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ON SIGNLESS LAPLACIAN SPECTRUM OF THE ZERO DIVISOR GRAPHS OF THE RING $\mathbb{Z}_n$

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Abstract. For a finite commutative ring R with identity $1 \neq 0$, the zero divisor graph $\Gamma(R)$ is a simple connected graph having vertex set as the set of nonzero zero divisors of R, where two vertices x and y are adjacent if and only if $xy = 0$. We find the signless Laplacian spectrum of the zero divisor graphs $\Gamma(\mathbb{Z}_n)$ for various values of n. Also, we find signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ for $n = p^z, z \geq 2$, in terms of signless Laplacian spectrum of its components and zeros of the characteristic polynomial of an auxiliary matrix. Further, we characterise n for which zero divisor graph $\Gamma(\mathbb{Z}_n)$ are signless Laplacian integral.

1. Introduction

Throughout this paper, we consider only connected, undirected, simple and finite graphs. A graph is denoted by $G = G(V(G), E(G))$, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ is its vertex set and $E(G)$ is its edge set. $|V(G)| = n$ is the *order* and $|E(G)| = m$ is the *size* of G. The *neighborhood* of a vertex v, denoted by $N(v)$, is the set of vertices of G adjacent to v. The *degree* of v, denoted by $d_G(v)$ (we simply d_v) is the cardinality of $N(v)$. A graph is said to be *regular* if each of its vertex has the same degree. The adjacency matrix $A = (a_{ij})$ of G is a $(0,1)$-square matrix of order n, whose (i,j)-entry is equal to 1, if v_i is adjacent to v_j and equal to 0, otherwise. Let $Deg(G) = diag(d_1, d_2, \ldots, d_n)$ be the diagonal matrix of vertex degrees $d_i = d_G(v_i)$, $i = 1, 2, \ldots, n$ associated to G. The matrices $L(G) = Deg(G) - A(G)$ and $Q(G) = Deg(G) + A(G)$ are respectively the Laplacian and the signless Laplacian matrices. Their spectrum are respectively the Laplacian spectrum and signless Laplacian spectrum of the graph G. These matrices are real symmetric and positive semi-definite having real eigenvalues which can be ordered as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$ and $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_n(G)$ respectively. More about Laplacian and signless Laplacian matrices can be seen in [7, 8, 11–13, 15] and the references therein.

Let R be a commutative ring with multiplicative identity $1 \neq 0$. A nonzero element $x \in R$ is called a zero divisor of R if there exists a nonzero element $y \in R$ such that $xy = 0$. The zero divisor graphs of commutative rings were first introduced by

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Beck [4], in the definition he included the additive identity and was interested mainly in coloring of commutative rings. Later Anderson and Livingston [2] modified the definition of zero divisor graphs and excluded the additive identity of the ring in the zero divisor set. Zero divisor graphs are simple, connected and undirected graphs having vertex set as the set of nonzero zero divisors, in which two vertices x and y are connected by an edge if and only if $xy = 0$. The zero divisor graph of $\mathbb{Z}_n$ is of order $n - \phi(n) - 1$, where ϕ is Euler's totient function. Adjacency and Laplacian spectral analysis has been done in [6,16,19]. More literature about zero divisor graphs can be found in [1,2,10] and the references therein.

For any graph G, we write $Spec(G)$ to represent the spectrum of G which contains its eigenvalues including multiplicities. If vertices x and y are adjacent in G, we write $x \sim y$. We use standard notations, K_n and $K_{a,b}$, for complete graph and complete bipartite graph, respectively. Other undefined notations and terminology from algebraic graph theory, algebra and matrix theory can be found in [3,7,9,14].

The rest of the paper is organized as follows. In Section 2, we mention some basic definitions and results. In Section 3, we discuss the signless Laplacian spectrum of the zero divisor graph $\Gamma(\mathbb{Z}_n)$ for some values of $n \in \{pq, p^2q, (pq)^2\}$. We find signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ for $n = p^z,\ z \geq 2$ in terms of signless Laplacian spectrum of components of $\Gamma(\mathbb{Z}_n)$ and zeros of characteristic polynomial of an auxiliary matrix and show that $\Gamma(\mathbb{Z}_n)$ is signless Laplacian integral, for $n \in \{p^2, pq\}$. We have used computational software Wolfram Mathematica for computing approximate eigenvalues and characteristic polynomials of various matrices.

2. Preliminaries

We start the section with the definitions and previously known results which are used in proving the main results of the next section.

Definition 2.1. Let $G(V, E)$ be a graph of order n having vertex set $\{1, 2, \ldots, n\}$ and $G_i = G_i(V_i, E_i)$ be disjoint graphs of order n_i, $1 \leq i \leq k$. The graph $G[G_1, G_2, \ldots, G_n]$ is formed by taking the graphs $G_1, G_2, \ldots, G_n$ and joining each vertex of G_i to every vertex of G_j whenever i and j are adjacent in G.

This graph operation $G[G_1, G_2, \ldots, G_n]$ is called generalized join graph operation [5], or generalized composition [17] or G-join operation [7]. If $G = K_2$, the K_2-join is the usual join operation, namely $G_1 \nabla G_2$. Herein, we follow the later name with notation $G[G_1, G_2, \ldots, G_n]$ and call it G-join. Schwenk [17] determined the adjacency spectra of G-join of regular graphs. In [5], Laplacian spectra of G-join of arbitrary graphs has been determined and in [18] normalized Laplacian and signless Laplacian spectra of the G-join of regular graphs is computed.

