A NOTE ON $\delta$-QUASI FUZZY SUBNEAR-RINGS AND IDEALS

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Abstract. In this paper, we discuss in detail some of the properties of the new kind of $(\in, \in \lor q^0)$-fuzzy ideals in Near-ring. The concept of $(\in, \in \lor q^0_0)$-fuzzy ideal of Near-ring is introduced and some of its related properties are investigated.

1. Introduction

The notion of a fuzzy set was introduced by L.A Zadeh [17], and since then this concept have been applied to various algebraic structure. Rosenfeld [16] applied this concept and introduced fuzzy subgroup. The notions of fuzzy subnear-ring and fuzzy ideals of near-rings were introduced by Abou Zaid [1]. The concept of quasi-coincidence of a fuzzy point with a fuzzy subset was introduced by P. Ming and Y. Ming [15]. Using the idea of quasi-coincidence of a fuzzy point with a fuzzy set S. Bhakat [2] defined different types of fuzzy subgroup called $(\alpha, \beta)$-fuzzy subgroups. In particular, he introduced $(\in, \in \lor q)$-fuzzy subgroup as the only non trivial generalization of a fuzzy subgroup defined by Rosenfeld. The notions of $(\in, \in \lor q)$-fuzzy subrings and $(\in, \in \lor q)$-fuzzy ideals of a ring were introduced by S.K. Bhakat and P. Das [4]. A. Narayanan and
T. Manikantan [14] defined \((\in, \in \lor)\)-fuzzy subnear-rings and \((\in, \in \lor)\)-fuzzy ideals of a near-ring. Y.B. Jun and M.A. Ozturk [10] introduced the concepts of \((\in, \in \lor)\)-fuzzy subrings and \((\in, \in \lor)\)-fuzzy ideals in a ring. In this paper, the notions of \((\in, \in \lor)\)-fuzzy subnear-rings and \((\in, \in \lor)\)-fuzzy ideals of a near-ring are introduced and related properties are investigated.

2. Preliminaries

We first recall the definition of near-ring. A non-empty subset \(N\) with two binary operation \(\cdot^+\) (addition) and \(\cdot\) (multiplication) is called a near-ring if it satisfies the following axioms:

i) \((N,+)\) is a group;
ii) \((N,\cdot)\) is a semigroup;
iii) \((x + y) \cdot z = x \cdot z + y \cdot z\) for all \(x, y, z \in N\).

It is a right near-ring because it satisfies the right distributive law. If it satisfies left distributive law it is called left near-ring. Unless otherwise stated, we shall consider only right near-rings throughout this paper.

**Definition 2.1.** Let \(N\) be a near-ring. A normal subgroup \(I\) of \((N,+)\) is called

i) a right ideal if \(IN \subseteq I\)
ii) a left ideal if \(n(m + i) - nm \in I\) for all \(n, m \in N\) and \(i \in I\)
iii) an ideal if it is both right and left ideal.

**Definition 2.2.** [15] A fuzzy set \(\mu\) in a set \(X\) of the form

\[
\mu(y) = \begin{cases} 
  t \in (0, 1] & \text{if } y = x; \\
  0 & \text{if } y \neq x.
\end{cases}
\]

is said to be a fuzzy point with support \(x\) and value \(t\) and is denoted by \(x_t\).

**Definition 2.3.** [15] For a fuzzy point \(x_t\) and a fuzzy set \(\mu\) in a set \(X\), we say that

i) \(x_t \in \mu\) (resp. \(x_t \lor q\mu\)) if \(\mu(x) \geq t\) (resp. \(\mu(x) + t > 1\)),
ii) \(x_t \in \lor q\mu\) if \(x_t \in \mu\) or \(x_t \lor q\mu\).
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Definition 2.4. [2],[3] A fuzzy set \( \mu \) of a group \( G \) is said to be an \((\in, \in \lor q)\)-fuzzy subgroup of \( G \) if for all \( x, y \in G \) and \( t, r \in (0, 1] \),

i) \( x_t, y_r \in \mu \Rightarrow (xy)_{\min\{t, r\}} \in \lor q \mu \) and

ii) \( x_t \in \mu \Rightarrow (-x)_t \in \lor q \mu \).

Definition 2.5. [14] A fuzzy set \( \mu \) is said to be an \((\in, \in \lor q)\)-fuzzy subnear-ring of \( N \) if \( \forall x, y \in N \) and \( t, r \in (0, 1] \)

i) \( x_t, y_r \in \mu \Rightarrow (x + y)_{\min\{t, r\}} \in \lor q \mu \).

ii) \( x_t \in \mu \Rightarrow (-x)_t \in \lor q \mu \).

iii) \( x_t, y_r \in \mu \Rightarrow (xy)_{\min\{t, r\}} \in \lor q \mu \).

Definition 2.6. [14] A fuzzy set \( \mu \) of a near-ring \( N \) is said to be an \((\in, \in \lor q)\)-fuzzy ideal of \( N \) if

i) \( \mu \) is an \((\in, \in \lor q)\)-fuzzy subnear-ring of \( N \),

ii) \( x_t \in \mu \) and \( y \in N \Rightarrow (y + x - y)_t \in \lor q \mu \),

iii) \( x_t \in \mu \) and \( y \in N \Rightarrow (xy)_t \in \lor q \mu \),

iv) \( a_t \in \mu \) and \( x, y \in N \Rightarrow (y(x + a) - yx)_t \in \lor q \mu \) \( \forall x, y, a \in N \).

Definition 2.7. [9] Let \( \mu \) be a fuzzy set of \( G \). Then \( \forall t \in (0, 1] \), the set \( \mu_t = \{ x \in G; \mu(x) \geq t \} \) is called level subset of \( \mu \).

Definition 2.8. [5] The subset \( \overline{\mu}_t = \{ x \in X; \mu(x) \geq t \) or \( \mu(x) + t > 1 \} \) is called \((\in \lor q)\)-level subset of \( X \) determined by \( \mu \) and \( t \).

Jun et al [11] generalized a quasi-coincident fuzzy point. Let \( \delta \in (0, 1] \). For a fuzzy point \( x_t \) and a fuzzy set \( \mu \) in a set \( X \), we say that

\( x_t \) is a \( \delta \)-quasi-coincident with \( \mu \), written as \( x_{t \in \lor q \delta \mu} \), if \( \mu(x) + t > \delta \).

\( x_t \in \lor q \delta \mu \), if \( x_t \in \mu \) or \( x_{t \in \lor q \delta \mu} \).

If \( \delta = 1 \), then the \( \delta \)-quasi-coincident with \( \mu \) is the quasi-coincident with \( \mu \), i.e. \( x_{t \in \lor \mu} = x_{t \in q \mu} \).

Definition 2.9.[11] Let \( \mu \) be a fuzzy set of \( N \). Then the subset \( \overline{\mu_t^\delta} = \{ x \in N; \mu(x) \geq t \) or \( \mu(x) + t > \delta \} \) is called \((\in \lor q \delta)\)-level subset of \( N \).

Definition 2.10. [11] For a subset \( A \) of \( N \), a fuzzy set \( \chi_A^\delta \) in \( N \) defined by

\( \chi_A^\delta : N \rightarrow [0, \delta] \) as

\[
\chi_A^\delta(x) = \begin{cases} 
\delta & \text{if } x \in A; \\
0 & \text{otherwise.}
\end{cases}
\]
is called a $\delta$-characteristic fuzzy set of $A$ in $N$.

