& \ (x,y) \in R^c) \to x \in A)$$
$$\text{iff } \vdash (\forall x \in X)(x \in [\langle S^{-1}\rangle](B) \to x \in A)$$
$$\text{iff } [\langle S^{-1}\rangle](B) \subset A.$$

□

Theorem 3.6. *Let (e_X, R, S, e_Y) be a dual Galois frame.*
(1) *If we define $\langle R^{-1}\rangle^c : I(X) \to I(Y)$ and $\langle S^{-1}\rangle^c : I(Y) \to I(X)$ as*

$$y \in \langle R^{-1}\rangle^c(A) \text{ iff } (\exists x \in X)(x \in A^c \ \& \ (x,y) \in R)$$
$$x \in \langle S^{-1}\rangle^c(B) \text{ iff } (\exists y \in Y)(y \in B^c \ \& \ (y,x) \in S).$$

Then $(e_{I(X)}, \langle R^{-1}\rangle^c, \langle S^{-1}\rangle^c, e_{I(Y)})$ *is a dual Galois connection.*

(2) *If we define* $[[R^{-1}]] : I(X) \to I(Y)$ *and* $[[S^{-1}]] : I(Y) \to I(X)$ *as*

$$y \in [[R^{-1}]](A) \text{ iff } (\forall x \in X)(x \in A \to (x, y) \in R)$$
$$x \in [[S^{-1}]](B) \text{ iff } (\forall y \in Y)(y \in B \to (y, x) \in S).$$

Then $(e_{I(X)}, [[R^{-1}]], [[S^{-1}]], e_{I(Y)})$ *is a Galois connection.*

Proof. (1) Since $R \circ e_Y \subset e_X^{-1} \circ R \circ e_Y \subset R$, by Theorem 2.4 (8), $\langle R^{-1}\rangle^c(A) \in I(Y)$. Since $(e_X^{-1} \circ R \circ e_Y)^{-1} = e_Y^{-1} \circ S \circ e_X \subset S$, by Theorem 2.4 (8), $\langle S^{-1}\rangle^c(B) \in I(X)$. Hence $\langle R^{-1}\rangle^c$ and $\langle S^{-1}\rangle^c$ is well defined. Moreover, $(e_{I(X)}, \langle R^{-1}\rangle^c, \langle S^{-1}\rangle^c, e_{I(Y)})$ is a dual Galois connection from

$$\begin{aligned}
&\langle R^{-1}\rangle^c(A) \subset B \text{ iff } \vdash (\forall y \in Y)(y \in \langle R^{-1}\rangle^c(A) \to y \in B) \\
&\text{iff } \vdash (\forall y \in Y)((\exists x \in X)(x \in A^c \ \& \ (x, y) \in R \to y \in B)) \\
&\text{iff } \vdash (\forall y \in Y)(\forall x \in X)((x, y) \in R \to (x \in A^c \to y \in B)) \\
&\text{iff } \vdash (\forall y \in Y)(\forall x \in X)((x, y) \in R \to (y \in B^c \to x \in A)) \\
&\text{iff } \vdash (\forall x \in X)((\exists y \in Y)(y \in B^c \ \& \ (x, y) \in R) \to x \in A) \\
&\text{iff } \vdash (\forall x \in X)(x \in \langle S^{-1}\rangle^c(B) \to x \in A) \\
&\text{iff } \langle S^{-1}\rangle^c(B) \subset A.
\end{aligned}$$

