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ARITHMETIC PROPERTIES OF TRIANGULAR PARTITIONS

Byungchan Kim

Abstract. We obtain a two variable generating function for the number of triangular partitions. Using this generating function, we study arithmetic properties of a certain weighted count of triangular partitions. Finally, we introduce a rank-type function for triangular partitions, which gives a combinatorial explanation for a triangular partition congruence.

1. Introduction

In a recent paper [7], the author initiates the study on counting the number of orbits in the sets of various partitions under group actions. For example, for the set of tri-partitions $T = \{(\pi_1, \pi_2, \pi_3) : \pi_i \text{ are ordinary partitions}\}$ and the symmetric group S_3, a group action is defined by

$$\sigma \cdot (\pi_1, \pi_2, \pi_3) = \left(\pi_{\sigma(1)}, \pi_{\sigma(2)}, \pi_{\sigma(3)}\right)$$

for $\sigma \in S_3$. In [7], it is shown that

$$\begin{aligned}\sum_{n\geq 0} |T/S_3|(n)q^n &= \frac{1}{6}\left(\frac{1}{(q)_\infty^3} + \frac{3}{(q)_\infty (q^2;q^2)_\infty} + \frac{2}{(q^3;q^3)_\infty}\right) \\ &= 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + \cdots,\end{aligned}$$

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where $|T/S_3|(n)$ is the number of orbits having the weight n and the weight means that the number being partitioned. Here and in the sequel, we use the standard q-series notation:

$$(a)_n = (a;q)_n = \prod_{k=1}^{n} \left(1 - aq^{k-1}\right) \text{ for } n \in \mathbb{N}_0 \cup \{\infty\},$$
$$(a_1, a_2, \dots, a_k)_n = (a_1, a_2, \dots, a_k; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_k;q)_n.$$

In T/S_3, two partitions are considered to be same if they are in the same orbit. We may interpret $|T/S_3|(n)$ as there is a partition π_i on the vertex i of the regular triangle and we consider tri-partitions (π_1, π_2, π_3) are the same if they are invariant under the group action, i.e. plane isometries fixing the regular triangle. In this sense, the author [8] names $|T/S_3|(n)$ as the number of triangular partitions of n and extends it to regular polygons. For example, there are three triangular partitions of 2:

$$(2, \phi, \phi), \quad (1+1, \phi, \phi), \quad (1, 1, \phi).$$

For convenience, by abusing the notation, we do not discriminate a representative of the orbit and a triangular partitions. In the above example, $(2, \phi, \phi)$ is a representative of the orbit $\{(2, \phi, \phi), (\phi, 2, \phi), (\phi, \phi, 2)\}$. We define $\mathcal{T}$ be the set of triangular partitions and $t(n)$ be the number of triangular partitions of n.

One of most important results in the theory of partitions [1] is Euler's pentagonal number theorem

$$(q)_\infty = \sum_{n \in \mathbb{Z}} (-1)^n q^{n(3n-1)/2}.$$

By interpreting the left-hand side as a generating function for the difference between the number of partitions into even number of distinct parts and the number of partitions into odd number of distinct parts, Euler's pentagonal number theorem implies that the difference is almost always zero. Motivated from this, we define

$$t_w(n) = \sum_{m \geq 0} (-1)^m t(n, m),$$

where $t(n, m)$ is the number of triangular partitions with m parts. This is well-defined as the total number of parts is same in all partitions in the orbit. Thus, $t_w(n)$ is the difference between number of triangular partitions with even parts and the number of triangular partitions with odd parts. To study $t_w(n)$, we first find a two variable generating function for triangular partitions.

Proposition 1.1. *We have*

$$f_{\mathcal{T}}(z,q) := \sum_{n,m\geq 0} t(n,m)z^m q^n = 1 + zq + \left(2z^2+z\right)q^2 + \left(3z^3+2z^2+z\right)q^3 + \cdots$$
$$= \frac{1}{6}\left(\frac{1}{(zq)_\infty^3} + \frac{3}{(zq)_\infty (z^2q^2;q^2)_\infty} + \frac{2}{(z^3q^3;q^3)_\infty}\right).$$

Using the generating function and combinatorial arguments, we study arithmetic properties of $t_w(n)$. It is straightforward to see that

$$\sum_{n\geq 0} t_w(n)q^n = f_{\mathcal{T}}(-1,q)$$
$$= 1 - q + q^2 - 2q^3 + 3q^4 - 4q^5 + 5q^6 - 7q^7 + 10q^8 - 14q^9 + 17q^{10} + \cdots .$$

The first few coefficients looks sign-alternating and indeed this is the case.