3. Main Results

In this section, we defined the new kind of $\delta$-quasi-coincident with fuzzy set $\mu$ in a near-ring. The properties of $(\in, \in \lor q^\delta)$-fuzzy ideals in near-ring are discussed and some of these characterizations are explored. Here $\delta$ and $N$ denote an element of $(0,1]$ and a near-ring respectively unless otherwise specified.

**Definition 3.1.** A fuzzy set $\mu$ in $N$ is called an $(\in, \in \lor q^\delta)$-fuzzy subnear-ring of $N$ if for all $x, y \in N$ and $t, r \in (0, \delta]$,

i) $x_t \in \mu, y_r \in \mu \Rightarrow (x - y)_{\min\{t, r\}} \in \lor q^\delta \mu$ and

ii) $x_t \in \mu, y_r \in \mu \Rightarrow (xy)_{\min\{t, r\}} \in \lor q^\delta \mu$.

**Example 3.2.** Let $N = \{0, a, b, c\}$ with $(N, +)$ as Klien 4-group and $(N, \cdot)$ as defined in table by

<table>
<thead>
<tr>
<th>$\cdot$</th>
<th>0</th>
<th>a</th>
<th>b</th>
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</table>

Then, $(N, +, \cdot)$ is a right near-ring. Define a fuzzy set $\mu$ in $N$ by $\mu(0) = 0.8, \mu(a) = 0.7, \mu(b) = 0.48, \mu(c) = 0.45$. Then, $\mu$ is an $(\in, \in \lor q^\delta)$-fuzzy subnear-ring of $N$ with $\delta \in (0, 0.9]$. If $\delta = 0.95 \in (0.9, 1]$, then $a_{0.47} \in \mu, b_{0.46} \in \mu$ but $(a - b)_{\min\{0.47, 0.46\}} = c_{0.46} \notin \lor q^\delta \mu$.

Thus, $\mu$ is not an $(\in, \in \lor q^\delta)$-fuzzy subnear-ring of $N$ when $\delta \in (0.9, 1]$. Note that every $(\in, \in \lor q^\delta)$-fuzzy subnear-ring with $\delta = 1$ is an $(\in, \in q)$-fuzzy subnear-ring. If $\delta_1 > \delta_2$ in $(0,1]$, then every $(\in, \in \lor q^\delta)$-fuzzy subnear-ring of $N$ with $\delta = \delta_1$ is also an $(\in, \in \lor q^\delta)$-fuzzy subnear-ring of $N$ with $\delta = \delta_2$. But the converse is not true as seen in example 3.2.

So, every $(\in, \in q)$-fuzzy subnear-ring is an $(\in, \in \lor q^\delta)$-fuzzy subnear-ring, but the converse is not true.
Analogous to result in [7],[14], the necessary and sufficient condition for determining the fuzzy set to be \((\in,\in \in \in Vq_{0}^{\delta})\)-fuzzy subnear-ring is given here.

**Theorem 3.3.** For a fuzzy set \(\mu\) in \(N\), \(\mu\) is an \((\in,\in \in \in Vq_{0}^{\delta})\)-fuzzy subnear-ring of \(N\) if and only if \(\mu(x - y), \mu(xy) \geq \min\{\mu(x), \mu(y), \frac{\delta}{2}\}\)

**Proof.** Let \(\mu\) is an \((\in,\in \in \in Vq_{0}^{\delta})\)-fuzzy subnear-ring of \(N\). Suppose \(x, y \in N\) such that \(\mu(x - y) > \min\{\mu(x), \mu(y), \frac{\delta}{2}\}\)
choose \(t \in (0, \delta]\) such that \(\mu(x - y) < t \leq \min\{\mu(x), \mu(y), \frac{\delta}{2}\}\)
\[\Rightarrow x_{t} \in \mu, y_{t} \in \mu \text{ but } (x - y)_{t} \in \in Vq_{0}^{\delta}\mu \text{ which is a contradiction.}
\]
Therefore, \(\mu(x - y) \geq \min\{\mu(x), \mu(y), \frac{\delta}{2}\}\). for all \(x, y \in N\).
Similarly, \(\mu(xy) \geq \min\{\mu(x), \mu(y), \frac{\delta}{2}\}\). for all \(x, y \in N\).
Conversely, let us assume that \(\mu(x - y), \mu(xy) \geq \min\{\mu(x), \mu(y), \frac{\delta}{2}\}\). for all \(x, y \in N\).
Let \(x_{t} \in \mu \text{ and } y_{r} \in \mu \text{ for } x, y \in N \text{ and } t, r \in (0, \delta]\).
Then \(\mu(x) \geq t \text{ and } \mu(y) \geq r\).
Now, \(\mu(x - y) \geq \min\{\mu(x), \mu(y), \frac{\delta}{2}\} \geq \min\{t, r, \frac{\delta}{2}\}\)
\[
\Rightarrow \mu(x - y) \geq \left\{ \begin{array}{ll}
\min\{t, r\} & \text{if } t, r \leq \frac{\delta}{2}; \\
\frac{\delta}{2} & \text{if } \frac{\delta}{2} < t, r.
\end{array} \right.
\]
\[\Rightarrow (x - y)_{\min\{t, r\}} \in \in Vq_{0}^{\delta}\mu.
\]
and \(\mu(xy) \geq \min\{\mu(x), \mu(y), \frac{\delta}{2}\} \geq \min\{t, r, \frac{\delta}{2}\}\)
\[
\Rightarrow \mu(xy) \geq \left\{ \begin{array}{ll}
\min\{t, r\} & \text{if } t, r \leq \frac{\delta}{2}; \\
\frac{\delta}{2} & \text{if } \frac{\delta}{2} < t, r.
\end{array} \right.
\]
\[\Rightarrow (xy)_{\min\{t, r\}} \in \in Vq_{0}^{\delta}\mu.
\]
Therefore, \(\mu\) is an \((\in,\in \in \in Vq_{0}^{\delta})\)-fuzzy subnear-ring of \(N\). \(\square\)

**Corollary 3.4.** [7],[14] A fuzzy set \(\mu\) of \(N\) is an \((\in,\in \in \in Vq_{0}^{\delta})\)-fuzzy subnear-ring of \(N\) if and only if \(\mu(x - y), \mu(xy) \geq \min\{\mu(x), \mu(y), 0.5\}\forall x, y \in N\).

**Definition 3.5.** A fuzzy set \(\mu\) in \(N\) is called an \((\in,\in \in \in Vq_{0}^{\delta})\)-fuzzy ideal in \(N\) if,
\begin{enumerate}
  \item[(i)] it is an \((\in,\in \in \in Vq_{0}^{\delta})\)-fuzzy subnear-ring of \(N\),
  \item[(ii)] \(x_{t} \in \mu, y \in N \Rightarrow (y + x - y)_{t} \in \in Vq_{0}^{\delta}\mu,\)
  \item[(iii)] \(x_{t} \in \mu, y \in N \Rightarrow (xy)_{t} \in \in Vq_{0}^{\delta}\mu\) and
  \item[(iv)] \(a_{t} \in \mu, x, y \in \mu \Rightarrow (y(x + a) - yx)_{t} \in \in Vq_{0}^{\delta}\mu.\)
\end{enumerate}
A fuzzy set with condition \( i, ii, iii \) is called an \( (\in, \in \vee q \delta) \)-fuzzy right ideal of \( N \) and if it satisfies \( i, ii, iv \), then it is called an \( (\in, \in \vee q \delta) \)-fuzzy left ideal of \( N \).

