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INEQUALITIES FOR THE DERIVATIVE OF POLYNOMIALS WITH RESTRICTED ZEROS

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Abstract. For a polynomial $P(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ of degree n having all its zeros in $|z| \leq k, k \geq 1$, it was shown by Rather and Dar [13] that

$$\max_{|z|=1} |P'(z)| \geq \frac{1}{1+k^n}\left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|}\right) \max_{|z|=1} |P(z)|.$$

In this paper, we shall obtain some sharp estimates, which not only refine the above inequality but also generalize some well known Turán-type inequalities.

1. Introduction and Statement of results

Let $\mathcal{P}_n$ denote the class of all algebraic polynomials of the form $P(z) = \sum_{j=o}^{n} a_j z^j$ of degree $n \geq 1$. It was shown by P. Turán [17] that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{1}$$

Equality in (1) holds for $P(z) = az^n + b$, $|a| = |b| = 1$.

As an extension of (1), Govil [8] proved that if $P \in \mathcal{P}_n$ and $P(z)$ has

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all its zeros in $|z| \leq k, k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{1+k^n} \max_{|z|=1} |P(z)|. \tag{2}$$

The result is sharp as shown by the polynomial $P(z) = z^n + k^n$.

By involving the minimum modulus of $P(z)$ on $|z| = 1$, Aziz and Dawood [2], proved under the hypothesis of inequality (1) that

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2} \left\{ \max_{|z|=1} |P(z)| + \min_{|z|=1} |P(z)| \right\}. \tag{3}$$

Equality in (3) holds for $P(z) = az^n + b$, $|a| = |b| = 1$.

Dubinin [7] obtained a refinement of (1) by involving some of the coefficients of polynomial $P \in \mathcal{P}_n$ in the bound of inequality (1). More precisely, proved that if all the zeros of the polynomial $P \in \mathcal{P}_n$ lie in $|z| \leq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{1}{2} \left(n + \frac{|a_n| - |a_0|}{|a_n| + |a_0|} \right) \max_{|z|=1} |P(z)|. \tag{4}$$

Rather and Dar [13] generalized this inequality and proved that if $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k, k \geq 1$, then

$$\max_{|z|=1} |P'(z)| \geq \frac{1}{1+k^n} \left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|} \right) \max_{|z|=1} |P(z)|. \tag{5}$$

The result is sharp and equality holds for $P(z) = z^n + k^n$.
In literature, there exist several generalizations and extensions of (1), (2), (3) and (4) (see [1]- [5], [10], [12]- [16]). In this paper, we are interested in estimating the lower bound for the maximum modulus of $P'(z)$ on $|z| = 1$ for $P \in \mathcal{P}_n$ not vanishing in the region $|z| > k$ where $k \geq 1$ and establish some refinements and generalizations of the inequalities (1), (2), (3), (4) and (5). We begin by proving the following refinement of inequality (5):

Theorem 1.1. *If all the zeros of polynomial $P \in \mathcal{P}_n$ of degree $n \geq 2$ lie in $|z| \leq k, k \geq 1$, then*

(6)
$$\max_{|z|=1} |P'(z)| \geq \frac{1}{1+k^n} \left(n + \frac{k^n|a_n| - |a_0|}{k^n|a_n| + |a_0|} \right) \left(\max_{|z|=1} |P(z)| + \frac{|a_{n-1}|\phi(k)}{k} \right) + |a_1|\psi(k),$$

where $\phi(k) = \left(\frac{k^{n}-1}{n} - \frac{k^{n-2}-1}{n-2}\right)$ *or* $\frac{(k-1)^2}{2}$ *and* $\psi(k) = (1 - 1/k^2)$ or $(1 - 1/k)$ *according as* $n > 2$ *or* $n = 2$.
The result is best possible and equality in (6) *holds for* $P(z) = z^n + k^n$.

Remark 1.2. Since $\phi(k)$ and $\psi(k)$ are non-negative, hence it clearly follows that inequality (6) refines inequality (5). Further for $k = 1$, inequality (6) reduces to inequality (4).