Theorem 1.2. *For all non-negative integers n, $(-1)^n t_w(n)$ is positive.*

Moreover, the numeric shows that $|t_w(n)|$ is monotone and this is also true.

Theorem 1.3. *For all positive integers n,*

$$|t_w(n)| = |O_3/S_3|(n) + a(n),$$

where $O_3 = \{(\pi_1,\pi_2,\pi_3) : \ \pi_i$ are partitions into distinct odd parts$\}$ and $a(n)$ is the number of partitions into parts $\not\equiv 2 \pmod 4$ with odd number of even parts. Moreover, for all positive integers $n > 1$,

$$|t_w(n+1)| > |t_w(n)|.$$

In [7], it is shown that

$$t(3n+2) \equiv 0 \pmod 3.$$

To give a combinatorial explanation on Ramanujan's partition congruences, Dyson [5] defined a rank of partition as the size of the largest part minus the number of parts. Let $N(i,m,n)$ be the number of partitions of n with rank $\equiv i \pmod m$, then Atkin and Swinnerton-Dyer [3] confirmed Dyson's conjecture:

$$N(0,5,5n+4) = N(1,5,5n+4) = \cdots = N(4,5,5n+4)$$
$$N(0,7,7n+5) = N(1,7,7n+5) = \cdots = N(6,7,7n+5),$$

which implies that $p(5n+4) \equiv 0 \pmod 5$ and $p(7n+5) \equiv 0 \pmod 7$, where $p(n)$ is the number of ordinary partitions of n. Motivated from the partition rank, to give a combinatorial interpretation for triangular partitions, here we define a polynomial rank for a triangular partition $\pi = (\pi_1, \pi_2, \pi_3)$ as follows:

1. if all partitions π_1, π_2 and π_3 are different, then we define
$$\mathrm{rank}_{\mathcal{T}}(\pi; z) = \frac{1}{6} \sum_{\sigma \in S_3} z^{\#(\pi_{\sigma(2)}) - \#(\pi_{\sigma(3)})},$$
where $\#(\mu)$ is the number of parts in the partition μ. Note that there are six elements in the orbit of π, and thus for each π, $\mathrm{rank}_{\mathcal{T}}(\pi; 1) = 1$.
2. if exactly two of π_1, π_2, and π_3 are same, then we may choose the representative satisfying $\pi_2 = \pi_3$, so that we define
$$\mathrm{rank}_{\mathcal{T}}(\pi; z) = \frac{1}{6} \big(1 + z^{\#(\pi_1) - \#(\pi_2)} + z^{\#(\pi_2) - \#(\pi_1)} \\ + 3 z^{\text{the number of even parts in } \pi_1 \, - \, \#(\pi_2)} \big).$$
3. All three partitions π_1, π_2, and π_3 are same, then we define
$$\mathrm{rank}_{\mathcal{T}}(\pi; z) = 1.$$

To illustrate, here we list polynomial ranks in Table 1.

Employing the same argument used in the proof of Proposition 1.1 and the inclusion-exclusion principle, we find the polynomial rank generating function.

Theorem 1.4. *Let $|\pi|$ be the sum of parts in $\pi = (\pi_1, \pi_2, \pi_3) \in \mathcal{T}$. Then,*

$$\begin{aligned} R_{\mathcal{T}}(z, q) &:= \sum_{\pi \in \mathcal{T}} \mathrm{rank}_{\mathcal{T}}(\pi, z) q^{|\pi|} \\ &= 1 + (z + z^{-1} + 4)\frac{q}{6} + (z^2 + z^{-2} + 5z + 5z^{-1} + 6)\frac{q^2}{6} \\ &\quad + (z^3 + z^{-3} + 2z^2 + 2z^{-2} + 8z + 8z^{-1} + 14)\frac{q^3}{6} + \cdots \\ &= \frac{1}{6}\left(\frac{1}{(q, zq, z^{-1}q)_\infty} + \frac{3}{(q, zq^2, z^{-1}q^2; q^2)_\infty} + \frac{2}{(q^3; q^3)_\infty} \right). \end{aligned} \tag{1.1}$$

