THE CHROMATIC POLYNOMIAL FOR CYCLE GRAPHS

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Abstract. Let $P(G, \lambda)$ denote the number of proper vertex colorings of $G$ with $\lambda$ colors. The chromatic polynomial $P(C_n, \lambda)$ for the cycle graph $C_n$ is well-known as

$$P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)$$

for all positive integers $n \geq 1$. Also its inductive proof is widely well-known by the deletion-contraction recurrence. In this paper, we give this inductive proof again and three other proofs of this formula of the chromatic polynomial for the cycle graph $C_n$.

1. Introduction

The number of proper colorings of a graph with finite colors was introduced only for planar graphs by George David Birkhoff [1] in 1912, in an attempt to prove the four color theorem, where the formula for this number was later called by the chromatic polynomial. In 1932, Hassler Whitney [3] generalized Birkhoff’s formula from the planar graphs to general graphs. In 1968, Ronald Cedric Read [2] introduced the concept of chromatically equivalent graphs and asked which polynomials are the chromatic polynomials of some graph, that remains open.
Chromatic polynomial. For a graph $G$, a coloring means almost always a \textit{(proper) vertex coloring}, which is a labeling of vertices of $G$ with colors such that no two adjacent vertices have the same colors. Let $P(G, \lambda)$ denote the number of (proper) vertex colorings of $G$ with $\lambda$ colors and $\chi(G)$ the least number $\lambda$ satisfying $P(G, \lambda) > 0$, where $P(G, \lambda)$ and $\chi(G)$ are called a \textit{chromatic polynomial} and \textit{chromatic number} of $G$, respectively.

In fact, it is clear that the number of $\lambda$-colorings is a polynomial in $\lambda$ from a deletion-contraction recurrence.

\textbf{Proposition 1} (Deletion-contraction recurrence). For a given a graph $G$ and an edge $e$ in $G$, we have

$$P(G, \lambda) = P(G - e, \lambda) - P(G/e, \lambda),$$

(1)

where $G - e$ is a graph obtained by deletion the edge $e$ and $G/e$ is a graph obtained by contraction the edge $e$.

\textbf{Example}. The chromatic polynomials of graphs in Figure 1 are

$$P(G, \lambda) = \lambda(\lambda - 1)^2(\lambda - 2),$$

$$P(G - e, \lambda) = \lambda^2(\lambda - 1)(\lambda - 2),$$

and

$$P(G/e, \lambda) = \lambda(\lambda - 1)(\lambda - 2).$$

It is confirmed that (1) is true for the graph $G$ and the edge $e$ in Figure 1.

\textbf{Cycle graph}. A \textit{cycle graph} $C_n$ is a graph that consists of a single cycle of length $n$, which could be drawn by a $n$-polygonal graph in a plane. The chromatic polynomial for cycle graph $C_n$ is well-known as follows.
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Theorem 2. For a positive integer \( n \geq 1 \), the chromatic polynomial for cycle graph \( C_n \) is

\[
P(C_n, \lambda) = (\lambda - 1)^n + (-1)^n(\lambda - 1)
\]  

(2)

Example. For an integer \( n \leq 3 \), it is easily checked that the chromatic polynomials of \( C_n \) are from (2) as follows.

\[
P(C_1, \lambda) = (\lambda - 1) + (-1)(\lambda - 1) = 0,
\]

\[
P(C_2, \lambda) = (\lambda - 1)^2 + (-1)^2(\lambda - 1) = \lambda(\lambda - 1),
\]

\[
P(C_3, \lambda) = (\lambda - 1)^3 + (-1)^3(\lambda - 1) = \lambda(\lambda - 1)(\lambda - 2).
\]

As shown in Figure 2, the cycle graph \( C_1 \) is a graph with one vertex and one loop and \( C_1 \) cannot be colored, that means \( P(C_1, \lambda) = 0 \). The cycle graph \( C_2 \) is a graph with two vertices, where two edges between two vertices, and \( C_2 \) can have colorings by assigning two vertices with different colors, that means \( P(C_2, \lambda) = \lambda(\lambda - 1) \). The cycle graph \( C_3 \) is drawn by a triangle and \( C_3 \) can have colorings by assigning all three vertices with different colors, that means \( P(C_3, \lambda) = \lambda(\lambda - 1)(\lambda - 2) \).

2. Four proofs of Theorem 2

In this section, we show the formula (2) in four different ways.