An integer d is called a proper divisor of n if d divides n, $1 < d < n$ and is written as $d|n$. Let $d_1, d_2, \ldots, d_t$ be the distinct proper divisors of n. Let Υ_n be the simple graph with vertex set $\{d_1, d_2, \ldots, d_t\}$, in which two distinct vertices are connected by an edge if and only if $n|d_id_j$. If n has the prime power factorization $p_1^{n_1}p_2^{n_2}\ldots p_r^{n_r}$, where $r, n_1, n_2, \ldots, n_r$ are positive integers and $p_1, p_2, \ldots, p_r$ are distinct prime numbers, the order of the Υ_n is given by

$$|V(\Upsilon_n)| = \prod_{i=1}^{r}(n_i + 1) - 2.$$

This Υ_n is connected [6] and plays a fundamental role in the sequel. For $1 \leq i \leq t$, we consider the following sets

$$A_{d_i} = \{x \in \mathbb{Z}_n : (x, n) = d_i\},$$

where (x, n) represents greatest common divisor of x and n. We see that $A_{d_i} \cap A_{d_j} = \emptyset$, when $i \neq j$, implying that the sets $A_{d_1}, A_{d_2}, \dots, A_{d_t}$ are pairwise disjoint and partitions the vertex set of $\Gamma(\mathbb{Z}_n)$ as

$$V(\Gamma(\mathbb{Z}_n)) = A_{d_1} \cup A_{d_2} \cup \cdots \cup A_{d_t}.$$

From the definition of A_{d_i}, a vertex of A_{d_i} is adjacent to the vertex of A_{d_j} in $\Gamma(\mathbb{Z}_n)$ if and only if n divides $d_i d_j$, for $i, j \in \{1, 2, \dots, t\}$ [6].
The following result [19] gives the cardinality of A_{d_i}.

Lemma 2.2. *Let d divides n. Then $|A_{d_i}| = \phi\left(\frac{n}{d_i}\right)$, for $1 \leq i \leq t$.*

The next lemma [6] shows the that induced subgraphs $\Gamma(A_{d_i})$ of $\Gamma(\mathbb{Z}_n)$ are either cliques or their complements.

Lemma 2.3. *The following hold.*

(i) *For $i \in \{1, 2, \dots, t\}$, the induced subgraph $\Gamma(A_{d_i})$ of $\Gamma(\mathbb{Z}_n)$ on the vertex set A_{d_i} is either the complete graph $K_{\phi\left(\frac{n}{d_i}\right)}$ or its complement $\overline{K}_{\phi\left(\frac{n}{d_i}\right)}$. Indeed, $\Gamma(A_{d_i})$ is $K_{\phi\left(\frac{n}{d_i}\right)}$ if and only n divides d_i^2.*

(ii) *For $i, j \in \{1, 2, \dots, t\}$ with $i \neq j$, a vertex of A_{d_i} is adjacent to either all or none of the vertices in A_{d_j} of $\Gamma(\mathbb{Z}_n)$.*

The following lemma shows that $\Gamma(\mathbb{Z}_n)$ is a G-join of certain complete graphs and null graphs.

Lemma 2.4. [6] *Let $\Gamma(A_{d_i})$ be the induced subgraph of $\Gamma(\mathbb{Z}_n)$ on the vertex set A_{d_i} for $1 \leq i \leq t$. Then $\Gamma(\mathbb{Z}_n) = \Upsilon_n[\Gamma(A_{d_1}), \Gamma(A_{d_2}), \dots, \Gamma(A_{d_t})]$.*

Next, we mention the statement of a result of [18] that gives the signless Laplacian spectrum of G-join of graphs in terms of signless Laplacian spectrum of its components and eigenvalues of an auxiliary matrix.

Theorem 2.5. *[18] Let H be a graph with $V(H) = \{1, 2, \dots, t\}$, and G_i's be r_i-regular graphs of order n_i $(i = 1, 2, \dots, t)$. If $G = H[G_1, G_2, \cdots, G_t]$, then signless Laplacian spectrum of G can be computed as follows.*

$$Spec_Q(G) = \left(\bigcup_{i=1}^{t} \Big(N_i + \big(Spec_Q(G_i) \backslash \{2r_i\} \big) \Big) \right) \bigcup Spec(C_Q(H)),$$

where

$$N_i = \begin{cases} \displaystyle\sum_{j \in N_H(i)} n_j, & N_H(i) \neq \emptyset \\ 0, & otherwise \end{cases},$$

and

$$C_Q(H) = (c_{ij})_{t\times t} = \begin{cases} 2r_i + N_i, & i = j, \\ \sqrt{n_i n_j}, & ij \in E(H), \\ 0 & otherwise. \end{cases}$$

The next observation is a consequence of Theorem 2.5 and the proof follows trivially.