Table 1. rank polynomials for triangular partitions for $n = 2$ and $n = 4$

triangular partition $\pi = (\pi_1, \pi_2, \pi_3)$	$\mathrm{rank}_{\mathcal{T}}(\pi, z)$
$(2, \phi, \phi)$	$\frac{1}{6}\left(1 + z + z^{-1} + 3z\right)$
$(1+1, \phi, \phi)$	$\frac{1}{6}\left(1 + z^2 + z^{-2} + 3\right)$
$(\phi, 1, 1)$	$\frac{1}{6}\left(1 + z^{-1} + z^1 + 3z^{-1}\right)$
$(4, \phi, \phi)$	$\frac{1}{6}\left(1 + z + z^{-1} + 3z\right)$
$(3+1, \phi, \phi)$	$\frac{1}{6}\left(1 + z^2 + z^{-2} + 3\right)$
$(2+2, \phi, \phi)$	$\frac{1}{6}\left(1 + z^2 + z^{-2} + 3z^2\right)$
$(2+1+1, \phi, \phi)$	$\frac{1}{6}\left(1 + z^3 + z^{-3} + 3z\right)$
$(1+1+1+1, \phi, \phi)$	$\frac{1}{6}\left(1 + z^4 + z^{-4} + 3\right)$
$(3, 1, \phi)$	$\frac{1}{6}\left(z + z^{-1} + 1 + z^{-1} + 1 + z\right)$
$(\phi, 2, 2)$	$\frac{1}{6}\left(1 + z^{-1} + z + 3z^{-1}\right)$
$(2+1, 1, \phi)$	$\frac{1}{6}\left(z + z^{-2} + z + z^{-1} + z^{-1} + z^2\right)$
$(2, 1+1, \phi)$	$\frac{1}{6}\left(z^2 + z^{-1} + z^{-1} + z^{-2} + z + z\right)$
$(2, 1, 1)$	$\frac{1}{6}\left(1 + 1 + 1 + 3\right)$
$(1+1+1, 1, \phi)$	$\frac{1}{6}\left(z + z^{-3} + z^2 + z^{-1} + z^{-2} + z^3\right)$
$(\phi, 1+1, 1+1)$	$\frac{1}{6}\left(1 + z^{-2} + z^2 + 3z^{-2}\right)$
$(1+1, 1, 1)$	$\frac{1}{6}\left(1 + z + z^{-1} + 3z^{-1}\right)$

Let

$$R_{\mathcal{T}}(z, q) = \sum_{n \geq 0} t_n(z) q^n,$$

where $t_n(z)$ is a polynomial in $\mathbb{Z}[z, z^{-1}]$. We also define

$$r_t(i, m, n) = \sum_{j \equiv i \pmod{m}} [z^j] t_n(z),$$

where $[z^k]g(z)$ is the coefficient of z^k in $g(z)$. Then, we prove that $r_t(i, 3, n)$ works as a rank function for triangular partitions.

Theorem 1.5. *For all non-negative integers n,*

$$r_t(0, 3, 3n+2) = r_t(1, 3, 3n+2) = r_t(2, 3, 3n+2).$$

Since $\sum_{j=0}^{2} r_t(j, 3, 3n+2) = t(3n+2)$, we have

Corollary 1.6. *For all non-negative integers n,*

$$t(3n+2) \equiv 0 \pmod{3}.$$

2. Proofs of Results

We start with the proof of Proposition 1.1.

Proof of Proposition 1.1. As in [7], we can employ Burnside Lemma to count the number of orbits. It is well-known that

$$\frac{1}{(zq)_\infty} = \sum_{n,m\geq 0} p(n,m) z^m q^n,$$

where $p(n,m)$ is the number of ordinary partitions of n with m parts. Since $S_3 = D_3$, the dihedral group of order 6, we can see that the fixed partition under the three reflections are generated by

$$\frac{1}{(zq)_\infty (z^2q^2;q^2)_\infty},$$

and the fixed partition under the two rotations are generated by

$$\frac{1}{(z^3q^3;q^3)_\infty}.$$

In summary, we have shown that

$$f_{\mathcal{T}}(z,q) = \sum_{n,m\geq 0} t(n,m) z^m q^n = \frac{1}{6}\left(\frac{1}{(zq)_\infty^3} + \frac{3}{(zq)_\infty (z^2q^2;q^2)_\infty} + \frac{2}{(z^3q^3;q^3)_\infty}\right)$$

as desired. □

Now we prove Theorem 1.2

Proof of Theorem 1.2. We first observe that

$$f_{\mathcal{T}}(-1,-q) = \sum_{n,m\geq 0} (-1)^m t(n,m)(-q)^n = \sum_{n\geq 0} (-1)^n t_w(n) q^n.$$