Theorem 1.3. *If all the zeros of polynomial* $P \in \mathcal{P}_n$ *of degree* $n \geq 2$ *lie in* $|z| \leq k$ *where* $k \geq 1$ *and* $m = \min_{|z|=k} |P(z)|$, *then for* $0 \leq l < 1$

$$\begin{aligned}\max_{|z|=1} |P'(z)| \geq{}& \frac{n}{1+k^n}\big(\max_{|z|=1} |P(z)| + lm\big) + \psi(k)|a_1| \\ &+ \frac{1}{k^n(1+k^n)}\left\{\left(\frac{k^n|a_n| - lm - |a_0|}{k^n|a_n| - lm + |a_0|}\right)\big(k^n \max_{|z|=1} |P(z)| - lm\big)\right. \\ &\left.+ k^{n-1}|a_{n-1}|\phi(k)\left(n + \frac{k^n|a_n| - lm - |a_0|}{k^n|a_n| - lm + |a_0|}\right)\right\},\end{aligned} \tag{7}$$

where $\phi(k)$ *and* $\psi(k)$ *are same as defined in Theorem 1.1.*

The result is sharp and equality in (7) holds for $P(z) = z^n + k^n$.

Remark 1.4. As before, it can be easily seen that Theorem 1.3 is a refinement of Theorem 1.1. Moreover, for $k = 1$, we get the following refinement of inequality (4).

Corollary 1.5. *If all the zeros of* $P \in \mathcal{P}_n$ *of degree* $n \geq 2$, *lie in* $|z| \leq 1$ *and* $m_1 = \min_{|z|=1} |P(z)|$, *then for* $0 \leq l < 1$

$$\max_{|z|=1} |P'(z)| \geq \frac{n}{2}\left\{\max_{|z|=1} |P(z)| + lm_1\right\} + \frac{1}{2}\left(\frac{|a_n| - lm_1 - |a_0|}{|a_n| - lm_1 + |a_0|}\right)\big(\max_{|z|=1} |P(z)| - lm_1\big), \tag{8}$$

The result is sharp and equality holds for $P(z) = (z^n + 1)$.

2. Lemmas

For the proof of these theorems, we need the following lemmas. The first Lemma is due to Erdös and Lax [9]

Lemma 2.1. *If* $P \in \mathcal{P}_n$ *does not vanish in* $|z| < 1$, *then*

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)|. \tag{9}$$

Next Lemma is a special case of a result due to Aziz and Rather [3,4].

LEMMA 2.2. *If $P \in \mathcal{P}_n$ and $P(z)$ has its all zeros in $|z| \leq 1$ and $Q(z) = z^n\overline{P(1/\overline{z})}$, then for $|z| = 1$,*

$$|Q'(z)| \leq |P'(z)|.$$

The following result is due to Frappier, Rahman and Ruscheweyh [6].

LEMMA 2.3. *If $P \in \mathcal{P}_n$ is a polynomial of degree $n \geq 1$, then for $R \geq 1$,*

$$\max_{|z|=R}|P(z)| \leq R^n \max_{|z|=1}|P(z)| - (R^n - R^{n-2})|P(0)| \qquad \text{if} \quad n > 1 \tag{10}$$

and

$$\max_{|z|=R}|P(z)| \leq R \max_{|z|=1}|P(z)| - (R - 1)|P(0)| \qquad \text{if} \quad n = 1. \tag{11}$$

From above lemma, we deduce:

LEMMA 2.4. *If $P \in \mathcal{P}_n = a_n\prod_{j=1}^{n}(z - z_j)$ is a polynomial of degree $n \geq 2$ having no zeros in $|z| < 1$, then for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and $R \geq 1$,*

$$\begin{aligned}\max_{|z|=R}|P(z)| \leq \frac{R^n+1}{2}\max_{|z|=1}|P(z)| - |\alpha|\frac{R^n-1}{2}\min_{|z|=1}|P(z)| \\ - \left(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right)|P'(0)| \qquad \text{if} \quad n > 2\end{aligned} \tag{12}$$

and

$$\begin{aligned}\max_{|z|=R}|P(z)| \leq \frac{R^2+1}{2}\max_{|z|=1}|P(z)| - |\alpha|\frac{R^2-1}{2}\min_{|z|=1}|P(z)| \\ - \frac{(R-1)^2}{2}|P'(0)| \qquad \text{if} \quad n = 2.\end{aligned} \tag{13}$$