Example 3.2 is also an example of \( (\in, \in \vee q \delta) \)-fuzzy ideal for \( \delta \in (0, 0.9] \) but not \( (\in, \in \vee q \delta) \)-fuzzy ideal when \( \delta \in (0.9, 1] \). Note that every \( (\in, \in \vee q \delta) \)-fuzzy ideal with \( \delta = 1 \) is an \( (\in, \in \vee q) \)-fuzzy ideal.

If \( \delta_1 > \delta_2 \) in \( (0, 1] \), then every \( (\in, \in \vee q \delta) \)-fuzzy ideal of \( N \) with \( \delta = \delta_1 \) is also an \( (\in, \in \vee q \delta) \)-fuzzy ideal of \( N \) with \( \delta = \delta_2 \). But the converse is not true as seen in example 3.2.

So, every \( (\in, \in \vee q) \)-fuzzy ideal is an \( (\in, \in \vee q \delta) \)-fuzzy ideal, but the converse is not true.

**Example 3.6** Let \( N = \{(a, b)|a, b \in Z\} \), where \( Z \) is the integers. Then \( (N, +, \cdot) \) is a near-ring under the additive operation and multiplication operation defined as follows:

\[
(a, b) + (c, d) = (a + c, b + d) \quad \text{and} \quad (a, b) \cdot (c, d) = (a, b) \quad \text{for all} \quad (a, b), (c, d) \in N.
\]

Define a fuzzy set \( \mu \) in \( N \) as

\[
\mu(x) = \begin{cases} 
0.88 & \text{if } x = (1, 8), \\
0.44 & \text{if } x \in A, \\
0.33 & \text{if } x \in B, \\
0.22 & \text{otherwise}.
\end{cases}
\]

where \( A = \{(a, 4b)|a, b \in Z\} \setminus \{(1, 8)\} \) and \( B = \{(a, 2b)|a, b \in Z\} \setminus \{(a, 4b)|a, b \in Z\} \). Then, \( \mu \) is an \( (\in, \in \vee q \delta) \)-fuzzy subnear-ring for all \( \delta \in (0, 1] \). It is not an \( (\in, \in \vee q \delta) \)-fuzzy ideal since \( (1, 8)_{0.45} \in \mu, (1, 2), (1, 3) \in N \) but

\[
((1, 2) \cdot ((1, 3) + (1, 8)) - (1, 2) \cdot (1, 3))_{0.45} = ((1, 2) \cdot (2, 11) - (1, 2))_{0.45} = ((1, 2) - (1, 2))_{0.45} = (0, 0)_{0.45} \in \vee q \mu^\delta_0 \text{ when } \delta = 0.9.
\]

**Theorem 3.7.** Let \( \mu \) be a fuzzy set of a near-ring \( N \). Then \( \mu \) is an \( (\in, \in \vee q \delta) \)-fuzzy ideal of \( N \) if and only if

i) \( \mu(x - y) \geq \min\{\mu(x), \mu(y), \frac{\delta}{2}\} \)

ii) \( \mu(xy) \geq \min\{\mu(x), \mu(y), \frac{\delta}{2}\} \)

iii) \( \mu(y + x - y) \geq \min\{\mu(x), \frac{\delta}{2}\} \)
iv) \( \mu(xy) \geq \min\{\mu(x), \frac{\delta}{2}\} \)

v) \( \mu(y(x+a) - yx) \geq \min\{\mu(a), \frac{\delta}{2}\} \) for all \( x, y, a \in N \)

**Proof.** The proof is similar to the proof of theorem 3.3. \( \Box \)

Note: If \( \mu \) is an \((\varepsilon, \in \vee q_0^\delta)\)-fuzzy ideal of \( N \) then,
\[
\mu(x) = \mu(-y + y + x - y + y) \geq \min\{\mu(y + x - y), \frac{\delta}{2}\} \quad \text{[by condition iii]}
\]
\[\Rightarrow \mu(x) \geq \min\{\mu(y + x - y), \frac{\delta}{2}\} \] for all \( x, y \in N \).

As discussed in [7], the properties of characteristic function of subset \( A \) of \( N \) is now replaced by the \( \delta \)-characteristic function of \( A \).

**Theorem 3.8.** A non-empty subset \( A \) of \( N \) is a subnear-ring(ideal) of \( N \) if and only if \( \chi_A^\delta \) is an \((\varepsilon, \in \vee q_0^\delta)\)-fuzzy subnear-ring(ideal) of \( N \).

**Proof.** Let \( A \) be an ideal of \( N \), and let \( x, y \in N \), if \( x, y \in A \) then \( x - y, xy \in A \). Therefore, \( \chi_A^\delta(x - y) = \delta > \min\{\chi_A^\delta(x), \chi_A^\delta(y), \frac{\delta}{2}\} \)

and \( \chi_A^\delta(xy) = \delta > \min\{\chi_A^\delta(x), \chi_A^\delta(y), \frac{\delta}{2}\} \). If at least one of \( x, y \not\in A \), then \( \chi_A^\delta(x - y) \geq 0 = \min\{\chi_A^\delta(x), \chi_A^\delta(y), \frac{\delta}{2}\} \)

and \( \chi_A^\delta(xy) \geq 0 = \min\{\chi_A^\delta(x), \chi_A^\delta(y), \frac{\delta}{2}\} \)

Let \( x \in A \), then \( y + x - y \in A \) and so \( \chi_A(y + x - y) = \delta > \min\{\chi_A^\delta(x), \frac{\delta}{2}\} \)

and if \( x \not\in A \), then \( \chi_A^\delta(y + x - y) \geq 0 = \min\{\chi_A^\delta(x), \frac{\delta}{2}\} \)

Let \( u, v \in N \), if \( x \in A \) then \( xu, uv + x - \nu \in A \). Therefore,
\[\chi_A^\delta(xu) = \delta > \min\{\chi_A^\delta(x), \frac{\delta}{2}\} \quad \text{and} \quad \chi_A^\delta(uv + x - \nu) = \delta > \min\{\chi_A^\delta(x), \frac{\delta}{2}\} \]

If \( x \not\in A \), then \( \chi_A^\delta(xu) \geq 0 = \min\{\chi_A^\delta(x), \frac{\delta}{2}\} \)

and \( \chi_A^\delta(uv + x - \nu) \geq 0 = \min\{\chi_A^\delta(x), \frac{\delta}{2}\} \)

Hence, \( \chi_A^\delta \) is an \((\varepsilon, \in \vee q_0^\delta)\)-fuzzy subnear-ring(ideal) of \( N \).

Conversely, Let \( \chi_A^\delta \) is an \((\varepsilon, \in \vee q_0^\delta)\)-fuzzy subnear-ring(ideal) of \( N \). Let \( x, y \in A \), Now \( \chi_A^\delta(x - y) \geq \min\{\chi_A^\delta(x), \chi_A^\delta(y), \frac{\delta}{2}\} = \min\{\delta, \frac{\delta}{2}\} = \frac{\delta}{2} \neq 0 \)

so, \( x - y \in A \). Similarly, we can show that \( u + x - u, xu, u(v + x) - uv \in A \) for all \( x, y \in A \) and \( u, v \in N \). Therefore, \( A \) is an ideal of \( N \). \( \Box \)

The level sets have important aspects in respect to the connection of the fuzzy sets and crisp sets. As discussed in [5], the \((\varepsilon)\)-level set \( \tilde{\mu}_t \) is a generalized level set of \( \mu_t \). It was found that \( \mu_t \) is subnear-ring(ideal) if \( t \in (0, 0.5) \) and \( \tilde{\mu}_t \) is subnear-ring(ideal) if \( t \in (0, 1) \). Here we attempt
to develop this kind of connections in regard to the level set $\tilde{\mu}^\delta_t$ as well.