PROPOSITION 2.6. *The G-join graph is signless Laplacian integral if and only if each of G_i is signless Laplacian integral and the matrix $C_Q(G)$ is integral.*

3. Main results

We recall that $\Gamma(\mathbb{Z}_n)$ is a complete graph if and only if $n = p^2$ for some prime p. Further the signless Laplacian spectrum of K_ω and $\overline{K}_\omega$ on ω vertices are $\{2\omega - 2, (\omega-2)^{[\omega-1]}\}$ and $\{0^{[\omega]}\}$, respectively. By Lemma 2.3, $\Gamma(A_{d_i})$ is either $K_{\phi\left(\frac{n}{d_i}\right)}$ or its complement $\overline{K}_{\phi\left(\frac{n}{d_i}\right)}$ for $1 \le i \le t$. So by Theorem 2.5, out of $n-\phi(n)-1$ number of signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$, $n-\phi(n)-1-t$ of them are known to be non-negative integers. The remaining t signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ will be calculated from the zeros of the characteristic polynomial of the matrix $C_Q(H)$.

We start with an example of $\Gamma(\mathbb{Z}_n)$, for $n = 30$ and find its signless Laplacian spectrum with the help of Theorem 2.5.

EXAMPLE 3.1. Signless Laplacian spectrum of $\Gamma(\mathbb{Z}_{30})$.

Let $n = 30$. Then $2, 3, 5, 6, 10$ and 15 are the proper divisors of n and Υ_n is the graph $G_6 : 3 \sim 10 \sim 6 \sim 5$,$10 \sim 15 \sim 2$ and $6 \sim 15$, that is, Υ_n is a triangle having pendent vertex at each vertex of the triangle as shown in Figure (1). Ordering the vertices by increasing divisor sequence and applying Lemma 2.4, we have

$$\Gamma(\mathbb{Z}_{30}) = \Upsilon_{30}[\overline{K}_8, \overline{K}_4, \overline{K}_{24}, \overline{K}_4, \overline{K}_2, \overline{K}_1].$$

By Theorem 2.5, the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_{30})$ consists of the eigenvalues $\{1^{[7]}, 2^{[4]}, 3^{[3]}, 11\}$ and the remaining eigenvalues are given by

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \sqrt{8} \\ 0 & 2 & 0 & 0 & \sqrt{8} & 0 \\ 0 & 0 & 4 & \sqrt{8} & 0 & 0 \\ 0 & 0 & \sqrt{8} & 5 & \sqrt{8} & 2 \\ 0 & \sqrt{8} & 0 & \sqrt{8} & 9 & \sqrt{2} \\ \sqrt{8} & 0 & 0 & 2 & \sqrt{2} & 14 \end{pmatrix}.$$

The characteristic polynomial of above matrix is

$$x^6 - 35x^5 + 413x^4 - 1917x^3 + 3098x^2 - 1624x + 256$$

and its approximated zeros are

$$\{15.6845, 10.4343, 6.39444, 1.70695, 0.483479, 0.29642\}.$$

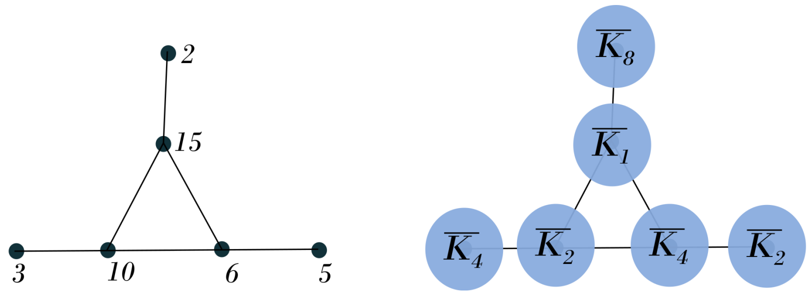

FIGURE 1. Proper divisor graph Υ_{30} and zero divisor graph $\Gamma(\mathbb{Z}_{30})$

Now, we discuss the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ for $n \in \{pq, p^2q, (pq)^2, p^3, p^4, p^z, z \geq 2\}$, with the help of Theorem 2.5. Let $n = pq$, where $p < q$ are primes. Then, by Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned} \Gamma(\mathbb{Z}_{pq}) &= \Upsilon_{pq}[\Gamma(A_p), \Gamma(A_q)] = K_2[\overline{K}_{\phi(p)}, \overline{K}_{\phi(q)}] \\ &= \overline{K}_{\phi(p)} \nabla \overline{K}_{\phi(q)} = K_{\phi(p),\phi(q)}. \end{aligned} \tag{1}$$

In the next lemma, we find the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ for $n = pq$ with $p < q$.

LEMMA 3.2. *The signless Laplacian spectrum of* $\Gamma(\mathbb{Z}_{pq})$ *is* $\{0, (q-1)^{[p-2]}, (p-1)^{[q-2]}, p+q-2\}$.

Proof. Let $n = pq$, where p and q $(p < q)$ are primes. The proper divisors of n are p and q, and so Υ_{pq} is K_2. By Theorem 2.5, $(N_1, N_2) = (q-1, p-1)$. From equation (1), the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ consists of the eigenvalue $q-1$ with multiplicity $p-2$, the eigenvalue $p-1$ with multiplicity $q-2$ and remaining two eigenvalues are given by the matrix

$$\begin{pmatrix} q-1 & \sqrt{(p-1)(q-1)} \\ \sqrt{(p-1)(q-1)} & p-1 \end{pmatrix}.$$

□

PROPOSITION 3.3. *The signless Laplacian spectrum of* $\Gamma(\mathbb{Z}_{p^2q})$ *is*

$$\left\{(q-1)^{[p^2-p-1]}, (p^2-1)^{[q-2]}, (p-1)^{[pq-p-q]}, (pq+p-3)^{[p-2]}, x_1, x_2, x_3, x_4\right\}$$

where $x_1 \geq x_2 \geq x_3 \geq x_4$ *are the zeros of the characteristic polynomial of the matrix* $C_Q(P_4)$.