2.1. Inductive proof. This inductive proof is widely well-known. A path graph \( P_n \) is a connected graph in which \( n-1 \) edges connect \( n \) vertices of vertex degree at most 2, which could be drawn on a single straight line. The chromatic polynomial for path graph \( P_n \) is easily obtained by coloring all vertices \( v_1, \ldots, v_n \) where \( v_i \) and \( v_{i+1} \) have different colors for \( i = 1, \ldots, n-1 \).
Lemma 3. For a positive integer \( n \geq 1 \), the chromatic polynomial for path graph \( P_n \) is

\[
P(P_n, \lambda) = \lambda(\lambda - 1)^{n-1}.
\]  \hspace{1cm} (3)

We use an induction on the number \( n \) of vertices by the deletion-contraction recurrence and the above lemma for path graph: It is already shown that (2) is true for \( n \leq 3 \) by the example in Section 1. Assume that (2) is true for a positive integer \( n \). Using (1) and (3), we have

\[
P(C_{n+1}, \lambda) = P(P_{n+1}, \lambda) - P(C_n, \lambda)
\]

by (1)

\[
= \lambda(\lambda - 1)^n - ((\lambda - 1)^n + (-1)^n(\lambda - 1))
\]

by (3)

\[
= (\lambda - 1)^{n+1} + (-1)^{n+1}(\lambda - 1).
\]

Thus, (2) is true for all positive integers \( n \geq 1 \).

2.2. Proof by inclusion-exclusion principle. The inclusion-exclusion principle is a technique of counting the size of the union of finite sets.

Proposition 4 (Inclusion-exclusion principle). Let \( A_1, A_2, \ldots, A_n \) be subsets of a finite set \( U \). Then number of elements excluding their union is as follows

\[
\left| \bigcap_{i=1}^{n} \overline{A_i} \right| = \sum_{I \subseteq [n]} (-1)^{|I|} \left| \bigcap_{i \in I} A_i \right|
\]

\[
= |U| - \sum_{i=1}^{n} |A_i| + \sum_{i<j} |A_i \cap A_j| - \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|
\]

where \( \overline{A} \) is the complement of \( A \) in \( U \).
Considering every condition to assign different colors to two adjacent vertices, for each edge $e$, we define a finite sets of arbitrary (including improper) colorings to assign same color to two adjacent vertices by the edge $e$.

Let $A_i$ be a set of colorings such that two vertices $v_i$ and $v_{i+1}$ are of same color, where $v_{n+1}$ is regarded as $v_1$. Applying the inclusion-exclusion principle, we can write the following

$$P(C_n, \lambda) = |U| - \sum_{i=1}^{n} |A_i| + \sum_{i<j} |A_i \cap A_j| + \cdots + (-1)^n |A_1 \cap \cdots \cap A_n|$$

$$= \lambda^n - \binom{n}{1}\lambda^{n-1} + \binom{n}{2}\lambda^{n-2} + \cdots + (-1)^n \lambda$$

$$= (\lambda - 1)^n - (-1)^n + (-1)^n \lambda$$

$$= (\lambda - 1)^n + (-1)^n(\lambda - 1).$$

Thus, (2) is true for all positive integers $n \geq 1$.

**2.3. Algebraic proof.** Let us consider a case of $n = 5$ and $\lambda = 4$, that is, to assign the vertices of $C_5$ in four colors: red, blue, yellow, and green. Also let us consider a complete graph $K_4$ with vertex names red, blue, yellow, and green, see Figure 4.

When red-blue-red-yellow-green is assigned in order from the vertex $v_1$ to the vertex $v_5$ in $C_5$, it is corresponding to a closed walk of length 5 in $K_4$ which begins and ends at red, that is, it is red-blue-red-yellow-green-red in $K_4$. By generalizing it, we have a correspondence between
\( \lambda \)-colorings of \( C_n \) and closed walks of length \( n \) in \( K_\lambda \). By this correspondence, it is enough to count the number of closed walks of length \( n \) in \( K_\lambda \), instead of the number of \( \lambda \)-colorings of \( C_n \).

For a graph \( G \) with vertex set \( \{v_1, \ldots, v_n\} \), the adjacency matrix of \( G \) is an \( n \times n \) square matrix \( A \) such that its element \( A_{ij} \) is one when there is an edge between two vertices \( v_i \) and \( v_j \), and zero when there is no edge between \( v_i \) and \( v_j \).

The following related to an adjacency matrix is well-known.

**Proposition 5.** Let \( A \) be the adjacency matrix of the graph \( G \) on \( n \) vertices \( v_1, \ldots, v_n \). Then the \((i, j)\)th entry of the matrix \( A^n \) is the number of the walk of length \( n \) beginning at \( v_i \) and ending at \( v_j \).