**THEOREM 3.9.** A fuzzy set $\mu$ in $N$ is an $(\varepsilon, \in \mathcal{V}q_0^\delta)$-fuzzy subnear-ring(ideal) of $N$ if and only if the $(\varepsilon, \in \mathcal{V}q_0^\delta)$-level subset $\tilde{\mu}^\delta_t$ is a subnear-ring(ideal) of $N$ for all $t \in (0, \delta]$ and $\delta \in (0, 1]$. 

*Proof.* Let $\mu$ be an $(\varepsilon, \in \mathcal{V}q_0^\delta)$-fuzzy subnear-ring(ideal) of $N$ and let $x, y \in \tilde{\mu}^\delta_t$ for $t \in (0, \delta]$. Then, $x_t \in \mathcal{V}q_0^\delta \mu$ or $y_t \in \mathcal{V}q_0^\delta \mu$ that is, $\mu(x) \geq t$ or $\mu(x) + t > \delta$ and $\mu(y) \geq t$ or $\mu(y) + t > \delta$.

Since $\mu$ is an $(\varepsilon, \in \mathcal{V}q_0^\delta)$-fuzzy subnear-ring(ideal) of $N$, we have $\mu(x - y) \geq \min\{\mu(x), \mu(y), \frac{\delta}{2}\}$.

Case 1. $\mu(x) \geq t$ and $\mu(y) \geq t$.

a) if $t > \frac{\delta}{2}$, then $\mu(x - y) \geq \min\{\mu(x), \mu(y), \frac{\delta}{2}\} \geq \min\{t, \frac{\delta}{2}\} = \frac{\delta}{2}$

thus, $\mu(x - y) + t > \delta \Rightarrow (x - y)_t \in \mathcal{V}q_0^\delta \mu$.

b) if $t \leq \frac{\delta}{2}$, then $\mu(x - y) \geq \min\{\mu(x), \mu(y), \frac{\delta}{2}\} \geq \min\{t, \frac{\delta}{2}\} = t$

$\Rightarrow (x - y)_t \in \mu$.

Case 2. Let $\mu(x) \geq t$ and $\mu(y) + t > \delta$ or $\mu(x) + t > \delta$ and $\mu(y) \geq t$.

a) if $t > \frac{\delta}{2}$, then $\mu(x - y) \geq \min\{\mu(x), \mu(y), \frac{\delta}{2}\} > \min\{t, \delta - t, \frac{\delta}{2}\} = \delta - t$,

$\Rightarrow \mu(x - y) + t > \delta \Rightarrow (x - y)_t \in \mathcal{V}q_0^\delta \mu$.

b) if $t \leq \frac{\delta}{2}$, then $\mu(x - y) \geq \min\{\mu(x), \mu(y), \frac{\delta}{2}\} > \min\{t, \delta - t, \frac{\delta}{2}\} = t$

$\Rightarrow (x - y)_t \in \mu$. Thus, in all cases, we have $(x - y)_t \in \mathcal{V}q_0^\delta \mu \Rightarrow x - y \in \tilde{\mu}^\delta_t$.

Similarly, we can show that $a + x - a, xa, a(b + x) - ab \in \tilde{\mu}^\delta_t$ for all $a, b, x \in N$.

Thus, $\tilde{\mu}^\delta_t$ is a subnear-ring(ideal) of $N$ for all $t \in (0, \delta]$ and $\delta \in (0, 1]$.

Conversely, let $\tilde{\mu}^\delta_t$ is an ideal of $N$.

Suppose, $\mu(x - y) < t \leq \min\{\mu(x), \mu(y), \frac{\delta}{2}\}$, then $\mu(x) \geq t$ and $\mu(y) \geq t$

$\Rightarrow x_t \in \mu, y_t \in \mu \Rightarrow x, y \in \tilde{\mu}^\delta_t \Rightarrow x - y \in \tilde{\mu}^\delta_t$ [since $\tilde{\mu}^\delta_t$ is an ideal],

which is a contradiction to $\mu(x - y) < t \leq \frac{\delta}{2}$.

Hence, $\mu(x - y) \geq \min\{\mu(x), \mu(y), \frac{\delta}{2}\}$.

Similarly, we can show that $\mu(xy) \geq \min\{\mu(x), \mu(y), \frac{\delta}{2}\}$, $\mu(a + x - a) \geq \min\{\mu(x), \frac{\delta}{2}\}$ and $\mu(xy) \geq \min\{\mu(x), \frac{\delta}{2}\}$.
\( \mu(a(b + x) - ab) \geq \min\{\mu(x), \frac{\delta}{2}\} \) for all \( a, b, x, y \in N \).

Hence, \( \mu \) is an \( (\epsilon, \in \vee q_0^\delta) \)-fuzzy subnear-ring(ideal) of \( N \). \( \square \)

**Corollary 3.10.** [11] A fuzzy set \( \mu \) in a group \( N \) is an \( (\epsilon, \in \vee q)^\delta \)-fuzzy subgroup of \( N \) if and only if the \( (\epsilon, \in \vee q)^\delta \)-level subset \( \bar{\mu}_t^\delta \) is a subgroup of \( N \) for all \( t \in (0, \delta] \).

**Corollary 3.11.** [5] A fuzzy set \( \mu \) in a group \( N \) is an \( (\epsilon, \in \vee q)^\delta \)-fuzzy subgroup of \( N \) if and only if the \( (\epsilon, \in \vee q)^\delta \)-level subset \( \bar{\mu}_t^\delta \) is a subgroup of \( N \) for all \( t \in (0, 1] \).

**Corollary 3.12.** [8],[12]. A fuzzy set \( \mu \) of \( N \) is an \( (\epsilon, \in \vee q)^\delta \)-fuzzy subnear-ring(ideal) of \( N \) if and only if the \( (\epsilon, \in \vee q)^\delta \)-level subset \( \bar{\mu}_t^\delta \) is a subnear-ring(ideal) of \( N \) for all \( t \in (0, 1] \).

**Theorem 3.13.** A fuzzy set \( \mu \) in \( N \) is an \( (\epsilon, \in \vee q_0^\delta) \)-fuzzy subnear-ring(ideal) of \( N \) if and only if the \( (\epsilon, \in \vee q_0^\delta) \)-level subset \( \bar{\mu}_t^\delta \) is a subnear-ring(ideal) of \( N \) for all \( t \in (0, \delta/2] \) and \( \delta \in (0, 1] \).

**Proof.** Assume that \( \mu \) is an \( (\epsilon, \in \vee q_0^\delta) \)-fuzzy subnear-ring(ideal) of \( N \). Let \( x, y \in \bar{\mu}_t^\delta \). Then, \( x_t \in \vee q_\mu \) or \( y_t \in \vee q_\mu \)

that is, \( \mu(x) \geq t \) or \( \mu(x) + t > 1 \) and \( \mu(y) \geq t \) or \( \mu(y) + t > 1 \).

\( \Rightarrow \mu(x) \geq t \) and \( \mu(y) \geq t \) [since if \( \mu(x) < t \leq \frac{\delta}{2} \leq 0.5 \Rightarrow \mu(x) + t < 1 \) and \( \mu(y) < t \leq \frac{\delta}{2} \leq 0.5 \Rightarrow \mu(y) + t < 1 \Rightarrow x, y \notin \bar{\mu}_t^\delta \), which is a contradiction].