Moreover, by a q-series manipulation, we find that

$$
\begin{aligned}
&f_{\mathcal{T}}(-1,-q)\\
&=\frac{1}{6}\left(\frac{1}{(q;-q)_\infty^3}+\frac{3}{(q;-q)_\infty(q^2;q^2)_\infty}+\frac{2}{(q^3;-q^3)_\infty}\right)\\
&=\frac{1}{6}\left(\frac{1}{(q;q^2)_\infty^3(-q^2;q^2)_\infty^3}+\frac{3}{(q;q^2)_\infty(-q^2;q^2)_\infty(q^2;q^2)_\infty}+\frac{2}{(q^3;q^6)_\infty(-q^6;q^6)_\infty}\right)\\
&=\frac{1}{6}\left(\frac{(q^2;q^2)_\infty^3}{(q;q^2)_\infty^3(q^4;q^4)_\infty^3}+\frac{3}{(q;q^2)_\infty(q^4;q^4)_\infty}+2\frac{(q^6;q^6)_\infty}{(q^3;q^6)_\infty(q^{12};q^{12})_\infty}\right)\\
&=\frac{1}{6}\left(\frac{1}{(q^4;q^4)_\infty^3}\left(\sum_{n\geq 0}q^{n(n+1)/2}\right)^3+\frac{3}{(q;q^2)_\infty(q^4;q^4)_\infty}+\frac{2}{(q^{12};q^{12})_\infty}\sum_{n\geq 0}q^{3n(n+1)/2}\right),
\end{aligned}
$$

where we have applied Gauss identity

$$\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}=\sum_{n\geq 0}q^{n(n+1)/2}.$$

From Gauss Eureka theorem, we know that every positive integers can be represented as a sum of three triangular numbers. Let $\Delta_3(n)$ be the number of representations of n into three triangular numbers (see [4, Chapter 3] for example). Since

$$\left(\sum_{n\geq 0}q^{n(n+1)/2}\right)^3=\sum_{n\geq 0}\Delta_3(n)q^n,$$

we can conclude that $\frac{1}{(q^4;q^4)_\infty^3}\sum_{n\geq 0}\Delta_3(n)q^n$ has positive coefficients, which gives the desirable positivity of $(-1)^n t_w(n)$. $\square$

Before proceeding further, we remark that $(-1)^n t_w(n)=|t_w(n)|$ from Theorem 1.2 and

$$\frac{1}{(q^4;q^4)_\infty}\sum_{n\geq 0}q^{n(n+1)/2}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty(q^4;q^4)_\infty}=\frac{(q^2;q^4)_\infty}{(q;q^2)_\infty}=(-q;q^2)_\infty.$$

Using these two facts, we will rewrite $f_{\mathcal{T}}(-1,-q)$ to interprets $|t_w(n)|$.

Proof. We first rewrite $f_{\mathcal{T}}(-1,-q)$.

$$
\begin{aligned}
&f_{\mathcal{T}}(-1,-q)\\
&=\sum_{n\geq 0}|t_w(n)|q^n\\
&=\frac{1}{6}\left(\frac{1}{(q^4;q^4)_\infty^3}\left(\sum_{n\geq 0}q^{n(n+1)/2}\right)^3+\frac{3}{(q;q^2)_\infty(-q^4;q^4)_\infty}+\frac{2}{(q^{12};q^{12})_\infty}\sum_{n\geq 0}q^{3n(n+1)/2}\right)\\
&\qquad+\frac{1}{2(q;q^2)_\infty}\left(\frac{1}{(q^4;q^4)_\infty}-\frac{1}{(-q^4;q^4)_\infty}\right)\\
&=\frac{1}{6}\left((-q;q^2)_\infty^3+3(-q;q^2)_\infty(-q^2;q^4)_\infty+2(-q^3;q^6)_\infty\right)\\
&\qquad+\frac{1}{2(q;q^2)_\infty}\left(\frac{1}{(q^4;q^4)_\infty}-\frac{1}{(-q^4;q^4)_\infty}\right),
\end{aligned}
$$

where we use

$$
\frac{1}{(q;q^2)_\infty(-q^4;q^4)_\infty}=\frac{(-q)_\infty}{(-q^4;q^4)_\infty}=(-q;q^2)_\infty(-q^2;q^4)_\infty.
$$