By Proposition 5, we can calculate the number of closed walk of length \( n \) in the complete graph \( K_\lambda \): Let \( A \) be an adjacency matrix of \( K_\lambda \). Then \( A \) is a \( \lambda \times \lambda \) matrix as follows

\[
A = (a_{ij}) = \begin{pmatrix}
0 & 1 & \cdots & 1 & 1 \\
1 & 0 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 0 & 1 \\
1 & 1 & \cdots & 1 & 0
\end{pmatrix},
\]

where \( a_{ij} = 0 \) if \( i = j \), and otherwise \( a_{ij} = 1 \). So the number of closed walks of length \( n \) in \( K_\lambda \) is enumerated by \( \text{tr}(A^n) \), which equals the sum of all eigenvalues of \( A^n \). Also let all eigenvalues of the matrix \( A \) be denoted
by $u_1, \ldots, u_\lambda$, then all eigenvalues of the matrix $A^n$ are $u_1^n, \ldots, u_\lambda^n$.

$$A = \begin{pmatrix}
0 & 1 & \cdots & 1 & 1 \\
1 & 0 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 0 & 1 \\
1 & 1 & \cdots & 1 & 0
\end{pmatrix} \sim \begin{pmatrix}
\lambda - 1 & 0 & \cdots & 0 & 0 \\
0 & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -1 & 0 \\
0 & 0 & \cdots & 0 & -1
\end{pmatrix}$$

Since the matrix $A$ have $\lambda$ eigenvalues $u_1 = \lambda - 1$ and $u_2 = \cdots = u_\lambda = -1$, we have

$$\text{tr}(A^n) = \sum_{i=1}^{\lambda} u_i^n = (\lambda - 1)^n + (-1)^n + \cdots + (-1)^n.$$  

Thus, (2) is true for all positive integers $n \geq 1$.

2.4. Bijective proof. Let $X_n$ denote the set of $\lambda$-colorings of $C_n$ and $[\lambda - 1]^n$ be the set of $n$-tuples of positive integers less than $\lambda$, where $[\lambda - 1]$ means $\{1, \ldots, \lambda - 1\}$. We consider a mapping $\varphi$ from $\lambda$-colorings of $C_n$ in $X_n$ to $n$-tuples in $[\lambda - 1]^n$.

A mapping $\varphi$ from $X_n$ to $[\lambda - 1]^n$. The mapping $\varphi : X_n \rightarrow [\lambda - 1]^n$ is defined as follows: Let $\omega$ be a $\lambda$-coloring of $C_n$ in $X_n$, we write $\omega = (\omega_1, \ldots, \omega_n)$ where $\omega_i$ is the color of $v_i$ in $C_n$ and it is obvious that $\omega_i \neq \omega_{i+1}$ for $1 \leq i \leq \lambda$, where $\omega_{n+1}$ is regarded as $\omega_1$. An entry $\omega_i$ is called a cyclic descent of $C$ if $\omega_i > \omega_{i+1}$ for $1 \leq i \leq \lambda$. Then we define $\varphi(\omega) = \sigma = (\sigma_1, \ldots, \sigma_n)$ with

$$\sigma_i = \begin{cases} 
\omega_i - 1, & \text{if } \omega_i \text{ is a cyclic descent} \\
\omega_i, & \text{otherwise.}
\end{cases}$$

Given a $\lambda$-coloring $\omega$, if $\omega_i = \lambda$ then $\omega_{i+1} < \lambda$, so $\omega_i = \lambda$ should be a cyclic descent. Thus we have $\sigma_i < \lambda$ for all $1 \leq i \leq n$ and $\varphi(\omega)$ belongs to $[\lambda - 1]^n$.

For example, in a case of $n = 9$ and $\lambda = 4$, $\omega = (1, 2, 1, 3, 2, 3, 1, 4, 2) \in X_9$ is given as an example of 4-colorings of $C_9$. Here $\omega_2 = 2$, $\omega_4 = 3$, $\omega_6 = 3$, $\omega_8 = 4$, and $\omega_9 = 2$ are cyclic descents of $\omega$. So we have

$$\varphi(\omega) = \sigma = (1, 1, 1, 2, 2, 1, 3, 1) \in [3]^9.$$
A mapping $\psi$ as the inverse of $\varphi$. Let $Z_n$ be the set of $n$-tuples $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ in $[\lambda - 1]^n$ with

$$\sigma_1 = \sigma_2 = \cdots = \sigma_n$$

and it is obvious that the size of $Z_n$ is $\lambda - 1$.