(2) Since $R \circ e_Y \subset e_X^{-1} \circ R \circ e_Y \subset R$, by Theorem 2.4 (8), $[[R^{-1}]](A) \in I(X)$. Since $S \circ e_X \subset R$, by Theorem 2.4 (8), $[[S^{-1}]](B) \in I(X)$. Then $(e_{I(X)}, [[R^{-1}]], [[S^{-1}]], e_{I(Y)})$ is a Galois connection from

$$\begin{aligned}
&B \subset [[R^{-1}]](A) \text{ iff } \vdash (\forall y \in Y)(y \in B \to y \in [[R^{-1}]](A)) \\
&\text{iff } \vdash (\forall y \in Y)(y \in B \to (\forall x \in X)(x \in A \to (x, y) \in R)) \\
&\text{iff } \vdash (\forall y \in Y)(\forall x \in X)(y \in B \to (x \in A \to (x, y) \in R)) \\
&\text{iff } \vdash (\forall y \in Y)(\forall x \in X)(x \in A \to (y \in B \to (x, y) \in R)) \\
&\text{iff } \vdash (\forall x \in X)(x \in A \to x \in [[S^{-1}]](B)) \\
&\text{iff } A \subset [[S^{-1}]](B).
\end{aligned}$$

□

THEOREM 3.7. *Let* (e_X, R, S, e_Y) *be a residuated frame.*

(1) *If we define* $\langle R^{-1}\rangle : I(X) \to I(Y)$ *and* $[S^{-1}] : I(Y) \to I(X)$ *as*

$$y \in \langle R^{-1}\rangle(A) \text{ iff } (\exists x \in X)((x,y) \in R \mathbin{\&} x \in A)$$
$$x \in [S^{-1}](B) \text{ iff } (\forall y \in Y)((y,x) \in S \to y \in B)$$

Then $(e_{I(X)}, \langle R^{-1}\rangle, [S^{-1}], e_{I(Y)})$ *is a residuated connection.*

(2) *If we define* $[(R^c)^{-1}] : I(X) \to I(Y)$ *and* $\langle (S^c)^{-1} : I(Y) \to I(X)$ *as*

$$y \in [(R^c)^{-1}](A) \text{ iff } (\forall x \in X)((x,y) \in R^c \to x \in A)$$
$$x \in \langle (S^c)^{-1}\rangle(B) \text{ iff } (\exists y \in Y)(y \in B \mathbin{\&} (y,x) \in S^c).$$

Then $(e_{I(X)}, [(R^c)^{-1}], \langle (S^c)^{-1}\rangle, e_{I(Y)})$ *is a dual residuated connection.*

Proof. (1) Since $R \circ e_Y \subset e_X \circ R \circ e_Y \subset R$, by Theorem 2.4 (8), $\langle R^{-1}\rangle(A) \in I(Y)$. Since $S \circ e_X^{-1} \subset (e_X \circ R \circ e_Y)^{-1} = e_Y^{-1} \circ S \circ e_X^{-1} \subset S$, by Theorem 2.4 (9), $[S^{-1}](B) \in I(X)$. Hence $\langle R^{-1}\rangle$ and $[S^{-1}]$ is well defined. Moreover, $(e_{I(X)}, \langle R^{-1}\rangle, [S^{-1}], e_{I(Y)})$ is a residuated connection from

$$\begin{aligned}
&\langle R^{-1}\rangle(A) \subset B \text{ iff } \vdash (\forall y \in Y)(y \in \langle R^{-1}\rangle(A) \to y \in B)\\
&\text{iff } \vdash (\forall y \in Y)((\exists x \in X)(x \in A \mathbin{\&} (x,y) \in R) \to y \in B)\\
&\text{iff } \vdash (\forall y \in Y)(\forall x \in X)(x \in A \to ((x,y) \in R \to y \in B))\\
&\text{iff } \vdash (\forall x \in X)(x \in A \to (\forall y \in Y)((x,y) \in R \to y \in B))\\
&\text{iff } A \subset [S^{-1}](B).
\end{aligned}$$