Proof. Let $n = p^2q$, where p and q are distinct primes. Since proper divisors of n are p, q, pq, p^2, so Υ_{p^2q} is the path $P_4 : q \sim p^2 \sim pq \sim p$. By Lemma 2.4, we have

$$\begin{aligned} \Gamma(\mathbb{Z}_{p^2q}) &= \Upsilon_{p^2q}[\Gamma(A_q), \Gamma(A_{p^2}), \Gamma(A_{pq}), \Gamma(A_p)] \\ &= P_4[\overline{K}_{\phi(p^2)}, \overline{K}_{\phi(q)}, K_{\phi(p)}, \overline{K}_{\phi(pq)}]. \end{aligned}$$

Now, by Theorem 2.5, $(N_1, N_2, N_3, N_4) = (q-1, p^2-1, pq-p, p-1)$ and the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^2q})$ consists of the eigenvalue $q-1$ with multiplicity p^2-p-1, the eigenvalue p^2-1 with multiplicity $q-2$, the eigenvalue $p-1$ with multiplicity

$pq-p-q$, the eigenvalue $N_3+2r_3=2p-4+pq-p=pq+p-4$ with multiplicity $p-2$ and the remaining four eigenvalues are given by the matrix $C_Q(P_4)$

$$\begin{pmatrix} q-1 & \sqrt{(p^2-p)(q-1)} & 0 & 0 \\ \sqrt{(p^2-p)(q-1)} & p^2-1 & \sqrt{(p-1)(q-1)} & 0 \\ 0 & \sqrt{(p-1)(q-1)} & pq+p-4 & b \\ 0 & 0 & b & p-1 \end{pmatrix},$$

where $b=\sqrt{(p-1)(pq-p-q+1)}$. □

PROPOSITION 3.4. *The signless Laplacian spectrum of* $\Gamma(\mathbb{Z}_{(pq)^2})$ *is*

$$\begin{aligned}&\{(p-1)^{[\phi(pq^2)-1]},(p^2-1)^{[\phi(q^2)-1]},(q-1)^{[\phi(p^2q)-1]},(q^2-1)^{[\phi(p^2)-1]},\\&(p(q-1)-2)^{[\phi(pq)-1]},((q-1)(pq+1)+p-3)^{[\phi(p)]},\\&((p-1)(pq+q)+q-3)^{[\phi(q)-1]}\}\end{aligned}$$

and the zeros of the characteristic polynomial of the matrix $C_Q(G_7)$ *in* (3).

Proof. Let $n=(pq)^2$, where p and q $(p<q)$ are distinct primes. Since proper divisors of n are p,p^2,q,q^2,pq,pq^2,p^2q, so $\Upsilon_{(pq)^2}$ is the graph $G_7: q\sim p^2q\sim q^2\sim p^2\sim pq^2\sim p,\ p^2q\sim pq\sim pq^2\sim p^2q$. By Lemma 2.4, we have

$$\begin{aligned}\Gamma(\mathbb{Z}_{(pq)^2})=&\Upsilon_{(pq)^2}[\Gamma(A_q),\Gamma(A_{p^2q}),\Gamma(A_{q^2}),\Gamma(A_{p^2}),\Gamma(A_{pq^2}),\Gamma(A_p),\Gamma(A_{pq})]\\=&G_7[\overline{K}_{\phi(p^2q)},K_{\phi(q)},\overline{K}_{\phi(p^2)},\overline{K}_{\phi(q^2)},K_{\phi(p)},\overline{K}_{\phi(pq^2)},K_{\phi(pq)}].\end{aligned}$$

We name the vertices in G_7 according to the proper divisor sequence so that $n_1=\phi(pq^2),\ n_2=\phi(q^2),\ n_3=\phi(p^2q),\ n_4=\phi(p^2),\ n_5=\phi(pq),\ n_6=\phi(p)$ and $n_7=\phi(q)$. Also, we have

$$\begin{aligned}(2)\qquad &\big(N_1,N_2,N_3,N_4,N_5,N_6,N_7\big)\\&=\big(p-1,p^2-1,q-1,q^2-1,p+q-2,p(q^2-1),(p-1)(pq-p)\big).\end{aligned}$$

By theorem 2.5, the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_{(pq)^2})$ consists of the eigenvalue $N_1=p-1$ with multiplicity $\phi(pq^2)-1$, the eigenvalue $N_2=p^2-1$ with multiplicity $\phi(q^2)-1$, the eigenvalue $N_3=q-1$ with multiplicity $\phi(p^2q)-1$, the eigenvalue $N_4=q^2-1$ with multiplicity $\phi(p^2)-1$, the eigenvalues $N_5+\phi(pq)-2=pq-3$ with multiplicity $\phi(pq)-1$, the eigenvalue $N_6+\phi(p)-2=pq^2-3$ with multiplicity $\phi(p)-1$, the eigenvalue $N_7+\phi(q)-2=(p-1)(pq+q)+q-3$ with multiplicity $\phi(q)-1$ and the remaining seven eigenvalues are the eigenvalues of matrix $C_Q(G_7)$ given in (3).