By employing the same argument in the proof of Proposition 1.1, we see that

$$
\sum_{n\geq 0}|O_3/S_3|(n)q^n=\frac{1}{6}\left((-q;q^2)_\infty^3+3(-q;q^2)_\infty(-q^2;q^4)_\infty+2(-q^3;q^6)_\infty\right)
$$

and it is immediate to see that

$$
\sum_{n\geq 0}a(n)q^n=\frac{1}{2(q;q^2)_\infty}\left(\frac{1}{(q^4;q^4)_\infty}-\frac{1}{(-q^4;q^4)_\infty}\right)
$$

from $1/(zq)_\infty=\sum_{n,m\geq 0}p(n,m)z^mq^n$.

Now we turn to prove the monotonicity. We start with noting that $a(n)$ is monotone since

$$
\begin{aligned}
(1-q)\sum_{n\geq 0}a(n)q^n&=\frac{1}{2(q^3;q^2)_\infty}\left(\frac{1}{(q^4;q^4)_\infty}-\frac{1}{(-q^4;q^4)_\infty}\right)\\
&=\frac{1}{(q^3;q^2)_\infty}\left(q^4+q^8+2q^{12}+\cdots\right)
\end{aligned}
$$

has non-negative coefficients. Moreover $a(n)$ is strictly increasing for $n\geq 7$ as

$$
7=4+3,\quad 8=8,\quad 9=5+4
$$

and we may add 3 as many as we want. To prove that $|O_3/S_3|(n)$ is monotone, we construct an injection. To this end, let TD be a set of partitions counted by O_3/S_3 and $TD(n)$ be the set of partitions counted by $|O_3/S_3|(n)$. We partition the set $TD(n)$ into the subsets $TD_{j,k}(n)$ consisting of partitions $(\pi, \mu, \lambda) \in TD(n)$ satisfying that the number of partitions in the orbit containing (π, μ, λ) is j and the total number of appearances of one as a part in the partitions π, μ, and λ is k. For the partitions into distinct odd parts we say the partition π is larger then the partition λ if $|\pi| > |\lambda|$ or there is a J satisfying that $\pi_J > \lambda_J$ and $\pi_i = \lambda_i$ for all $i < J$ if $|\pi| = |\lambda|$, where we list parts in the partition in non-increasing order. We denote $\pi > \lambda$ if the partition π is larger than the partition λ. We note that if $(\pi, \mu, \lambda) \in TD_{6,k}$, then we may list partitions to be $\pi > \lambda > \mu$. This is feasible since π, μ, and λ are partitions into distinct odd parts. If $(\pi, \mu, \lambda) \in TD_{3,k}$, we assume that μ and λ are the same partition.

For $n > 3$, we define a map $\xi : TD(n) \to TD(n+1)$ according to $(\pi, \mu, \lambda) \in TD_{j,k}(n)$.

- I-1. If $(\pi, \mu, \lambda) \in TD_{6,0}(n)$, then we insert the part 1 to the largest partition. The resulting partition is in the set $TD_{6,3}(n+1)$.
- I-2. If $(\pi, \mu, \lambda) \in TD_{3,0}(n)$ or $TD_{1,0}(n)$, then we insert the part 1 to π, which implies that the resulting partition is in the set $TD_{3,1}(n+1)$.
- II-1. Suppose that $(\pi, \mu, \lambda) \in TD_{6,1}(n)$. Let π be the partition with the part one. Then we delete the part one from π and insert the part one to the other two partitions μ and λ. The resulting partition is in the set $TD_{6,2}(n+1)$.
- II-2. Suppose that $(\pi, \mu, \lambda) \in TD_{3,1}(n)$. Note that π should have one as a part. We delete one from π and insert the part one to the other two partitions μ and λ. The resulting partition is in the set $TD_{3,2}(n+1)$.
- III-1. Suppose that $(\pi, \mu, \lambda) \in TD_{6,2}(n)$. If there is a part 1 in the largest partition, then we delete one from the partition and increase the largest part by 2 for that partition. The resulting partition is in the set $TD_{6,1}(n+1)$, but this is not overlapped with the case (I-1) as the resulting partition has the part 1 in other than the largest partition. If there is no part 1 in the largest partition, then we append the part 1 in that partition. The resulting partition is now in the set $TD_{6,3}(n+1)$.

