ITERATES OF WEIGHTED BEREZIN TRANSFORM
UNDER INVARIANT MEASURE IN THE UNIT BALL

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Abstract. We focus on the interactions of the weighted Berezin transform $T_\alpha$ on $L^p(\tau)$, where $\tau$ is the invariant measure on the complex unit ball $B_n$. Iterations of $T_\alpha$ on $L^1_R(\tau)$ the space of radial integrable functions played important roles in proving $M$-harmonicity of bounded functions with invariant mean value property. Here, we introduce more properties on iterations of $T_\alpha$ on $L^1_R(\tau)$ and observe differences between the iterations of $T_\alpha$ on $L^1(\tau)$ and $L^p(\tau)$ for $1 < p < \infty$.

1. Introduction

Let $B_n$ be the unit ball of $\mathbb{C}^n$ with norm $|z| = (\langle z, z \rangle)^{1/2}$ where $\langle \ , \ \rangle$ is the Hermitian inner product, and let $\nu$ be the Lebesgue measure on $\mathbb{C}^n$ normalized to $\nu(B_n) = 1$.

For $\alpha > -1$, we define a positive measure $\nu_\alpha$ on $B_n$ by

d$\nu_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha \, d\nu(z),$

where

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)}.$$


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is a normalizing constant so that \( \nu_\alpha(B_n) = 1 \). For such \( \alpha \) and \( f \in L^1(B_n, \nu_\alpha) \), the weighted Berezin transform \( T_\alpha f \) on \( B_n \) is defined by

\[
(T_\alpha f)(z) = \int_{B_n} f(\varphi_z(w)) \, d\nu_\alpha(w) \quad \text{for} \quad z \in B_n,
\]

where \( \varphi_\alpha \in \text{Aut}(B_n) \) is the canonical automorphism given by

\[
\varphi_\alpha(z) = \frac{a - Pz - (1 - |a|^2)^{1/2}Qz}{1 - \langle z, a \rangle}
\]

where \( P \) is the projection into the space spanned by \( a \in B_n \) and \( Q_z = z - Pz \). Equivalently we can write

\[
(1.1) \quad (T_\alpha f)(z) = \int_{B_n} f(w) \frac{(1 - |z|^2)^{n+1+\alpha}}{|1 - \langle z, w \rangle|^{2n+2+2\alpha}} \, d\nu_\alpha(w).
\]

The invariant Laplacian \( \hat{\Delta} \) is defined for \( f \in C^2(B_n) \) by

\[
(\hat{\Delta} f)(z) = \Delta(f \circ \varphi_z)(0).
\]

The \( \mathcal{M} \)-harmonic functions in \( B_n \) are those for which \( \hat{\Delta} f = 0 \). If a function \( f \in L^1(B_n, \nu_\alpha) \) is \( \mathcal{M} \)-harmonic, then \( f \circ \psi \) is also \( \mathcal{M} \)-harmonic for every \( \psi \in \text{Aut}(B_n) \). Thus for every given \( \alpha > -1 \), bounded \( \mathcal{M} \)-harmonic function \( f \) satisfies an invariant mean value property

\[
\int_{B_n} (f \circ \psi) \, d\nu_\alpha = f(\psi(0)) \quad \text{for every} \quad \psi \in \text{Aut}(B_n),
\]

which is equivalent to saying that \( (T_\alpha f)(z) = f(z) \) for every \( z \in B_n \).

Conversely, Furstenberg ([2],[3]) provided abstract proofs that on any dimensional symmetric domain, a bounded function which is invariant under a weighted Berezin transform is harmonic with respect to the intrinsic metric, which implies that \( f \in L^\infty(B_n) \) satisfying \( T_\alpha f = f \) is \( \mathcal{M} \)-harmonic. In 1993, Ahern, Flores and Rudin ([1]) gave an analytic proof that \( f \in L^\infty(B_n) \) satisfying \( T_0 f = f \) is \( \mathcal{M} \)-harmonic, and \( f \in L^1(B_n, \nu_\alpha) \) satisfying \( T_0 f = f \) has to be \( \mathcal{M} \)-harmonic if and only if \( n \leq 11 \).