Proof of Lemma 2.4. By hypothesis all the zeros of $P(z)$ lie in $|z| \geq 1$. Let $m = \min_{|z|=1}|P(z)|$, then $m \leq |P(z)|$ for $|z| = 1$. Applying Rouche's theorem, it follows that the polynomial $G(z) = P(z) + \alpha m z^n$ has all its zeros in $|z| \geq 1$ for every α with $|\alpha| < 1$ (this is trivially true for $m = 0$.) Now for each θ, $0 \leq \theta < 2\pi$, we have

$$G(Re^{i\theta}) - G(e^{i\theta}) = \int_1^R e^{i\theta}G'(te^{i\theta})dt. \tag{14}$$

This gives with the help of (10) of Lemma 2.3 and Lemma 2.1 for $n > 2$,

$$\begin{aligned}
&\left|G(Re^{i\theta}) - G(e^{i\theta})\right| \\
&\leq \int_1^R |G'(te^{i\theta})|dt \\
&\leq \frac{n}{2}\left(\int_1^R t^{n-1}dt\right)\max_{|z|=1}|G(z)| - \int_1^R \left(t^{n-1} - t^{n-3}\right)dt|G'(0)| \\
&= \frac{R^n - 1}{2}\max_{|z|=1}|G(z)| - \left(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right)|P'(0)|,
\end{aligned}$$

so that for $n > 2$ and $0 \leq \theta < 2\pi$, we have

$$\begin{aligned}
\left|G(Re^{i\theta})\right| &\leq \left|G(Re^{i\theta}) - G(e^{i\theta})\right| + \left|G(e^{i\theta})\right| \\
&= \frac{R^n+1}{2}\max_{|z|=1}|G(z)| - \left(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right)|P'(0)|.
\end{aligned}$$

Replacing $G(z)$ by $P(z) + \alpha m z^n$, we get for $|z| = 1$,

$$\begin{aligned}
&|P(Rz) + \alpha m R^n z^n| \\
&\leq \frac{R^n+1}{2}\max_{|z|=1}|P(z) + \alpha m z^n| - \left(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right)|P'(0)|.
\end{aligned} \tag{15}$$

Choosing argument of α in the left hand side of (15) suitably, we obtain for $n > 2$ and $|z| = 1$,

$$\begin{aligned}
&|P(Rz)| + |\alpha| m R^n \\
&\leq \frac{R^n+1}{2}\left\{\max_{|z|=1}|P(z)| + |\alpha| m\right\} - \left(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right)|P'(0)|,
\end{aligned}$$

equivalently for $n > 2$, $|\alpha| < 1$ and $|z| = 1$, we have

$$\begin{aligned}
|P(Rz)| \leq& \frac{R^n+1}{2}\max_{|z|=1}|P(z)| - |\alpha|\frac{R^n-1}{2}\min_{|z|=1}|P(z)| \\
&- \left(\frac{R^n-1}{n} - \frac{R^{n-2}-1}{n-2}\right)|P'(0)|,
\end{aligned}$$

which proves inequality (12) for $n > 2$ and $|\alpha| < 1$. Similarly we can prove inequality (13) for $n = 2$ by using (11) of Lemma 2.3 instead of (10). For $|\alpha| = 1$, the result follows by continuity. This completes the proof of Lemma 2.4. □

Finally we also need the Lemma due to Osserman [11], known as boundary Schwarz lemma.

Lemma 2.5. *If*

(a) $f(z)$ *is analytic for* $|z| < 1$,

(b) $|f(z)| < 1$ *for* $|z| < 1$,

(c) $f(0) = 0$,

(d) *for some* b *with* $|b| = 1, f(z)$ *extends continuously to* b, $|f(b)| = 1$ *and* $f'(b)$ *exists.*