We would like to describe a mapping $\psi : ([\lambda - 1]^n \setminus Z_n) \to X_n$ in order to satisfy $\varphi \circ \psi$ is the identity on $[\lambda - 1]^n \setminus Z_n$ as follows: Given a $\sigma \in [\lambda - 1]^n \setminus Z_n$, we define $\overline{\sigma} = (\overline{\sigma}_1, \ldots, \overline{\sigma}_n)$ with

$$\overline{\sigma}_i = \begin{cases} 
\sigma_i + 1, & \text{if } \sigma_i \text{ is a cyclic descent} \\
\sigma_i, & \text{otherwise.}
\end{cases}$$

Since $\overline{\sigma}$ may have consecutive same entries, we define $\psi(\sigma) = \omega = (\omega_1, \ldots, \omega_n)$ from $\overline{\sigma}$ with $\omega_i = \overline{\sigma}_i + 1$ for any entry $\overline{\sigma}_i$ of $\overline{\sigma}$ with a finite positive even integer $\ell$ satisfying

$$\overline{\sigma}_i = \overline{\sigma}_{i+1} = \cdots = \overline{\sigma}_{i+\ell-1} \neq \overline{\sigma}_{i+\ell},$$

where $\overline{\sigma}_{n+k}$ is regarded as $\overline{\sigma}_k$ for $1 \leq k \leq n$, and $\omega_i = \overline{\sigma}_i$, otherwise. Thus $\omega$ has no consecutive same entries and $1 \leq \omega_i \leq \lambda$ for all $1 \leq i \leq n$, so $\psi(\sigma) = \omega$ belongs to $X_n$. Moreover, it is obvious that $\sigma_i \leq \omega_i \leq \sigma_i + 1$ for all $1 \leq i \leq n$ and if $\omega_i = \sigma_i + 1$ for some $1 \leq i \leq n$ then $\omega_i$ is a cyclic descent in $\omega$. Hence $\varphi(\omega) = \sigma$ and $\sigma \in [\lambda - 1]^n \setminus Z_n$ if and only if $\psi(\sigma) = \omega$.

In a previous example, $\sigma = (1, 1, 1, 2, 2, 1, 3, 1)$ is denoted as an example of 9-tuples in [3]9. Here $\sigma_6 = 2, \sigma_8 = 3$ are cyclic descents of $\sigma$ and we obtain $\overline{\sigma} = (1, 1, 1, 2, 2, 3, 1, 4, 1)$. And then there exist only three entries $\overline{\sigma}_2, \overline{\sigma}_4$, and $\overline{\sigma}_9$ in $\overline{\sigma}$ satisfying the following

$$k = 2 : \quad \overline{\sigma}_2 = \overline{\sigma}_3 \neq \overline{\sigma}_4 \quad (\ell = 2),$$

$$k = 4 : \quad \overline{\sigma}_4 = \overline{\sigma}_5 \neq \overline{\sigma}_6 \quad (\ell = 2), \text{ and}$$

$$k = 9 : \quad \overline{\sigma}_9 = \overline{\sigma}_1 = \overline{\sigma}_2 = \overline{\sigma}_3 \neq \overline{\sigma}_4 \quad (\ell = 4),$$

so we get $\omega_2 = \overline{\sigma}_2 + 1 = 2, \omega_4 = \overline{\sigma}_4 + 1 = 3, \omega_9 = \overline{\sigma}_9 + 1 = 2$, and

$$\psi(\sigma) = \omega = (1, 2, 1, 3, 2, 3, 1, 4, 2) \in X_9.$$
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When $n$ is even, for any $1 \leq i \leq \lambda - 1$, there exist only two $n$-tuples in $X_n$

$\omega = (i + 1, i, i + 1, i, \ldots, i + 1, i)$ and $\omega = (i, i + 1, i, i + 1, \ldots, i, i + 1)$

satisfying $\varphi(\omega) = (i, i, \ldots, i) \in \mathbb{Z}_n$. If $n$ is even, the size of $Y_n$ is equal to $2(\lambda - 1)$ and we obtain

$$P(C_n, \lambda) = |X_n| = |X_n \setminus Y_n| + |Y_n|$$

$$= [(\lambda - 1)^n - (\lambda - 1)] + 2(\lambda - 1).$$

(4)

When $n$ is odd, there is no $n$-tuples satisfying $\varphi(\omega) \in \mathbb{Z}_n$ and the set $Y_n$ is empty. If $n$ is odd, we obtain

$$P(C_n, \lambda) = |X_n| = |X_n \setminus Y_n| + |Y_n|$$

$$= [(\lambda - 1)^n - (\lambda - 1)] + 0.$$

(5)

Therefore, (2) yields from (4) and (5) for all positive integers $n \geq 1$.

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