(2) Since $R\circ e_Y \subset e_X \circ R \circ e_Y \subset R$ and $R \circ e_Y \subset R$ iff $R^c \circ e_Y^{-1} \subset R^c$, by Theorem 2.4 (8), $[(R^c)^{-1}](A) \in I(Y)$. Since $S \circ e_X^{-1} \subset (e_X \circ R \circ e_Y)^{-1} = e_Y^{-1} \circ S \circ e_X^{-1} \subset S$, by Theorem 2.4 (9), $\langle (S^c)^{-1}\rangle(B) \in I(X)$. So, $[(R^c)^{-1}]$ and $\langle (S^c)^{-1}\rangle$ is well defined. Thus, $(e_{I(X)}, \langle R^{-1}\rangle, \langle (S^c)^{-1}\rangle, e_{I(Y)})$ is a dual residuated connection from

$$\begin{aligned}
&B \subset [(R^c)^{-1}](A) \text{ iff } \vdash (\forall y \in Y)(y \in B \to y \in [(R^c)^{-1}](A))\\
&\text{iff } \vdash (\forall y \in Y)(y \in B \to (\forall x \in X)((x,y) \in R^c \to x \in A)\\
&\text{iff } \vdash (\forall y \in Y)(\forall x \in X)(y \in B \to ((x,y) \in R^c \to x \in A)\\
&\text{iff } \vdash (\forall y \in Y)(\forall x \in X)((y \in B \mathbin{\&} (x,y) \in R^c) \to x \in A)\\
&\text{iff } \vdash (\forall x \in X)((\exists y \in Y)(y \in B \mathbin{\&} (x,y) \in R^c) \to x \in A)\\
&\text{iff } \langle (S^c)^{-1}\rangle(B) \subset A.
\end{aligned}$$

□

THEOREM 3.8. *Let (e_X, R, S, e_Y) be a dual residuated frame.*

(1) *If we define $\langle (R^c)^{-1}\rangle : I(X) \to I(Y)$ and $[(S^c)^{-1}] : I(Y) \to I(X)$ as*

$$y \in \langle (R^c)^{-1}\rangle(A) \text{ iff } (\exists x \in X)((x,y) \in R^c \ \& \ x \in A)$$

$$x \in [(S^c)^{-1}](B) \text{ iff } (\forall y \in Y)((y,x) \in S^c \to y \in B)$$

Then $(e_{I(X)}, \langle (R^c)^{-1}\rangle, [(S^c)^{-1}], e_{I(Y)})$ is a residuated connection.

(2) *If we define $[R^{-1}] : I(X) \to I(Y)$ and $\langle S^{-1}\rangle : I(Y) \to I(X)$ as*

$$y \in [R^{-1}](A) \text{ iff } (\forall x \in X)((x,y) \in R \to x \in A)$$

$$x \in \langle S^{-1}\rangle(B) \text{ iff } (\exists y \in Y)((y,x) \in S \ \& \ y \in B)$$

Then $(e_{I(X)}, [R^{-1}], \langle S^{-1}\rangle, e_{I(Y)})$ is a dual residuated connection.

Proof. (1) Since $R \circ e_Y^{-1} \subset e_X^{-1} \circ R \circ e_Y^{-1} \subset R$ and $R \circ e_Y^{-1} \subset R$ iff $R^c \circ e_Y \subset R^c$, by Theorem 2.4(9), $\langle (R^c)^{-1}\rangle(A) \in I(Y)$. Since $S \circ e_X \subset (e_X^{-1} \circ R \circ e_Y^{-1})^{-1} = e_Y \circ S \circ e_X \subset S$, by Theorem 2.4 (8), $[(S^c)^{-1}](B) \in I(X)$. Hence $\langle (R^c)^{-1}\rangle$ and $[(S^c)^{-1}]$ is well defined. Then $(e_{I(X)}, \langle (R^c)^{-1}\rangle, [(S^c)^{-1}], e_{I(Y)})$ is a residuated connection from:

$$\begin{aligned}
&\langle (R^c)^{-1}\rangle(A) \subset B \text{ iff } \vdash (\forall y \in Y)(y \in \langle (R^c)^{-1}\rangle(A)(A) \to y \in B) \\
&\text{iff } \vdash (\forall y \in Y)((\exists x \in X)(x \in A \ \& \ (x,y) \in R^c) \to y \in B) \\
&\text{iff } \vdash (\forall y \in Y)(\forall x \in X)(x \in A \to ((x,y) \in R^c \to y \in B)) \\
&\text{iff } \vdash (\forall y \in Y)(\forall x \in X)(x \in A \to (\forall y \in Y)((x,y) \in R^c \to y \in B)) \\
&\text{iff } A \subset [(S^c)^{-1}](B).
\end{aligned}$$

(2) Since $R \circ e_Y^{-1} \subset e_X^{-1} \circ R \circ e_Y^{-1} \subset R$, by Theorem 2.4 (9), $[R^{-1}](A) \in I(Y)$. Since $S \circ e_X \subset (e_X^{-1} \circ R \circ e_Y^{-1})^{-1} = e_Y \circ S \circ e_X \subset S$, by Theorem 2.4 (8), $\langle S^{-1}\rangle(B) \in I(X)$. Hence $[R^{-1}]$ and $\langle S^{-1}\rangle$ is well defined. We similarly proved as in (1).

□

EXAMPLE 3.9. Let $(X = \{a,b,c,d\}, e_X)$ and $(Y = \{x,y,z\}, e_Y)$ be posets with

$$e_X = \{(a,a),(a,b),(a,c),(a,d),(b,b),(b,d),(c,c),(c,d),(d,d)\}.$$

$$e_Y = \{(x,x),(x,y),(x,z),(y,y),(z,y),(z,z)\}.$$

Put a relation $R \subset X \times Y$ as

$$R = \{(a,x),(b,x),(b,y),(b,z),(c,x),(c,z),(d,x),(d,y),(d,z)\}.$$

Then $(e_X)_a = \{a,b,c,d\}, (e_X)_b = \{b,d\}, (e_X)_c = \{c,d\}, (e_X)_d = \{d\}$ are e_X-upper sets. Since $(a,b) \in e_X$ & $(b,y) \in R$, but $(a,y) \notin R$. Hence $e_X \circ R \not\subset R$. Since $e_X^{-1} \circ R = R$, $R_x^{-1} = \{a,b,c,d\}, R_y^{-1} = \{b,d\}, R_z^{-1} = \{b,c,d\}$ are e_X-upper sets. Since $R \circ e_Y^{-1} = R$, $R_a^c = \{y,z\}, R_b^c = \emptyset, R_c^c = \{y\}, R_d^c = \emptyset$ are e_Y-upper sets. Since $R \circ e_Y \not\subset R$, $R_a = \{x\}$ is not an e_Y-upper sets because $x \in R_a$ & $(x,y) \in e_Y$, but $y \notin R_a$.

Furthermore, $I(X) = \{\emptyset, X, \{d\}, \{b,d\}, \{c,d\}, \{b,c,d\}\}$ and $I(Y) = \{\emptyset, Y, \{y\}, \{y,z\}\}$.

(1) We obtain $\langle (R^c)^{-1}\rangle : I(X) \to I(Y)$ and $[(S^c)^{-1}] : I(Y) \to I(X)$ as

$$\langle (R^c)^{-1}\rangle(A) = \begin{cases} \{y,z\} & \text{if } A = X, \\ \{y\} & \text{if } A \in \{\{b,c,d\},\{c,d\}\} \\ \emptyset & \text{if } A \in \{\emptyset,\{d\},\{b,d\}\} \end{cases}$$

$$[(S^c)^{-1}](B) = \begin{cases} \{b,d\} & \text{if } B = \emptyset, \\ \{b,c,d\} & \text{if } B = \{y\} \\ X & \text{if } B \in \{Y,\{y,z\}\} \end{cases}$$

Then $(e_{I(X)}, \langle (R^c)^{-1}\rangle, [(S^c)^{-1}], e_{I(Y)})$ is a residuated connection.