$$(3)\qquad \begin{pmatrix} N_1 & 0 & 0 & 0 & 0 & \sqrt{n_1n_6} & 0 \\ 0 & N_2 & 0 & \sqrt{n_2n_6} & 0 & \sqrt{n_2n_6} & 0 \\ 0 & 0 & N_3 & 0 & 0 & 0 & \sqrt{n_3n_7} \\ 0 & \sqrt{n_2n_4} & 0 & N_4 & 0 & 0 & \sqrt{n_4n_7} \\ 0 & 0 & 0 & 0 & 2r_5+N_5 & \sqrt{n_5n_6} & \sqrt{n_5n_7} \\ \sqrt{n_1n_6} & \sqrt{n_2n_6} & 0 & 0 & \sqrt{n_5n_6} & 2r_6+N_6 & \sqrt{n_6n_7} \\ 0 & 0 & \sqrt{n_3n_7} & \sqrt{n_4n_7} & \sqrt{n_5n_7} & \sqrt{n_6n_7} & 2r_6+N_7 \end{pmatrix},$$

where, $2r_5+N_5=2(p-1)(q-1)+q-3,\ 2r_6+N_6=2(p-2)+(q-1)(pq+1),$ and $2r_7+N_7=2(q-1)+(p-1)(pq+q)$. □

Now, we determine the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_{p^z})$, where p is prime and z is positive integer.

THEOREM 3.5. *Let $n = p^z$ where $p > 2$ is prime and $z \geq 2$ is a positive integer. Then the following hold.*

(i) *If $z = 2$, then the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ is*

$$\{2p-4, (p-3)^{[p-2]}\}.$$

(ii) *If $n = p^{2m}$ for some positive integer $m \geq 2$, then the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ is*

$$\Big\{(p-1)^{\left[\phi\left(p^{2m-1}\right)-1\right]}, (p^2-1)^{\left[\phi\left(p^{2m-2}\right)-1\right]}, \dots, (p^{m-2}-1)^{\left[\phi\left(p^{m+2}\right)-1\right]},$$
$$(p^{m-1}-1)^{\left[\phi\left(p^{m+1}\right)-1\right]}\Big\} \bigcup \Big\{(p^m-3)^{[\phi(p^m)-1]}, (p^{m+1}-3)^{\left[\phi\left(p^{m-1}\right)-1\right]},$$
$$\dots, (p^{2m-2}-3)^{\left[\phi\left(p^2\right)-1\right]}, (p^{2m-1}-3)^{[\phi(p)-1]}\Big\}$$

and the remaining signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ are the zeros of the characteristic polynomial of the matrix given in (4).

(iii) *If $n = p^{2m+1}$ for some positive integer $m \geq 2$, then the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ is*

$$\Big\{(p-1)^{\left[\phi\left(p^{2m}\right)-1\right]}, (p^2-1)^{\left[\phi\left(p^{2m-1}\right)-1\right]}, \dots, (p^{m-1}-1)^{\left[\phi\left(p^{m+2}\right)-1\right]},$$
$$(p^m-1)^{\left[\phi\left(p^{m+1}\right)-1\right]}\Big\} \bigcup \Big\{(p^{m+1}-3)^{[\phi(p^m)-1]}, (p^{m+2}-3)^{\left[\phi\left(p^{m-1}\right)-1\right]},$$
$$\dots, (p^{2m-1}-3)^{\left[\phi\left(p^2\right)-1\right]}, (p^{2m}-3)^{[\phi(p)-1]}\Big\},$$

and the remaining signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ are the zeros of the characteristic polynomial of the matrix given in (5).

Proof. **(i)**. Since $\Gamma(\mathbb{Z}_{p^2}) = \Gamma(A_p)$ is the complete graph K_{p-1}, the result follows for $p > 2$.

(ii). Let z be even, that is, $z = 2m$, for some positive integer $m \geq 2$. Then the proper divisors of n are $p, p^2, \dots, p^{2m-1}$. We observe that the vertex p^i is adjacent to the vertex p^j in $\Upsilon_{p^{2m}}$, for each $j \geq 2m-i$ with $1 \leq i \leq 2m-1$ and $i \neq j$. For $i = 1, 2, \dots, 2m-2, 2m-1$, it is easy to see that $N_i = \sum\limits_{i=1}^{m-1} \phi(p^i)$. Using the fact that $\sum\limits_{i=1}^{r} \phi(p^r) = p^r - 1$, we have

$$\Big(N_1, N_2, \dots, N_{m-2}, N_{m-1}\Big) = \Big(p-1, p^2-1, \dots, p^{m-2}-1, p^{m-1}-1\Big).$$

Similarly, for $i = m, m+1, \dots, 2m-2, 2m-1$, we have

$$N_i = \sum_{j=1}^{i} \phi(p^j) - \phi(p^{2m-i}) = p^i - 1 - \phi(p^{2m-i}).$$

So,

$$\Big(N_m, N_{m+1}, \dots, N_{2m-2}, N_{2m-1}\Big) = \Big(p^{m-1}-1, p^{m+1}-1-p^{m-1}+p^{m-2}, \dots, p^{2m-2}-1-p^2+p, p^{2m-1}-p\Big)$$