Since \( \mu \) is an \( (\epsilon, \in \vee q_0^\delta) \)-fuzzy subnear-ring(ideal) of \( N \), we have

\( \mu(x - y) \geq \min\{\mu(x), \mu(y), \frac{\delta}{2}\} \geq \min\{t, \frac{\delta}{2}\} = t. \Rightarrow x - y \in \bar{\mu}_t^\delta \), and

\( \mu(xy) \geq \min\{\mu(x), \mu(y), \frac{\delta}{2}\} \geq \min\{t, \frac{\delta}{2}\} = t. \Rightarrow xy \in \bar{\mu}_t^\delta \).

Therefore, \( \bar{\mu}_t^\delta \) is a subnear-ring of \( N \) for all \( t \in (0, \delta/2] \). Let \( a, b \in N \). Then,

\( \mu(a + x - a) \geq \min\{\mu(x), \frac{\delta}{2}\} \geq \min\{t, \frac{\delta}{2}\} = t \),

\( \mu(xa) \geq \min\{\mu(x), \frac{\delta}{2}\} \geq \min\{t, \frac{\delta}{2}\} = t \) and

\( \mu(ab + x) - ab \geq \min\{\mu(x), \frac{\delta}{2}\} \geq \min\{t, \frac{\delta}{2}\} = t \).

Therefore, \( a + x - a, xa, a(b + x) - ab \in \bar{\mu}_t^\delta \) for all \( a, b \in N \) and for all \( x \in \bar{\mu}_t^\delta \).

Hence, \( \bar{\mu}_t^\delta \) is an ideal of \( N \) for all \( t \in (0, \delta/2] \).

Proof of the converse part is similar to theorem 3.9. \( \square \)
Theorem 3.14. A fuzzy set \( \mu \) in \( N \) is an \((\varepsilon, \in \forall q_0^\delta)\)-fuzzy subnear-ring\((\text{ideal})\) of \( N \) if and only if the set \( \mu_t = \{ x \in N \mid \mu(x) \geq t \} \) is a subnear-ring\((\text{ideal})\) of \( N \) for all \( t \in (0, \frac{\delta}{2}] \) and \( \delta \in (0, 1] \).

Proof. It is similar to the proof of theorem 3.13. \( \square \)

Remark 3.15. The above theorem 3.14. may not be true if \( t \in (\frac{\delta}{2}, 1] \).

In the example 3.2., \( \mu \) is an \((\varepsilon, \in \forall q_6^\delta)\)-fuzzy subnear-ring of \( N \) for \( \delta \in (0, 0.9] \).

Take \( \delta = 0.9 \) and let \( t = 0.46 \in (\frac{\delta}{2}, 1] \). Then \( \mu_t = \{0, a, b\} \).

Now \( a, b \in \mu_t \) but \( a - b = c \notin \mu_t \). Therefore \( \mu_t \) is not a subnear-ring of \( N \).

Corollary 3.16. [11] A fuzzy set \( \mu \) of a group \( N \) is an \((\varepsilon, \in \forall q_0^\delta)\)-fuzzy subgroup of \( N \) if and only if the set \( \mu_t = \{ x \in N \mid \mu(x) \geq t \} \) is a subgroup of \( N \) for all \( t \in (0, \frac{\delta}{2}] \).

Remark 3.17. [3], [14] A fuzzy set \( \mu \) of a group \( N \) is an \((\varepsilon, \in \forall q)\)-fuzzy subgroup of \( N \) if and only if the level subset \( \mu_t = \{ x \in N \mid \mu(x) \geq t \} \) is a subgroup of \( N \forall t \in (0, 0.5] \). But the level set \( \mu_t, t \in (0.5, 1] \) may not be a subgroup of \( N \).

Theorem 3.18. Let \( A \) be a non-empty subset of \( N \) and \( \mu_A \) be a fuzzy set in \( N \) defined by

\[
\mu_A(x) = \begin{cases} 
\frac{\delta}{2}, & \text{if } x \in A; \\
\frac{t}{2}, & \text{otherwise.}
\end{cases}
\]

for all \( x \in N \) and \( t < \frac{\delta}{2} \). Then \( \mu_A \) is a \((\varepsilon, \in \forall q_0^\delta)\)-fuzzy ideal of \( N \) if and only if \( A \) is an ideal of \( N \).

Proof. Let \( \mu_A \) be an \((\varepsilon, \in \forall q_0^\delta)\)-fuzzy ideal of \( N \) and let \( x, y \in A \).

Then
\[
\mu_A(x - y) \geq \min\{\mu_A(x), \mu_A(y), \frac{\delta}{2}\} = \frac{\delta}{2} \Rightarrow x - y \in A
\]
\[
\mu_A(xy) \geq \min\{\mu_A(x), \mu_A(y), \frac{\delta}{2}\} = \frac{\delta}{2} \Rightarrow xy \in A.
\]

Let \( x \in A \). Now \( \mu_A(y + x - y) \geq \min\{\mu_A(x), \frac{\delta}{2}\} = \frac{\delta}{2} \) and
\[
\mu_A(xy) \geq \min\{\mu_A(x), \frac{\delta}{2}\} = \frac{\delta}{2} \text{ for any } y \in N. \Rightarrow y + x - y, xy \in A.
\]

Let \( x \in A \) and \( u, v \in N \). Now, \( \mu_A(u(v + x) - uv) \geq \min\{\mu_A(x), \frac{\delta}{2}\} = \frac{\delta}{2} \)
\[
\Rightarrow u(v + x) - uv \in A. \text{ Therefore, } A \text{ is an ideal of } N.\]
Conversely, Let $A$ be an ideal of $N$. If $x, y \in A$ then $x - y, xy \in A$ so, $\mu_A(x - y) = \frac{\delta}{2} = \min\{\mu_A(x), \mu_A(y), \frac{\delta}{2}\}$
$\mu_A(xy) = \frac{\delta}{2} = \min\{\mu_A(x), \mu_A(y), \frac{\delta}{2}\}$
If at least one of $x$ and $y$ does not belong to $A$, Then
$\mu_A(x - y) \geq t = \min\{\mu_A(x), \mu_A(y), \frac{\delta}{2}\}$ and
$\mu_A(xy) \geq t = \min\{\mu_A(x), \mu_A(y), \frac{\delta}{2}\}$. Let $x \in A$ and $u, v \in N$ then $u + x - u, xu, u(v + x) - uv \in A$. so,
$\mu_A(u + x - u) = \frac{\delta}{2} = \min\{\mu_A(x), \frac{\delta}{2}\}$
$\mu_A(xu) = \frac{\delta}{2} = \min\{\mu_A(x), \frac{\delta}{2}\}$
and $\mu_A(u(v + x) - uv) = \frac{\delta}{2} = \min\{\mu_A(x), \frac{\delta}{2}\}$.
If $x \notin A$, then $\mu_A(u + x - u) \geq t = \min\{\mu_A(x), \frac{\delta}{2}\}$,
$\mu_A(xu) \geq t = \min\{\mu_A(x), \frac{\delta}{2}\}$
and $\mu_A(u(v + x) - uv) \geq t = \min\{\mu_A(x), \mu_A(y), \frac{\delta}{2}\}$
Hence, $\mu_A$ is an $(\epsilon, \in \vee q_0^\delta)$-fuzzy ideal of $N$. 