(2) We obtain $[R^{-1}] : I(X) \to I(Y)$ and $\langle S^{-1}\rangle : I(Y) \to I(X)$ as

$$[R^{-1}](A) = \begin{cases} Y & \text{if } A = X, \\ \{y,z\} & \text{if } A = \{b,c,d\}, \\ \{y\} & \text{if } A \in \{\{b,d\},\{a,b,d\}\} \\ \emptyset & \text{otherwise} \end{cases}$$

$$\langle S^{-1}\rangle(B) = \begin{cases} X & \text{if } x \in B, \\ \{b,c,d\} & \text{if } B \in \{\{y,z\},\{z\}\}, \\ \{b,d\} & \text{if } B = \{y\}, \\ \emptyset & \text{if } B = \emptyset. \end{cases}$$

Then $(e_{I(X)}, [R^{-1}], \langle S^{-1}\rangle, e_{I(Y)})$ is a dual residuated connection.

(3) $(e_{I(X)}, [R^{-1}]^c, [S^{-1}]^c, e_{I(Y)})$ is not a Galois connection because $[S^{-1}](\{y\}^c) = \{a,c,d\} \notin I(X)$.

(4) $(e_{I(X)}, [\langle R^{-1}\rangle], [\langle S^{-1}\rangle], e_{I(Y)})$ is not a dual Galois connection because $[\langle S^{-1}\rangle](\{y\}) = \{a\} \notin I(X)$.

(5) $(e_{I(X)}, \langle R^{-1}\rangle^c, \langle S^{-1}\rangle^c, e_{I(Y)})$ is not a dual Galois connection because $\langle R^{-1}\rangle^c(\{b,d\}) = \{x,z\} \notin I(Y)$.

(6) $(e_{I(X)}, [[R^{-1}]], [[S^{-1}]], e_{I(Y)})$ is not a Galois connection because $[[R^{-1}]](\{c,d\}) = \{x,z\} \notin I(Y)$.

(7) $(e_{I(X)}, [(R^c)^{-1}], \langle (S^c)^{-1}\rangle, e_{I(Y)})$ is not a dual residuated connection because $[(R^c)^{-1}](\{d\}) = \{x\} \notin I(Y)$.

References

[1] R. Bělohlávek, *Lattices of fixed points of Galois connections*, Math. Logic Quart. **47** (2001), 111–116.

[2] G. Georgescu, A. Popescue, *Non-dual fuzzy connections*, Arch. Math. Logic **43** (2004), 1009–1039.

[3] J. Järvinen, M. Kondo, J. Kortelainen,, *Logics from Galois connections*, Internat. J. Approx. Reason. **49** (2008), 595–606.

[4] Y.C. Kim, Y.S. Kim, *Relations and upper sets on partially ordered sets*, Int. Math. Forum **6 (19)** (2011), 899–908.

[5] Ewa. Orlowska, I. Rewitzky, *Algebras for Galois-style connections and their discrete duality*, Fuzzy Sets and Systems **161** (2010), 1325–1342.

[6] Z. Pawlak, *Rough sets*, Int. J. Comput. Inf. Sci. **11** (1982), 341–356.

[7] G. Qi, W. Liu, *Rough operations on Boolean algebras*, Inform Sciences **173** (2005), 49–63.

[8] R. Wille, *Restructuring lattice theory; an approach based on hierarchies of concept, in: 1. Rival(Ed.), Ordered Sets, Reidel, Dordrecht, Boston*, 1982.

Department of Mathematics
Natural Science
Gangneung-Wonju National University
Gangneung, Gangwondo
210-702, Korea

E-mail: yck@gwnu.ac.kr

Department of Mathematics
Natural Science
Gangneung-Wonju National University
Gangneung, Gangwondo
210-702, Korea
E-mail: jmko@gwnu.ac.kr