To mention some previous works related to weighted Berezin transform and harmonicity, in 2008 ([4]) the author proved that for any \( 1 \leq p < \infty \) and \( c_1, c_2 > -1 \), a function \( f \in L^p(\nu_{c_1} \times \nu_{c_2}) \) on the bidisc which is invariant under the weighted Berezin transform; \( T_{c_1,c_2} f = f \) needs not be \( 2 \)-harmonic. Properties of such functions on the bidisc is mentioned in the recent work [6]. And in 2010, the author([5]) gave
an analytic proof that for every given $\alpha > -1$, $f \in L^\infty(B_n)$ satisfying $T_\alpha f = f$ is $M$-harmonic. In [5], the author used the spectral theory and iteration of $T_\alpha$ on the commutative Banach algebra $L^1_k(\tau)$, the space of all radial function $f$ on $B_n$ integrable with respect to the invariant measure $\tau$ defined by $d\tau(z) = (1 - |z|^2)^{-n-1} d\nu(z)$.

This paper, we focus on the iteration of the weighted Berezin transform $T_\alpha$ on $L^p(B_n, \tau)$, which has not been done any previous researches. Our motivation comes from Lemma 2.1 of [5] which plays a crucial role in the proof of the main result of that paper. Here, we develop further theory and results which follow from Lemma 2.1 of [5] and observe the major difference between the iterations of $T_\alpha$ on $L^1(\tau)$ and $L^p(\tau)$ for $1 < p < \infty$.

In section 2, we introduce some preliminaries on weighted Berezin transform $T_\alpha$ and invariant measure $\tau$ on $B_n$. In section 3, we propose a lemma and three new propositions about iterations of $T_\alpha$ on $L^p(\tau)$ for $1 \leq p < \infty$. Throughout the paper $\alpha$ is an arbitrarily given real number with $\alpha > -1$.

2. Preliminaries

Here, we introduce some preliminaries on weighted Berezin transform $T_\alpha$ and invariant measure $\tau$ on $B_n$ details of which are explained in [5] and [7]. We focus on the invariant measure $\tau$ on $B_n$ defined by $d\tau(z) = (1 - |z|^2)^{-n-1} d\nu(z)$, which satisfies

$$\int_{B_n} f \ d\tau = \int_{B_n} (f \circ \psi) \ d\tau$$

for every $f \in L^1(\tau)$ and $\psi \in \text{Aut}(B_n)$. Even though $\tau$ is not a finite measure on $B_n$ so that a non-zero constant does not belong to $L^1(\tau)$, $T_\alpha$ on $L^\infty(B_n)$ is the adjoint of $T_\alpha$ on $L^1(\tau)$ in the sense that

$$\int_{B_n} (T_\alpha f) \cdot g \ d\tau = \int_{B_n} f \cdot (T_\alpha g) \ d\tau$$

for $f \in L^1(\tau)$ and $g \in L^\infty(B_n)$. Since $L^\infty(B_n) = L^1(\tau)^*$, the spectrum of $T_\alpha$ on $L^\infty(B_n)$ is the same as the spectrum of $T_\alpha$ on $L^1(B_n, \tau)$. Moreover from the expression (1.1), we can easily see that the operator $T_\alpha$ on $L^\infty(B_n)$ is a positive contraction, which means that $T_\alpha$ is also a positive contraction on $L^1(B_n, \tau)$ so that we can iterate $T_\alpha$ on $L^1(\tau)$.
For $1 \leq p \leq \infty$, we denote $L^p_R(\tau)$ as the subspace of $L^p(B_n, \tau)$ which consists of radial functions, which means that $f \in L^p_R(\tau)$ if and only if $f \in L^p(\tau)$ and $f(z) = f(|z|)$ for all $z \in B_n$. In this case, $T_\alpha$ is a contraction on $L^1_R(\tau)$ which is a commutative Banach algebra under the convolution
\[(f * g)(z) = \int_{B_n} f(\varphi_z(w))g(w) \, d\tau(w)\]
for $f, g \in L^1_R(\tau)$. Hence if $f \in L^1_R(\tau)$, we can write $T_\alpha f = f * h_\alpha$ where $h_\alpha(z) = c_\alpha(1 - |z|^2)^{n+1+\alpha} \in L^1_R(\tau)$.

In [5], the key step to the proof of the main theorem is Lemma 2.1 which states that
\[(2.2) \quad \lim_{k \to \infty} \| T^k_\alpha (I - T_\alpha) \| = 0 \quad \text{on} \quad L^1_R(\tau).\]

We start section 3 by introducing a lemma on iteration of $T_\alpha$ on $L^1_R(\tau)$ which is a direct result of (2.2). Then we extend this lemma to a more general case.