Then

$$|f'(b)| \geq \frac{2}{1 + |f'(0)|}. \tag{16}$$

3. Proof of the Theorems

Proof of Theorem 1.1. Let $g(z) = P(kz)$. Since all the zeros of $P(z) = a_n \prod_{j=1}^{n}(z - z_j)$ lie in $|z| \leq k$ where $k \geq 1$, $g(z)$ has all its zeros in $|z| \leq 1$ and hence all the zeros of the conjugate polynomial $g^*(z) = z^n \overline{g(1/\bar{z})}$ lie in $|z| \geq 1$.
Therefore, the function

$$F(z) = \frac{g(z)}{z^{n-1}\overline{g(1/\bar{z})}} = z\frac{a_n}{\bar{a_n}} \prod_{j=1}^{n} \left(\frac{kz - z_j}{k - z\bar{z}_j} \right) \tag{17}$$

is analytic in $|z| < 1$ with $F(0) = 0$ and $|F(z)| = 1$ for $|z| = 1$. Further for $|z| = 1$, this gives

$$\frac{zF'(z)}{F(z)} = 1 - n + \frac{zg'(z)}{g(z)} + \overline{\left(\frac{zg'(z)}{g(z)} \right)}$$

so that

$$Re\left(\frac{zF'(z)}{F(z)} \right) = 1 - n + 2Re\left(\frac{zg'(z)}{g(z)} \right). \tag{18}$$

Also, we have from (17)

$$\frac{zF'(z)}{F(z)} = 1 + \sum_{j=1}^{n} \left(\frac{k^2 - |z_j|^2}{|kz - z_j|^2} \right) > 0 \text{ for } |z| = 1,$$

as such,

$$\frac{zF'(z)}{F(z)} = \left| \frac{zF'(z)}{F(z)} \right| = |F'(z)| \qquad \text{for} \quad |z| = 1.$$

Using this fact in (18), we get for points z on $|z|=1$ with $g(z)\neq 0$,

$$1-n+2Re\left(\frac{zg'(z)}{g(z)}\right)=|F'(z)|. \tag{19}$$

Applying lemma 2.5 to $F(z)$, we obtain for all points z on $|z|=1$ with $g(z)\neq 0$,

$$1-n+2Re\left(\frac{zg'(z)}{g(z)}\right)\geq\frac{2}{1+|F'(0)|},$$

that is, for $|z|=1$ with $g(z)\neq 0$,

$$Re\left(\frac{zg'(z)}{g(z)}\right)\geq\frac{1}{2}\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right).$$

This implies

$$\left|\frac{zg'(z)}{g(z)}\right|\geq\frac{1}{2}\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\qquad\text{for } |z|=1,\ g(z)\neq 0,$$

and hence,

$$|g'(z)|\geq\frac{1}{2}\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)|g(z)|\qquad\text{for } |z|=1. \tag{20}$$

Replacing $g(z)$ by $P(kz)$, we get for $|z|=1$,

$$k|P'(kz)|\geq\frac{1}{2}\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)|P(kz)|,$$

or equivalently,

$$2k\max_{|z|=k}|P'(z)|\geq\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\max_{|z|=k}|P(z)|. \tag{21}$$

Since $P'(z)$ is a polynomial of degree $n-1$, by (10) of Lemma 2.3 with $R=k\geq 1$, we have

$$k^{n-1}\max_{|z|=1}|P'(z)|-(k^{n-1}-k^{n-3})|a_1|\geq\max_{|z|=k}|P'(z)|,\qquad\text{if}\quad n>2.$$

Combining this inequality with (21), we get for $n>2$,

(22)

$$2k^n\max_{|z|=1}|P'(z)|-2(k^n-k^{n-2})|a_1|\geq\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\max_{|z|=k}|P(z)|.$$

Since all the zeros of polynomial $g^*(z) = z^n\overline{g(1/\bar{z})} = z^n\overline{P(k/\bar{z})}$ lie in $|z| \geq 1$, applying (12) of Lemma 2.4 with $R = k \geq 1$ and $\alpha = 0$ to the polynomial $g^*(z)$,we get

$$\max_{|z|=k}|g^*(z)| \leq \frac{k^n+1}{2}\max_{|z|=1}|g^*(z)| - \frac{|a_{n-1}|}{k}\left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)$$
$$\text{if} \quad n > 2.$$

That is,

$$k^n\max_{|z|=1}|P(z)| \leq \frac{k^n+1}{2}\max_{|z|=k}|P(z)| - |a_{n-1}|k^{n-1}\left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right)$$
$$\text{if} \quad n > 2,$$

or equivalently, we have for $n > 2$,

$$\max_{|z|=k}|P(z)| \geq \frac{2k^n}{k^n+1}\max_{|z|=1}|P(z)| + \frac{2k^{n-1}|a_{n-1}|}{k^n+1}\left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right).$$