Since n does not divide $(p^i)^2$, for $i = 1, 2, \dots, m-1$, therefore $G_i = \overline{K}_{\phi(p^{2m-i})}$ for $i = 1, 2, 3, \dots, m-1$ and $G_i = K_{\phi(p^{2m-i})}$ for $i = m, m+1, \dots, 2m-2, 2m-1$. This implies that $2r_i + N_i = p^i - 1$ for $i = 1, 2 \dots, m-1$, and $2r_i + N_i = p^i + \phi(p^{2m-i}) - 3$ for $i = m, \dots, 2m-2, 2m-1$. Also, order of G_i's are $n_i = \phi(p^{2m-i})$. Thus, by Theorem 2.5, we have

$$\begin{aligned} Spec_Q(\Gamma(\mathbb{Z}_n)) =& \Big\{(p-1)^{\left[\phi\left(p^{2m-1}\right)-1\right]}, (p^2-1)^{\left[\phi\left(p^{2m-2}\right)-1\right]}, \dots, \\ & (p^{m-2}-1)^{\left[\phi\left(p^{m+2}\right)-1\right]}, (p^{m-1}-1)^{\left[\phi\left(p^{m+1}\right)-1\right]}\Big\} \\ & \bigcup \left\{ \bigcup_{i=m}^{2m-1} \Big(N_i + \Big(Spec\left(K_{\phi(p^{2m-i})}\right) \setminus \{2r_i\}\Big)\Big) \right\} \end{aligned}$$

and the eigenvalues of matrix (4).

$$\begin{pmatrix} A_m & & B_{m\times(m-1)} & & \\ & c_{m+1} & \cdots & a_{m+1,2m-2} & a_{m+1,2m-1} \\ B^T & \vdots & \ddots & \vdots & \vdots \\ & a_{2m-2,m+1} & \cdots & c_{2m-2} & a_{2m-2,2m-1} \\ & a_{2m-1,m+1} & \cdots & a_{2m-1,2m-2} & c_{2m-1} \end{pmatrix}, \tag{4}$$

where $A_m = diag(N_1, N_2, \dots, N_{m-1}, c_m)$,

$$B = \begin{pmatrix} 0 & \dots & 0 & a_{1,2m-1} \\ 0 & \dots & a_{2,2m-2} & a_{2,2m-1} \\ \vdots & \ddots & \vdots & \vdots \\ a_{m-1,m+1} & \dots & a_{m-1,2m-2} & a_{m-1,2m-1} \\ a_{m,m+1} & \dots & a_{m,2m-2} & a_{m,2m-1} \end{pmatrix}$$

and $a_{i,j} = a_{j,i} = \sqrt{n_i n_j}$, for $1 \le i, j \le 2m-1$, $c_i = 2r_i + N_i$, for $i = m, m+1, \dots, 2m-1$.

Since the signless Laplacian spectrum of $K_{\phi(p^{2m-i})}$ is

$$\Big\{2\phi(p^{2m-i}) - 2, (\phi(p^{2m-i}) - 2)^{\phi(p^{2m-i})-1}\Big\}$$

and using $N_i = p^i - 1 - \phi(p^{2m-i})$ for $i = m, \dots, 2m-1$, we can easily see that

$$\begin{aligned} \bigcup_{i=m}^{2m-1} \Big(N_i + \Big(Spec\left(K_{\phi(p^{2m-i})}\right) \setminus \{2r_i\}\Big)\Big) =& \Big\{(p^m-3)^{[\phi(p^m)-1]}, \\ & (p^{m+1}-3)^{\left[\phi\left(p^{m-1}\right)-1\right]}, \dots, (p^{2m-2}-3)^{\left[\phi\left(p^2\right)-1\right]}, (p^{2m-1}-3)^{[\phi(p)-1]}\Big\}. \end{aligned}$$

(iii). Let $n = 2m+1$ be odd, where $m \ge 2$ is a positive integer. The proper divisors of n are $p, p^2, \dots, p^{2m}$. We observe that the vertex p^i is adjacent to the vertex p^j in

$\Upsilon_{p^{2m}}$ for each $j \geq 2m+1-i$ with $1 \leq i \leq 2m$ and $i \neq j$. For $i = 1, 2, \ldots, m-1, m$, it can be easily verified that $N_i = \sum\limits_{i=1}^{m} \phi(p^i)$, and using the fact that $\sum\limits_{i=1}^{r} \phi(p^r) = p^r - 1$, we have

$$\Big(N_1, N_2, \ldots, N_{m-1}, N_m\Big) = \Big(p-1, p^2-1, \ldots, p^{m-1}-1, p^m-1\Big).$$

For $i = m+1, m+2, \ldots, 2m-1, 2m$, we have

$$N_i = \sum_{j=1}^{i} \phi(p^j) - \phi(p^{2m+1-i}) = p^i - 1 - \phi(p^{2m+1-i}).$$

This further implies that

$$\Big(N_{m+1}, N_{m+2}, \ldots, N_{2m-1}, N_{2m}\Big) = \Big(p^{m+1} - 1 - p^m + p^{m-1},$$
$$p^{m+2} - 1 - p^{m-1} + p^{m-2}, \ldots, p^{2m-1} - 1 - p^2 + p, p^{2m} - p\Big).$$