3. The iterations of $T_\alpha$

We start this section by introducing Lemma 3.1 on iteration of $T_\alpha$ on $L^1_R(\tau)$ which is an application of (2.2).

**Lemma 3.1.** For $f \in L^1_R(\tau)$, we have
\[
\lim_{k \to \infty} \int_{B_n} | T^k_\alpha f | \, d\tau = 0 \quad \text{if and only if} \quad \int_{B_n} f \, d\tau = 0.
\]

**Proof.** Let $f \in L^1_R(\tau)$. By putting $g = 1$ in (2.1) we get
\[
\int_{B_n} T^k_\alpha f \, d\tau = \int_{B_n} f \, d\tau \quad \text{for every} \quad k \geq 0.
\]
Hence
\[
\lim_{k \to \infty} \int_{B_n} | T^k f | \, d\tau = 0 \quad \text{implies} \quad \int_{B_n} f \, d\tau = 0.
\]
To prove the converse, if we define
\[
D = \left\{ f \in L^1_R(\tau) \mid \int_{B_n} f \, d\tau = 0 \right\}.
\]
Then \((I - T)L^1_R(\tau) \subset E\). Now let \(\ell \in L^\infty_R(B_n)\) satisfy
\[
\int_{B_n} (f - T_\alpha f) \cdot \ell \, d\tau = 0 \quad \text{for every } f \in L^1_R(\tau).
\]
Then by (2.1)
\[
\int_{B_n} f \cdot (\ell - T_\alpha \ell) \, d\tau = 0 \quad \text{for every } f \in L^1_R(\tau).
\]
Hence \(T_\alpha \ell = \ell\), which means \(\ell\) is radial \(M\)-harmonic so that \(\ell\) is a constant. Hence we get
\[
\int_{B_n} g \cdot \ell \, d\tau = 0 \quad \text{for every } g \in D.
\]
By the Hahn-Banach theorem, this means \((I - T)L^1_R(\tau)\) is dense in \(D\).

Now from (2.2), we have
\[
\lim_{k \to \infty} \| T^k_\alpha (f - T_\alpha f) \|_{L^1(\tau)} = 0 \quad \text{for every } f \in L^1_R(\tau).
\]
Therefore, we conclude
\[
\lim_{k \to \infty} \int_{B_n} | T^k_\alpha f | \, d\tau = 0 \quad \text{for every } f \in D.
\]

Next proposition is a generalization of Lemma 3.1. Since non-zero constant does not belong to \(L^1(\tau)\), we can not simply apply Lemma 3.1 to the function \(f - \int_{B_n} f \, d\tau\).

**PROPOSITION 3.2.** If \(f \in L^1_R(\tau)\), then we have
\[
\lim_{k \to \infty} \int_{B_n} | T^k_\alpha f | \, d\tau = \left| \int_{B_n} f \, d\tau \right|.
\]

**Proof.** Let’s denote \(A = \{ \ell \in L^\infty_R(B_n) \mid \|\ell\|_\infty \leq 1 \}\), \(E_k = T^k_\alpha A\) and \(E = \bigcap_{k=1}^\infty E_k\). Then for \(f \in L^1_R(\tau)\),
\[
\int_{B_n} | T^k_\alpha f | \, d\tau = \sup \left\{ \left| \int_{B_n} (T^k_\alpha f) \cdot \ell \, d\tau \right| \mid \ell \in A \right\}
\]
\[
= \sup \left\{ \left| \int_{B_n} f \cdot (T^k_\alpha \ell) \, d\tau \right| \mid \ell \in A \right\}.
\]
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Hence

\[(3.1) \lim_{k \to \infty} \| T^k f \|_{L^1(\tau)} \geq \sup \left\{ \left| \int_{B_n} f \cdot h \, d\tau \right| \mid h \in E \right\}. \]

On the other hand, for every \( \varepsilon > 0 \) and \( k \geq 1 \) there exists \( h_k \in A \) with

\[
\| T^k f \|_{L^1(\tau)} \leq \left| \int_{B_n} (T^k f) \cdot h_k \, d\tau \right| + \varepsilon
\]

\[
= \left| \int_{B_n} f \cdot (T^k h_k) \, d\tau \right| + \varepsilon.
\]