Using above inequality in (22), we get for $n > 2$,

$$\begin{aligned}2k^n\max_{|z|=1}|P'(z)| - 2(k^n-k^{n-2})|a_1| \geq{} & \frac{2k^n}{1+k^n}\left(n + \frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\max_{|z|=1}|P(z)| \\ & + \frac{2k^{n-1}|a_{n-1}|}{1+k^n}\left(n + \frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right),\end{aligned}$$

consequently,

$$\begin{aligned}\max_{|z|=1}|P'(z)| \geq{} & \frac{1}{1+k^n}\left(n + \frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\max_{|z|=1}|P(z)| + (1-1/k^2)|a_1| \\ & + \frac{|a_{n-1}|}{k(1+k^n)}\left(n + \frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\left(\frac{k^n-1}{n} - \frac{k^{n-2}-1}{n-2}\right),\end{aligned}$$
$$\text{if} \quad n > 2,$$

which proves inequality (6) for the case $n > 1$. For the case $n = 2$, the result follows on similar lines in view of part second of Lemma 2.3 and Lemma 2.4 with $\alpha = 0$. This completes the proof of Theorem 1.1. $\square$

Proof of Theorem 1.3. By hypothesis $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq k, k \geq 1$. If $P(z)$ has a zero on $|z| = k$, then $m = 0$ and the result follows by Theorem 1.1. Henceforth, we assume that all the zeros of $P(z)$ lie in $|z| < k$, so that $m > 0$. Hence all the zeros of $h(z) = P(kz)$ lie in disk $|z| < 1$ and $m = \min_{|z|=k}|P(z)| = \min_{|z|=1}|h(z)|$. Therefore, we have $m \leq |h(z)|$ for $|z| = 1$. This implies for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ that

$$m|\lambda z^n| < |h(z)| \qquad \text{for} \quad |z| = 1.$$

Applying Rouche's theorem, it follows that all the zeros of the polynomial $H(z) = h(z) + \lambda m z^n$ lie in $|z| < 1$ for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. Now proceeding similarly as in the proof of Theorem 1.1 (with $g(z)$ replacing by $H(z)$), we obtain from (20)

$$|H'(z)| \geq \frac{1}{2}\left(n + \frac{|k^n a_n + \lambda m| - |a_0|}{|k^n a_n + \lambda m| + |a_0|}\right)|H(z)| \qquad \text{for } |z| = 1. \tag{23}$$

Using the fact that the function $t(x) = \frac{x-|a|}{x+|a|}$ is non-decreasing function of x and $|k^n a_n + \lambda m| \geq k^n|a_n| - |\lambda m|$, we get for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$ and $|z| = 1$,

$$|H'(z)| \geq \frac{1}{2}\left(n + \frac{k^n|a_n| - |\lambda m| - |a_0|}{k^n|a_n| - |\lambda m| + |a_0|}\right)|H(z)|. \tag{24}$$

Equivalently for $|z| = 1$ and $|\lambda| < 1$,

$$|h'(z) + nm\lambda z^{n-1}| \geq \frac{1}{2}\left(n + \frac{k^n|a_n| - |\lambda m| - |a_0|}{k^n|a_n| - |\lambda m| + |a_0|}\right)(|h(z)| - m|\lambda|). \tag{25}$$

Since all the zeros of $H(z) = h(z) + \lambda m z^n$ lie in $|z| < 1$, by Guass Lucas theorem it follows that all the zeros of $H'(z) = h'(z) + \lambda n m z^{n-1}$ lie in $|z| < 1$ for every $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. This implies

$$|h'(z)| \geq nm|z|^n \qquad \text{for } |z| \geq 1. \tag{26}$$

Choosing argument of λ in the left hand side of (25) such that

$$|h'(z) + nm\lambda z^{n-1}| = |h'(z)| - nm|\lambda| \qquad \text{for } |z| = 1,$$

which is possible by (26), we get

$$|h'(z)| - nm|\lambda| \geq \frac{1}{2}\left(n + \frac{k^n|a_n| - |\lambda m| - |a_0|}{k^n|a_n| - |\lambda m| + |a_0|}\right)(|h(z)| - m|\lambda|),$$

that is,

$$|h'(z)| \geq \frac{1}{2}\left(n + \frac{k^n|a_n| - |\lambda m| - |a_0|}{k^n|a_n| - |\lambda m| + |a_0|}\right)|h(z)| + \frac{1}{2}\left(n - \frac{k^n|a_n| - |\lambda m| - |a_0|}{k^n|a_n| - |\lambda m| + |a_0|}\right)|\lambda| m.$$