Also $G_i = \overline{K}_{\phi(p^{2m+1-i})}$ for $i = 1, 2, 3, \ldots, m$ and $G_i = K_{\phi(p^{2m+1-i})}$ for $i = m+1, \ldots, 2m-1, 2m$, which implies that $r_i + n_i = p^i - 1$ for $i = 1, 2, 3, \ldots, m$ and $2r_i + N_i = 2\phi(p^{2m+1-i}) - 2 + N_i = p^i + \phi(p^{2m+1-i}) - 3$ for $i = m+1, \ldots, 2m-1, 2m$. Thus, order of G_i's are $n_i = \phi(p^{2m+1-i})$. Therefore, by Theorem 2.5, we have

$$\begin{aligned} Spec_Q(\Gamma(\mathbb{Z}_n)) = &\Big\{(p-1)^{\left[\phi\left(p^{2m}\right)-1\right]}, (p^2-1)^{\left[\phi\left(p^{2m-1}\right)-1\right]}, \ldots, \\ &(p^{m-1}-1)^{\left[\phi\left(p^{m+2}\right)-1\right]}, (p^m-1)^{\left[\phi\left(p^{m+1}\right)-1\right]}\Big\} \\ &\bigcup \Big\{\bigcup_{i=m+1}^{2m} \Big(N_i + \Big(Spec\left(K_{\phi(p^{2m+1-i})}\right) \setminus \{2r_i\}\Big)\Big)\Big\}, \end{aligned}$$

and the eigenvalues of the following matrix

$$(5) \qquad \begin{pmatrix} A_{m+1} & & B_{(m+1)\times m} & & \\ & c_{m+2} & \cdots & a_{m+2,2m-1} & a_{m+2,2m} \\ B^T & \vdots & \ddots & \vdots & \vdots \\ & a_{2m-1,m+2} & \cdots & c_{2m-1} & a_{2m-1,2m} \\ & a_{2m,m+2} & \cdots & a_{2m,2m-1} & c_{2m} \end{pmatrix},$$

where $A_m = diag(N_1, N_2, \ldots, N_m, c_{m+1})$,

$$B = \begin{pmatrix} 0 & \ldots & 0 & a_{1,2m} \\ 0 & \ldots & a_{2,2m-1} & a_{2,2m} \\ \vdots & \ddots & \vdots & \vdots \\ a_{m,m+1} & \ldots & a_{m,2m-1} & a_{m,2m} \\ a_{m+1,m+1} & \ldots & a_{m+1,2m-1} & a_{m+1,2m} \end{pmatrix}$$

and $a_{i,j} = a_{j,i} = \sqrt{n_i n_j}$, for $1 \leq i, j \leq 2m$, $c_i = 2r_i + N_i$, for $i = m+1, m+2 \ldots, 2m$.

The signless Laplacian spectrum of $K_{\phi(p^{2m+1-i})}$ is

$$\{2\phi(p^{2m+1-i}) - 2, (\phi(p^{2m+1-i}) - 2)^{\phi(p^{2m+1-i})-1}\}.$$

Using $N_i = p^i - 1 - \phi(p^{2m+1-i})$ for $i = m, \dots, 2m-1$, we can easily verify that

$$\bigcup_{i=m+1}^{2m} \left(N_i + \left(Spec(K_{\phi(p^{2m+1-i})}) \setminus \{2r_i\}\right)\right) = \Big\{(p^{m+1}-3)^{[\phi(p^m)-1]},$$

$$(p^{m+2}-3)^{\left[\phi\left(p^{m-1}\right)-1\right]}, \dots, (p^{2m-1}-3)^{\left[\phi\left(p^2\right)-1\right]}, (p^{2m}-3)^{[\phi(p)-1]}\Big\}.$$

This completes the proof in both the cases.

□

The next two corollaries follow from Theorem 3.5 for particular values of n. These help in showing that $\Gamma(\mathbb{Z}_{p^z}), z > 2$ is not in general signless Laplacian integral.

Corollary 3.6. *If $n = p^3$, then the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ is*

$$\left\{(p-1)^{[p^2-p-1]}, (p^2-3)^{[p-2]}, \frac{1}{2}\left(p^2-3 \pm \sqrt{p^4-6p^2+8p+1}\right)\right\}.$$

Proof. Since proper divisors of n are p and p^2, therefore Υ_n is $K_2 : p \sim p^2$. By Lemma 2.4, we have

$$\Gamma(\mathbb{Z}_{p^3}) = \Upsilon_{p^3}[\Gamma(A_p), \Gamma(A_{p^2})] = K_2[\overline{K}_{\phi(p^2)}, \overline{K}_{\phi(p)}] = \overline{K}_{p(p-1)} \nabla K_{p-1}.$$

This implies that $\Gamma(\mathbb{Z}_{p^3})$ is a complete split graph of order $p^2 - 1$, having an independent set of cardinality $p(p-1)$ and a clique of size $p-1$. By Theorem 3.5, we have $(N_1, N_2) = (p-1, p^2-p)$, and

$$C_Q(K_2) = \begin{pmatrix} p-1 & \sqrt{(p-1)(p^2-p)} \\ \sqrt{(p-1)(p^2-p)} & p^2+p-2 \end{pmatrix}. \tag{6}$$

As $r_1 = 0$, so the signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$ consists of the eigenvalue $N_1 = p-1$ with multiplicity $n_1 - 1 = p^2 - p - 1$, the eigenvalue $N_2 + (Spec(K_{p-1}) \setminus \{2(p-2)\}) = p^2 - p + p - 3$ with multiplicity $p-2$ and the remaining two signless Laplacian eigenvalues are the zeros of the characteristic polynomial of the matrix in (6). □

Corollary 3.7. *The signless Laplacian spectrum of $\Gamma(\mathbb{Z}_n)$, where $n = p^4$ is*

$$\left\{(p-1)^{[p^3-p^2-1]}, (p^3-3)^{[p-2]}, (p^2-3)^{[p^2-p-1]}, x_1, x_2, x_3, \right\}$$

where $x_1 \geq x_2 \geq x_3$ are the zeros of the characteristic polynomial of the matrix $C_Q(P_3)$.