**Colloary 3.19.** Let $A$ be a non-empty subset of $N$ and $\mu_A$ be a fuzzy set in $N$ defined by

$$\mu_A(x) = \begin{cases} t, & \text{if } x \in A; \\ 0, & \text{otherwise.} \end{cases}$$

for all $x \in N$ with $t \in (0, \frac{\delta}{2}]$, Then $\mu_A$ is a $(\epsilon, \in \vee q_0^\delta)$-fuzzy ideal of $N$ if and only if $A$ is an ideal of $N$.

Let $x \in N$ be such that $\mu(x) \geq \frac{\delta}{2}$, then

$$\mu(0) = \mu(x - x) \geq \min\{\mu(x), \mu(x), \frac{\delta}{2}\} = \frac{\delta}{2}.$$ 
\[\Rightarrow \mu(0) \geq \frac{\delta}{2}.\]
Again if $\mu(0) < \frac{\delta}{2}$, then $\mu(x) < \frac{\delta}{2} \forall x \in N$ then $\mu$ is fuzzy subgroup in the sense of Rosenfeld. In order to see a nontrivial generalization of fuzzy subgroup, we assume that $\mu_{\frac{\delta}{2}} \neq \{0\}$.

Henceforth, unless otherwise mentioned by $(\epsilon, \in \vee q_0^\delta)$-fuzzy subnear-ring(ideal) of $N$, we shall mean an $(\epsilon, \in \vee q_0^\delta)$-fuzzy subnear-ring(ideal) of $N$ with $\mu_{\frac{\delta}{2}} \neq \{0\}$.

**Lemma 3.20.** Let $\mu$ be an $(\epsilon, \in \vee q_0^\delta)$-fuzzy subnear-ring of $N$. Let $x, y \in N$ be such that $\mu(x) < \mu(y)$, then

- i) $\mu(x + y) = \mu(y + x) = \mu(x)$ if $\mu(x) < \frac{\delta}{2}$.
- ii) $\mu(xy), \mu(yx) \geq \frac{\delta}{2}$ if $\mu(x) \geq \frac{\delta}{2}$. 


ii) Let \( a \) be induced by quasi \( \delta \) under isomorphism between two near-rings with respect to the structure \( \mu \). Hence, \( \mu \) is given if and only if \( \mu(x) < \mu(y) \) and \( \mu(x) \leq \frac{\delta}{2} \).

Then, \( \mu(x + y) = \mu(x - (-y)) \geq \min\{\mu(x), \mu(-y), \frac{\delta}{2}\} \geq \min\{\mu(x), \mu(y), \frac{\delta}{2}\} = \mu(x) \Rightarrow \mu(x + y) \geq \mu(x) \).

and \( \mu(x) = \mu(x + y - y) \geq \min\{\mu(x + y), \mu(y), \frac{\delta}{2}\} = \mu(x + y) \) since it is given \( \mu(x) < \mu(y) \) and \( \mu(x) < \frac{\delta}{2} \).

\( \Rightarrow \mu(x) \geq \mu(x + y) \). Therefore, \( \mu(x + y) = \mu(x) \).

Similarly, we can show that \( \mu(y + x) = \mu(x) \).

Lemma 3.21. Let \( \mu \) be an \((\in, \in \cap \delta^0)\)-fuzzy ideal of \( N \). Then \( \mu_a = \mu_b \) if and only if \( \mu(a - b), \mu(b - a) \geq \frac{\delta}{2} \forall a, b \in N \).

Proof. Suppose that \( \mu(a - b), \mu(b - a) \geq \frac{\delta}{2} \).

Let \( x \in N \), then \( \mu_a(x) = \min\{\mu(x - a), \frac{\delta}{2}\} = \min\{\mu((x - (a - b)) - b) - b), \frac{\delta}{2}\} \geq \min\{\mu(x - b), \mu(a - b), \frac{\delta}{2}\} \geq \min\{\mu(x - b), \frac{\delta}{2}\} = \mu_b(x) \) for all \( x \in N \).

\( \Rightarrow \mu_a \geq \mu_b \). Similarly, we can show that \( \mu_b \geq \mu_a \), thus \( \mu_a = \mu_b \).

Conversely, suppose that \( \mu_a = \mu_b \). Then \( \mu_a(a) = \mu_b(a) \)

\( \Rightarrow \min\{\mu(0), \frac{\delta}{2}\} = \min\{\mu(a - b), \frac{\delta}{2}\} \)

\( \Rightarrow \frac{\delta}{2} = \min\{\mu(a - b), \frac{\delta}{2}\} \Rightarrow \mu(a - b) \geq \frac{\delta}{2} \).

And \( \mu_a(b) = \mu_b(b) \Rightarrow \min\{\mu(b - a), \frac{\delta}{2}\} = \min\{\mu(0), \frac{\delta}{2}\} \)

\( \Rightarrow \min\{\mu(b - a), \frac{\delta}{2}\} = \frac{\delta}{2} \Rightarrow \mu(b - a) \geq \frac{\delta}{2} \). \( \square \)

4. Quasi \( \delta \)-fuzzy cosets

In this section, we introduce and discuss about quasi \( \delta \)-fuzzy cosets of a \((\in, \in \cap \delta^0)\)-fuzzy ideal in a near-ring \( N \) and prove fundamental theorem under isomorphism between two near-rings with respect to the structure induced by quasi \( \delta \)-fuzzy cosets.

Definition 4.1. Let \( \mu \) be an \((\in, \in \cap \delta^0)\)-fuzzy ideal in \( N \). Given \( a \in N \),
a fuzzy set $\mu_a$ in $N$ defined by $\mu_a(x) = \min\{\mu(x-a), \frac{\delta}{2}\}$ is called the $(\varepsilon, \in \vee_{q^\delta})$-fuzzy coset of $\mu$ in $N$ determined by $a$ and $\mu$.

**Definition 4.2.** Let $\mu$ be an $(\varepsilon, \in \vee_{q^\delta})$-fuzzy ideal of $N$ and $N^\mu = \{\mu_a | a \in N\}$ is the set of all $(\varepsilon, \in \vee_{q^\delta})$-fuzzy cosets of $\mu$ in $N$.

We provide two operations $\oplus$ and $\circ$ into $N^\mu$ as follows

$$\mu_x \oplus \mu_y = \mu_{x+y} \quad \text{and} \quad \mu_x \circ \mu_y = \mu_{xy} \quad \text{for all} \quad \mu_x, \mu_y \in N^\mu.$$

We first show that the compositions are well defined.

Let $a, b, x, y \in N$ be such that $\mu_a = \mu_x$ and $\mu_b = \mu_y$.

now, $\mu(a + b - y - x) = \mu(-(a + b - x)) = \mu((-a + x) + (b - y))$

$\geq \min\{\mu((-a + x)), \mu(b - y), \frac{\delta}{2}\} \geq \min\{\mu(a - x), \mu(y - b), \frac{\delta}{2}\}$

$\geq \frac{\delta}{2}$. [By lemma 3.21.]

$\Rightarrow \mu((a + b) - (x + y)) \geq \frac{\delta}{2}$.