Since \( E_k \) is weak * compact and \( E_k \downarrow E \), \( E \) is also weak * compact. If \( g \) is a weak* limit of a subsequence \( \{T^{k_j}_\alpha(h_{k_j})\} \) of \( \{T^k h_k\} \), then \( g \in E \) and

\[
\left| \int_{B_n} f \cdot g \, d\tau \right| = \lim_{j \to \infty} \left| \int_{B_n} f \cdot (T^{k_j}_\alpha h_{k_j}) \, d\tau \right| \\
\geq \lim_{j \to \infty} \| T^{k_j}_\alpha f \|_{L^1(\tau)} - \varepsilon.
\]

Hence we have

\[(3.2) \lim_{k \to \infty} \| T^k f \|_{L^1(\tau)} \leq \sup \left\{ \left| \int_{B_n} f \cdot h \, d\tau \right| \mid h \in E \right\}. \]

From (3.1), (3.2) we get

\[(3.3) \lim_{k \to \infty} \| T^k f \|_{L^1(\tau)} = \sup \left\{ \left| \int_{B_n} f \cdot h \, d\tau \right| \mid h \in E \right\}. \]

From (3.3) and Lemma 3.1, if \( u \in L^1_R(\tau) \) then for every \( h \in E \)

\[
\int_{B_n} f \, d\tau = 0 \quad \text{if and only if} \quad \int_{B_n} f \cdot h \, d\tau = 0.
\]

Therefore, we conclude

\[ E = \{ c \in \mathbb{C} \mid |c| \leq 1 \}, \]

and we can rewrite (3.3) as

\[
\lim_{k \to \infty} \| T^k f \|_{L^1(\tau)} = \sup \left\{ \left| \int_{B_n} cf \, d\tau \right| \mid |c| \leq 1 \right\} \\
= \left| \int_{B_n} f \, d\tau \right|.
\]

\[\square\]
Since $T_\alpha$ is a contraction on $L^1(\tau)$ and $L^\infty(B_n)$, it is also a contraction on $L^p(\tau)$ for $1 < p < \infty$. Next proposition says when $1 < p < \infty$, the iteration of $T_\alpha$ on $L^p(\tau)$ is much simpler in a way that

$$\lim_{k \to \infty} \| T_\alpha^k f \|_{L^p(\tau)} = 0 \quad \text{for every } f \in L^p(\tau).$$

**Proposition 3.3.** If $1 < p < \infty$ and $f \in L^p(\tau)$, then

$$\lim_{k \to \infty} \int_{B_n} | T_\alpha^k f |^p \, d\tau = 0.$$

**Proof.** Since $T_\alpha$ is a positive contraction on $L^p(\tau)$, by standard approximation, it is enough to prove the proposition when $f$ is a characteristic function $\chi_K$ for every compact subset $K$ of $B_n$.

First, we’ll show that $\lim_{k \to \infty} \| T_\alpha^k \chi_K \|_\infty = 0$. Choose $0 < r < 1$ such that $K \subset rB_n$, and define $u : [0, 1] \to \mathbb{R}$ by

$$u(t) = -1 \quad \text{for } 0 \leq t \leq r,$$

$$u(t) = \frac{t - 1}{1 - r} \quad \text{for } r \leq t \leq 1.$$

Then $v(z) = u(|z|)$ is subharmonic in $B_n$, which implies that $v \circ \varphi_a$ is subharmonic for each $a \in B_n$. Thus from the definition of $T_\alpha$ and submean value property, we get $T_\alpha v \geq v$. Since $T_\alpha$ is a positive operator, $\{T_\alpha^k v\}$ is increasing and uniformly bounded on $B_n$. Hence $\lim T_\alpha^k v = g$ exists and satisfies $T_\alpha g = g$. Since $g$ is bounded on $B_n$ satisfying $T_\alpha g = g$, $g$ is $\mathcal{M}$-harmonic. Thus we get $g = 0$ on $B_n$ since $g = 0$ on $\partial B_n$.

Therefore, by Dini’s theorem, $\{T_\alpha^k v\}$ converges uniformly to zero, which implies

$$\lim_{k \to \infty} \| T_\alpha^k \chi_K \|_\infty = 0.$$

since $T_\alpha^k v \leq -T_\alpha^k \chi_K \leq 0$.

Next, let $p = 1 + c$ for some $c > 0$. For a given $\varepsilon > 0$, we define $A_k = \{ z \in B_n \mid T_\alpha^k \chi_K > \varepsilon \}$. Then $\| T_\alpha^k \chi_K \|_\infty \leq 1$ for every $k$, and $A_k$ is empty for all $k$ sufficiently large.