Proof. As proper divisors of n are p, p^2 and p^3, so Υ_n is $P_3 : p \sim p^3 \sim p^2$. By Lemmas 2.2, 2.3 and 2.4, we have $\Gamma(A_p) = \overline{K}_{\phi(p^3)} = \overline{K}_{p^2(p-1)}, \Gamma(A_{p^2}) = K_{\phi(p^2)} = K_{p(p-1)}$ and $\Gamma(A_{p^3}) = K_{\phi(p)} = K_{p-1}$. Therefore

$$\begin{aligned}\Gamma(\mathbb{Z}_{p^4}) = \Upsilon_{p^3}[\Gamma(A_p), \Gamma(A_{p^3}), \Gamma(A_{p^2})] &= P_3[\overline{K}_{p^2(p-1)}, K_{p-1}, K_{p(p-1)}] \\ &= K_{p-1} \nabla (\overline{K}_{p^2(p-1)} \cup K_{p(p-1)}).\end{aligned}$$

Thus, by Theorem 3.5, we have $(N_1, N_2, N_3) = (p-1, p^3-p, p-1)$ and

$$C_Q(P_3) = \begin{pmatrix} p-1 & \sqrt{(p-1)(p^3-p^2)} & 0 \\ \sqrt{(p-1)(p^3-p^2)} & p^3+p-4 & \sqrt{(p-1)(p^2-p)} \\ 0 & \sqrt{(p-1)(p^2-p)} & 2p^2-p-3 \end{pmatrix}.$$

Now, by Theorem 3.5, it is clear that the signless Laplacian eigenvalues are as given in the statement. □

A graph G is said to an signless Laplacian integral if all signless Laplacian eigenvalues are integers. The next theorem gives a necessary and sufficient condition for a zero divisor graph $\Gamma(\mathbb{Z}_n)$ to be signless Laplacian integral.

THEOREM 3.8. *The zero divisor graph $\Gamma(\mathbb{Z}_n)$ is signless Laplacian integral if and only if the matrix $C_Q(H)$ of Theorem 2.5 is integral.*

As shown in [6], $\Gamma(\mathbb{Z}_n)$ is Laplacian integral when $n = p^z$ for every prime p and positive integer $z \geq 2$. While the answer is in negative for signless Laplacian matrix, however in general, $\Gamma(\mathbb{Z}_n)$ is integral for certain values of n.

THEOREM 3.9. *$\Gamma(\mathbb{Z}_n)$ is signless Laplacian integral if and only if $n \in \{p^2, 4q, pq\}$, where p and q are primes. Further in such cases, $\Gamma(\mathbb{Z}_n)$ is either a complete graph or a complete bipartite graph.*

Proof. If n is either prime power or product of two distinct primes, then by Lemma 3.2 and Theorem 3.5 part (i), we see that signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_n)$ are integers. Also, by Proposition 3.3, for $p = 2^2$, it is clear that $\Gamma(\mathbb{Z}_{4q})$ is the complete bipartite graph and its signless Laplacian eigenvalues are integers. Conversely, if n is a product of three primes, then by Example 3.1, we get at least 6 non integer signless Laplacian eigenvalues of $\Gamma(\mathbb{Z}_{pqr})$, where $p < q < r$ are primes. More generally, if $n = p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r}$, where $r, n_1, \ldots, n_r$ are non-negative integers and $p_i,\ i = 1, 2, \ldots, r$ are primes, then for $r \geq 3$, $\Gamma(\mathbb{Z}_n)$ contains the triangle $\left(\frac{n}{(p_3)^{n_3}}\right) \sim \left(\frac{n}{(p_2)^{n_2}}\right) \sim \left(\frac{n}{(p_1)^{n_1}}\right) \sim \left(\frac{n}{(p_3)^{n_3}}\right)$. This implies that $\Gamma(\mathbb{Z}_n)$ is not complete bipartite and cannot be signless Laplacian integral. Similarly, $\Gamma(\mathbb{Z}_{p_1^{n_1} p_2^{n_2}}),\ n_1, n_2 \geq 2$, contains the triangle $p_1^{n_1-1} p_2^{n_2} \sim p_1 p_2^{n_2-1} \sim p_1^{n_1} p_2^{n_2-1} \sim p_1^{n_1-1} p_1^{n_2}$. Therefore, its zero divisor graph is not complete bipartite. Again, for $n = p^2 q$ or $n = pq^2$, by Proposition 3.4, $C_Q(\Upsilon_n)$ is not integral. For $n = p^3, p^4$, by Corollaries 3.6 and 3.7, we can verify that the eigenvalues of $C_Q(\Upsilon_n)$ are not integers. For $n = p^{n_1},\ n_1 \geq 5$, we observe that $\Gamma(\mathbb{Z}_n)$ contains the triangle $p^{n_1-3} \sim p^{n_1-2} \sim p^{n_1-1} \sim p^{n_1-3}$ and is not bipartite, so its all signless Laplacian eigenvalues are not integers. Therefore, $\Gamma(\mathbb{Z}_n)$ is signless Laplacian integral only for $n = p^2, pq, 4q$, where p and q $(p < q)$ are primes. □

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