Therefore, by lemma 3.21., $\mu_{a+b} = \mu_{x+y} \Rightarrow \mu_a \oplus \mu_b = \mu_x \oplus \mu_y$. Again, $\mu(ab - xy) = \mu((ab - xy) + xy - x) = \mu((-a + x)b - (xy - xb))$

$\geq \min\{\mu((-a + x)b), \mu(xy - xb), \frac{\delta}{2}\} \geq \min\{\mu(a - x), \mu(x(b - y) - xb), \frac{\delta}{2}\}$

$\geq \min\{\mu(a - x), \mu(b - y), \frac{\delta}{2}\} \geq \frac{\delta}{2}$. [By lemma 3.21.]

Therefore, by lemma 3.21., $\mu_{ab} = \mu_{xy} \Rightarrow \mu_a \odot \mu_b = \mu_x \odot \mu_y$.

Hence, the composition are well defined.

**Theorem 4.3.** For any $(\varepsilon, \in \vee_{q^\delta})$-fuzzy ideal $\mu$ of $N$, the set of all $(\varepsilon, \in \vee_{q^\delta})$-fuzzy cosets of $\mu$ in $N$ i.e. $N^\mu = \{\mu_a | a \in N\}$ is a near-ring under operation $\oplus$ and $\circ$.

The Proof of Theorem 4.3 is straight foward.

For a fuzzy set $\mu$ in $N$, we define a fuzzy set $\bar{\mu}$ in $N^\mu$ by $\bar{\mu}(\mu_x) = \mu(x)$ for all $x \in N$.

**Theorem 4.4.** If $\mu$ is an $(\varepsilon, \in \vee_{q^\delta})$-fuzzy ideal of $N$, then $\bar{\mu}$ is an $(\varepsilon, \in \vee_{q^\delta})$-fuzzy ideal in $N^\mu$.

**Proof.** Suppose $\mu$ is an $(\varepsilon, \in \vee_{q^\delta})$-fuzzy ideal of $N$. Let $a, b \in N$.

Now,

$\bar{\mu}(\mu_a \ominus \mu_b) = \bar{\mu}(\mu_{a-b}) = \mu(a-b) \geq \min\{\mu(a), \mu(b), \frac{\delta}{2}\} = \min\{\bar{\mu}(\mu_a), \bar{\mu}(\mu_b), \frac{\delta}{2}\}$.

$\bar{\mu}(\mu_a \odot \mu_b) = \bar{\mu}(\mu_{ab}) = \mu(ab) \geq \min\{\mu(a), \mu(b), \frac{\delta}{2}\} = \min\{\bar{\mu}(\mu_a), \bar{\mu}(\mu_b), \frac{\delta}{2}\}$.

$\bar{\mu}(\mu_{a+b} \ominus \mu_a) = \bar{\mu}(\mu_{b-a}) = \mu(b-a) \geq \min\{\mu(b), \frac{\delta}{2}\} = \min\{\bar{\mu}(\mu_b), \frac{\delta}{2}\}$.

$\bar{\mu}(\mu_a \odot \mu_b) = \bar{\mu}(\mu_{ab}) = \mu(ab) \geq \min\{\mu(a), \mu(b), \frac{\delta}{2}\} = \min\{\bar{\mu}(\mu_a), \bar{\mu}(\mu_b), \frac{\delta}{2}\}$.

$\bar{\mu}(\mu_a \ominus (\mu_b \oplus \mu_c) \oplus (\mu_a \odot \mu_b)) = \bar{\mu}(\mu_{a+b+c} \ominus \mu_{ab}) = \bar{\mu}(\mu_{a+b+c} \ominus \mu_{ab}) = \bar{\mu}(\mu_{a+b+c} \ominus \mu_{ab}) =$
Therefore, \( \phi \) is called a near-ring homomorphism if \( \theta(x + y) = \theta(x) + \theta(y) \) and \( \theta(xy) = \theta(x)\theta(y) \) for all \( x, y \in N \).

**Theorem 4.6.** If \( \mu \) is an \((\in, \in \lor \in)_0\)-fuzzy ideal of \( N \), then the mapping \( f : N \to N_\delta^\mu \) as \( f(x) = \mu_x \) is a homomorphism with \( \ker f = \mu_2^\mu \).

**Proof.** Let \( x, y \in N \), now
\[
\begin{align*}
f(x + y) &= \mu_{x+y} = \mu_x \oplus \mu_y = f(x) \oplus f(y) \\
f(xy) &= \mu_{xy} = \mu_x \odot \mu_y = f(x) \odot f(y).
\end{align*}
\]
Therefore \( f \) is a homomorphism. And
\[
\ker f = \{ x \in N \mid f(x) = f(0) \} = \{ x \in N \mid \mu_x = \mu_0 \} = \{ x \in N \mid \mu_x(x) = \mu_0(x) \}
\]
\[
= \{ x \in N \mid \min \{ \mu(0), \frac{\delta}{2} \} = \min \{ \mu(x), \frac{\delta}{2} \} \} = \{ x \in N \mid \mu(x) \geq \frac{\delta}{2} \} = \mu_2^\mu.
\]

**Theorem 4.7.** For a near-ring homomorphism \( f : N \to N' \), Let \( \mu \) and \( \nu \) be \((\in, \in \lor \in)_0\)-fuzzy ideals of \( N \) and \( N' \) respectively. Then the mapping \( \phi : N_\delta^\mu \to N_\delta^\nu \) as \( \phi(\mu_x) = \nu_f(x) \) for \( x \in N \) is a homomorphism.

**Proof.** Let \( x, y \in N \), now
\[
\begin{align*}
\phi(\mu_x \oplus \mu_y) &= \phi(\mu_{x+y}) = \nu_f(x+y) = \nu_f(x) + \nu_f(y) = \phi(\mu_x) \oplus \phi(\mu_y) \\
\phi(\mu_x \odot \mu_y) &= \phi(\mu_{xy}) = \nu_f(xy) = \nu_f(x) \odot \nu_f(y) = \phi(\mu_x) \odot \phi(\mu_y).
\end{align*}
\]
Therefore, \( \phi \) is a homomorphism.

**Theorem 4.8.** If \( \mu \) is an \((\in, \in \lor \in)_0\)-fuzzy subnear-ring(ideal) of \( N \), then the fuzzy set \( \nu : N \to [0, \delta] \) as \( \nu(x) = \mu_2(\mu_x) \) is an \((\in, \in \lor \in)_0\)-fuzzy subnear-ring(ideal) of \( N \).

**Proof.** Let \( \mu \) is an \((\in, \in \lor \in)_0\)-fuzzy subnear-ring(ideal) of \( N \). Then by theorem 4.4., \( \mu \) is an \((\in, \in \lor \in)_0\)-fuzzy subnear-ring(ideal) of \( N_\delta^\mu \).

Let \( x, y \in N \), now
\[
\begin{align*}
\nu(x-y) &= \mu_2(\mu_{x-y}) = \mu_2(\mu_x \ominus \mu_y) \geq \min \{ \mu_2(\mu_x), \mu_2(\mu_y), \delta \} = \min \{ \nu(x), \nu(y), \delta \} \\
\nu(xy) &= \mu_2(\mu_{xy}) = \mu_2(\mu_x \odot \mu_y) \geq \min \{ \mu_2(\mu_x), \mu_2(\mu_y), \delta \} = \min \{ \nu(x), \nu(y), \delta \}.
\end{align*}
\]
\[ \nu(y + x - y) = \bar{\mu}(\mu_{y+x-y}) = \bar{\mu}(\mu_y \ominus \mu_x \ominus \mu_y) \geq \min\{\bar{\mu}(\mu_x), \frac{\delta}{2}\} = \min\{\nu(x), \frac{\delta}{2}\}. \]

\[ \nu(xy) = \bar{\mu}(\mu_{xy}) = \bar{\mu}(\mu_x \ominus \mu_y) \geq \min\{\bar{\mu}(\mu_x), \frac{\delta}{2}\} = \min\{\nu(x), \frac{\delta}{2}\}. \]

\[ \nu(y(x + a) - yx) = \bar{\mu}(\mu_y(x + a) - \mu_y) = \bar{\mu}\{\mu_y \ominus (\mu_x \ominus \mu_a) \ominus \mu_y\} \]

Therefore, \( \nu \) is an \((\epsilon, \bar{\epsilon}, \sqrt{q_0})\)-fuzzy subnear-ring(ideal) of \( N \).