Since

$$\int_{B_n} | T_\alpha^k \chi_K |^p \, d\tau = \int_{A_k} (T_\alpha^k \chi_K)(T_\alpha^k \chi_K)^c \, d\tau + \int_{B_n \setminus A_k} (T_\alpha^k \chi_K)(T_\alpha^k \chi_K)^c \, d\tau,$$

we have

$$\int_{B_n} | T_\alpha^k \chi_K |^p \, d\tau \leq \tau(A_k) + \tau(K)\varepsilon^c.$$

Therefore, we get the proof of the proposition by taking $k \to \infty$  \qed
Even though $T^k_\alpha f$ generally does not converges to zero in norm when $f \in L^1(\tau)$, next proposition implies that it converges pointwise to zero in $B_n$ and much more is true.

**Proposition 3.4.** If $f \in L^1(\tau)$ and $z \in B_n$, then

$$\sum_{k=0}^{\infty} |T^k_\alpha f(z)| < \infty.$$  

**Proof.** First, we prove that the function $u(z) = |z|^2 - 1$ satisfies $T_\alpha u > u$ on $B_n$.  

From the definition of weighted Berezin transform we get

$$(T_\alpha u)(z) = \int_{B_n} (|w|^2 - 1) \frac{(1 - |w|^2)^{n+\alpha}}{|1 - \langle z, w \rangle|^{2n+2+2\alpha}} \, d\nu_\alpha(w)$$

$$= -(1 - |z|^2)^{n+\alpha+1} \frac{\Gamma(n + \alpha + 1)}{n! \Gamma(\alpha + 1)} \int_{B_n} \frac{(1 - |w|^2)^{1+\alpha}}{|1 - \langle z, w \rangle|^{2n+2+2\alpha}} \, d\nu(w)$$

(3.4)

Using the binomial series identity

$$(1 - x)^{-\beta} = \sum_{k=0}^{\infty} \frac{\Gamma(k + \beta)}{k! \Gamma(\beta)} x^k$$

for $|x| < 1$, $\beta \geq 0$ and applying integration in polar coordinates (1.4.3 of [7]) together with Proposition 1.4.10 of [7], we get

$$\int_{B_n} \frac{(1 - |w|^2)^{1+\alpha}}{|1 - \langle z, w \rangle|^{2n+2+2\alpha}} \, d\nu(w) = \frac{n! \Gamma(\alpha + 2)}{\Gamma^2(n + \alpha + 1)} \sum_{k=0}^{\infty} \frac{\Gamma^2(n + k + \alpha + 1)}{k! \Gamma(k + n + \alpha + 2)} |z|^{2k}.$$  

Therefore, we have

$$(T_\alpha u)(z) = -(1 - |z|^2)^{n+\alpha+1} \frac{\alpha + 1}{\Gamma(n + \alpha + 1)} \sum_{k=0}^{\infty} \frac{\Gamma^2(n + k + \alpha + 1)}{k! \Gamma(k + n + \alpha + 2)} |z|^{2k}$$

$$> -(1 - |z|^2)^{n+\alpha+1} \sum_{k=0}^{\infty} \frac{\Gamma(n + k + \alpha + 1)}{k! \Gamma(k + n + \alpha + 2)} |z|^{2k}$$

(3.5)

$$= -(1 - |z|^2)^{n+\alpha+1} (1 - |z|^2)^{-n-\alpha} = u(z).$$

Next, since $u$ is a uniform limit of a sequence of functions on $C_c(B_n)$ and if $v \in C_c(B_n)$ then we can show exactly the same way as the proof of Proposition 3.3 that

$$\lim_{k \to \infty} \|T^k_\alpha v\|_{\infty} = 0.$$
Hence we get
\[
\lim_{k \to \infty} \| T^k_\alpha u \|_\infty = 0.
\]
Thus if we define \( g = T u - u \), then \( g > 0 \) and \( \|g\|_\infty \leq 2 \). Moreover,
\[
\sum_{k=0}^{m} T^k_\alpha g
\]
converges uniformly to \(-u\) as \( m \to \infty \). Combining this and (2.1), we get
\[
\int_{B_n} \left( \sum_{k=0}^{\infty} T^k_\alpha |f| \right) \cdot g \, d\tau = \int_{B_n} |f| \cdot \left( \sum_{k=0}^{\infty} T^k_\alpha g \right) \, d\tau
\]
\[
= \int_{B_n} |f| \cdot (-u) \, d\tau \leq \|f\|_{L^1(\tau)} \|u\|_\infty < \infty.
\]
Since \( g > 0 \), the proof is complete. \( \Box \)

References


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