Replacing $h(z)$ by $P(kz)$, we get

$$\begin{aligned}k\max_{|z|=k}|P'(z)| \geq & \frac{1}{2}\left(n+\frac{k^n|a_n|-|\lambda m|-|a_0|}{k^n|a_n|-|\lambda m|+|a_0|}\right)\max_{|z|=k}|P(z)| \\ & +\frac{1}{2}\left(n-\frac{k^n|a_n|-|\lambda m|-|a_0|}{k^n|a_n|-|\lambda m|+|a_0|}\right)|\lambda|m.\end{aligned} \tag{27}$$

Again as before, using (10) of Lemma 2.3 and (12) of lemma 2.4, we obtain for $0 \leq l < 1$ and $n > 2$,

$$\begin{aligned}& k^n\max_{|z|=1}|P'(z)|-(k^n-k^{n-2})|a_1| \\ & \geq \frac{1}{2}\left(n+\frac{k^n|a_n|-lm-|a_0|}{k^n|a_n|-lm+|a_0|}\right)\left\{\frac{2k^n}{1+k^n}\max_{|z|=1}|P(z)|+l\left(\frac{k^n-1}{k^n+1}\right)\min_{|z|=k}|P(z)|\right. \\ & \left.+\frac{2k^{n-1}|a_{n-1}|}{k^n+1}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)\right\}+\frac{1}{2}\left(n-\frac{k^n|a_n|-lm-|a_0|}{k^n|a_n|-lm+|a_0|}\right)lm,\end{aligned}$$

which on simplification yields for $0 \leq l < 1$ and $n > 2$,

$$\begin{aligned}& \max_{|z|=1}|P'(z)| \\ & \geq \frac{n}{1+k^n}\left(\max_{|z|=1}|P(z)|+lm\right)+\frac{n|a_{n-1}|}{k(1+k^n)}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right) \\ & +(1-1/k^2)|a_1|+\left(\frac{k^n|a_n|-lm-|a_0|}{k^n|a_n|-lm+|a_0|}\right)\left\{\frac{1}{k^n(1+k^n)}\left(k^n\max_{|z|=1}|P(z)|-lm\right)\right. \\ & \left.+\frac{|a_{n-1}|}{k(1+k^n)}\left(\frac{k^n-1}{n}-\frac{k^{n-2}-1}{n-2}\right)\right\}.\end{aligned}$$

The above inequality is equivalent to the inequality (7) for $n > 2$. For $n = 2$, the result follows on the similar lines by using inequality (11) of Lemma 2.3 and inequality (13) of Lemma 2.4 in the inequality (27). This proves Theorem 1.3. □

4. Concluding Remark

If we use Lemma 2.3 and Lemma 2.4 with $|\alpha| = 1$ in the proof of Theorem 1.1, we get the following refinement of inequalities (2) and (6).

THEOREM 4.1. *If $P \in \mathcal{P}_n$ has all its zeros in $|z| \leq k$ where $k \geq 1$, then*

(28)
$$\max_{|z|=1}|P'(z)| \geq \frac{1}{1+k^n}\left(n+\frac{k^n|a_n|-|a_0|}{k^n|a_n|+|a_0|}\right)\left(\max_{|z|=1}|P(z)|+\frac{k^n-1}{2k^n}\min_{|z|=k}|P(z)| + \frac{|a_{n-1}|}{k}\phi(k)\right)+|a_1|\psi(k)$$

where $\psi(k) = (1-1/k^2)$ *or* $(1-1/k)$ according as $n > 2$ *or* $n = 2$. The result is sharp and equality in (28) holds for $P(z) = z^n + k^n$.

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