**Definition 4.9.** [15] If \( \mu \) is a fuzzy set in \( N \) and \( f \) is a function defined on \( N \), then the fuzzy set \( \nu \) in \( f(N) \) defined by

\[ \nu(y) = \sup_{x \in f^{-1}(y)} \mu(x) \]

for all \( y \in f(N) \) is called the image of \( \mu \) under \( f \). Similarly, if \( \nu \) is a fuzzy set in \( f(N) \), then the fuzzy set \( \mu = f \circ \nu \) in \( N \) (that is, the fuzzy set defined by \( \mu(x) = \nu(f(x)) \)) for all \( x \in N \) is called the preimage of \( \nu \) under \( f \).

We say that a fuzzy set \( \mu \) in \( N \) has the sup property if for any subset \( T \) of \( N \), there exists \( t_0 \in T \) such that

\[ \mu(t_0) = \sup_{t \in T} \mu(t). \]

**Theorem 4.10.** A near-ring homomorphic preimage of an \((\epsilon, \bar{\epsilon}, \sqrt{q_0})\)-fuzzy ideal is an \((\epsilon, \bar{\epsilon}, \sqrt{q_0})\)-fuzzy ideal.

**Proof.** Let \( \theta : N \rightarrow N' \) be a near-ring homomorphism.

Let \( \nu \) be an \((\epsilon, \bar{\epsilon}, \sqrt{q_0})\)-fuzzy ideal of \( N' \) and \( \mu \) be the preimage of \( \nu \) under \( \theta \). Let \( x, y, a \in N \). Now

\[ \mu(x - y) = \nu(\theta(x - y)) = \nu(\theta(x) - \theta(y)) \geq \min\{\nu(\theta(x)), \nu(\theta(y)), \frac{\delta}{2}\} \]

\[ \mu(xy) = \nu(\theta(xy)) = \nu(\theta(x)\theta(y)) \geq \min\{\nu(\theta(x)), \nu(\theta(y)), \frac{\delta}{2}\} \]

\[ \mu(y + x - y) = \nu(\theta(y + x - y)) = \nu(\theta(y) + \theta(x) - \theta(y)) \geq \min\{\nu(\theta(x)), \frac{\delta}{2}\} \]

\[ \mu(xy) = \nu(\theta(xy)) = \nu(\theta(x)\theta(y)) \geq \min\{\nu(\theta(x)), \frac{\delta}{2}\} = \min\{\mu(x), \frac{\delta}{2}\} \]

\[ \mu(y(x + a) - yx) = \nu(\theta(y(x + a) - yx)) = \nu(\theta(y(x + a) - \theta(yx))) \]

Therefore, \( \mu \) is an \((\epsilon, \bar{\epsilon}, \sqrt{q_0})\)-fuzzy ideal.

**Theorem 4.11.** A near-ring homomorphic image of an \((\epsilon, \bar{\epsilon}, \sqrt{q_0})\)-fuzzy ideal having the sup property is an \((\epsilon, \bar{\epsilon}, \sqrt{q_0})\)-fuzzy ideal.
Proof. Let \( \theta : N \rightarrow N' \) be a near-ring homomorphism and \( \mu \) be an \((\in, \in \vee q^*_0)\)-fuzzy ideal of \( N \) having the sup property and \( \nu \) be the image of \( \mu \) under \( \theta \).
Let \( \theta(x), \theta(y) \in \theta(N) \) and \( x_0 \in \theta^{-1}(\theta(x)), y_0 \in \theta^{-1}(\theta(y)) \) be such that

\[
\mu(x_0) = \sup_{t \in \theta^{-1}(\theta(x))} \mu(t), \mu(y_0) = \sup_{t \in \theta^{-1}(\theta(y))} \mu(t)
\]

respectively. Then,

\[
\nu(\theta(x) - \theta(y)) = \sup_{t \in \theta^{-1}(\theta(x) - \theta(y))} \mu(t) \geq \mu(x_0 - y_0) \quad \text{[by sup property]}
\]

\[
\geq \min\{\mu(x_0), \mu(y_0), \frac{\delta}{2}\} = \min\{\sup_{t \in \theta^{-1}(\theta(x))} \mu(t), \sup_{t \in \theta^{-1}(\theta(y))} \mu(t), \frac{\delta}{2}\}
\]

\[
= \min\{\nu(\theta(x)), \nu(\theta(y)), \frac{\delta}{2}\}.
\]

\[
\nu(\theta(x)\theta(y)) = \sup_{t \in \theta^{-1}(\theta(x)\theta(y))} \mu(t) \geq \mu(x_0 y_0)
\]

\[
\geq \min\{\mu(x_0), \mu(y_0), \frac{\delta}{2}\} = \min\{\sup_{t \in \theta^{-1}(\theta(x))} \mu(t), \sup_{t \in \theta^{-1}(\theta(y))} \mu(t), \frac{\delta}{2}\}
\]

\[
= \min\{\nu(\theta(x)), \nu(\theta(y)), \frac{\delta}{2}\}.
\]

\[
\nu(\theta(y) + \theta(x) - \theta(y)) = \sup_{t \in \theta^{-1}(\theta(y) + \theta(x) - \theta(y))} \mu(t) \geq \mu(y_0 + x_0 - y_0).
\]

\[
\geq \min\{\mu(x_0), \frac{\delta}{2}\} = \min\{\sup_{t \in \theta^{-1}(\theta(x))} \mu(t), \frac{\delta}{2}\} = \min\{\nu(\theta(x)), \frac{\delta}{2}\}.
\]

\[
\nu(\theta(x)\theta(y)) = \sup_{t \in \theta^{-1}(\theta(x)\theta(y))} \mu(t) \geq \mu(x_0 y_0)
\]

\[
\geq \min\{\mu(x_0), \frac{\delta}{2}\} = \min\{\sup_{t \in \theta^{-1}(\theta(x))} \mu(t), \frac{\delta}{2}\} = \min\{\nu(\theta(x)), \frac{\delta}{2}\}.
\]

and \( \nu((\theta(x) + \theta(a))\theta(y) - \theta(x)\theta(y)) = \sup_{t \in \theta^{-1}((\theta(x) + \theta(a))\theta(y) - \theta(x)\theta(y))} \mu(t) \geq \mu((x_0 + a_0)y_0 - x_0 y_0) \geq \min\{\mu(a_0), \frac{\delta}{2}\} = \min\{\sup_{t \in \theta^{-1}(\theta(a))} \mu(t), \frac{\delta}{2}\}
\]

\[
= \min\{\nu(\theta(a)), \frac{\delta}{2}\}.
\]

Therefore, \( \nu \) is an \((\in, \in \vee q^*_0)\)-fuzzy ideal. 

\( \square \)
A note on δ-quasi fuzzy subnear-rings and ideals

References


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