III-2. If $(\pi, \mu, \lambda) \in TD_{3,2}(n)$, then we add one as a part to π. The resulting partition is in either $TD_{3,3}(n+1)$ or $TD_{1,3}(n+1)$.

IV-1. If $(\pi, \mu, \lambda) \in TD_{6,3}(n)$, then the largest partition has other parts beside the part 1 since $n > 3$. We first delete all ones from the partitions and we add 4 to the largest part of the unique largest partition. The resulting partition is in $TD_{6,0}(n+1)$.

IV-2. Suppose that $(\pi, \mu, \lambda) \in TD_{3,3}(n)$ or $TD_{1,3}(n)$ and $\pi \neq (1)$. First we delete all ones from the partitions and add 4 to the largest part of π. The resulting partition is in the set $TD_{3,0}(n+1)$.

IV-3. If $(\pi, \mu, \lambda) \in TD_{3,3}(n)$ with $\pi = (1)$, then after deleting all ones from the partitions, we add two to each of the largest part in μ and λ. Since $n > 3$ and $\pi = (1)$, μ and λ have parts other than 1. For this case, the resulting partition is again in the set $TD_{3,0}(n+1)$, but this is not overlapped with the case (IV-2) since the first component of the image is ϕ.

The map ξ is clearly reversible because we can determine the pre-image according to $\xi(\pi, \mu, \lambda) \in TD_{j,k}(n+1)$. Interested reader might want to check

`https://github.com/math-bkim/tri_ptn`

for the examples generated by Python. We have seen that $|O_3/S_3|(n+1) \geq |O_3/S_3|(n)$ for $n > 3$. For the strict inequality, we observe that following partitions in $TD(n+1)$ do not have a pre-image under ξ:

$$((2m+1, 2m-1), (3), (3)) \in TD_{3,0}(n+1) \quad \text{for } n = 4m+5 \geq 13,$$
$$((2m+1, 2m-1), (5,3), (3)) \in TD_{6,0}(n+1) \quad \text{for odd } n = 4m+10 \geq 22.$$
$$((2m+1, 2m-1), (5,3), (5,3)) \in TD_{3,0}(n+1) \quad \text{for } n = 4m+15 \geq 27,$$
$$((2m+1, 2m-1), (5,3), (5)) \in TD_{6,0}(n+1) \quad \text{for odd } n = 4m+12 \geq 24.$$

The above have no pre-image under ξ as the difference between the largest part and the second largest part is exactly 2 in the first component. Therefore, $|O_3/S_3|(n)$ is strictly increasing for $n \geq 24$. Thus, we can conclude that $|t_w(n)|$ is strictly increasing by checking up to q^{24} of the generating function $T(-1, -q)$. $\square$

Now we proceed to study a rank-type function for triangular partitions.

Proof of Theorem 1.4. By employing the inclusion-exclusion principle, we see that

$$\begin{aligned}
R_1(z,q) =& \frac{1}{6}\left(\frac{1}{(q,zq,z^{-1}q)_\infty} - \frac{1}{(q)_\infty(q^2;q^2)_\infty} - \frac{1}{(zq)_\infty(z^{-1}q^2;q^2)_\infty}\right.\\
&\left. - \frac{1}{(z^{-1}q)_\infty(zq^2;q^2)_\infty} + \frac{2}{(q^3;q^3)_\infty}\right)\\
R_2(z,q) =& \frac{1}{6}\left(\frac{1}{(q)_\infty(q^2;q^2)_\infty} + \frac{1}{(zq)_\infty(z^{-1}q^2;q^2)_\infty} + \frac{1}{(z^{-1}q)_\infty(zq^2;q^2)_\infty}\right.\\
&\left. + \frac{3}{(q,zq^2,z^{-1}q^2;q^2)_\infty} - \frac{6}{(q^3;q^3)_\infty}\right)\\
R_3(z,q) =& \frac{1}{(q^3;q^3)_\infty}
\end{aligned}$$

are generating functions corresponding to each case in the definition of rank polynomials for triangular partitions. It is immediate that

$$R_{\mathcal{T}}(z,q) = R_1(z,q) + R_2(z,q) + R_3(z,q).$$

□

Now we prove that $r_t(j,3,3n+2)$ are all same.

Proof of Theorem 1.5. We first observe that

$$6R_{\mathcal{T}}(\zeta,q) = \sum_{n\geq 0} 6\left(r_t(0,3,n) + r_t(1,3,n)\zeta + r_t(2,3,n)\zeta^2\right) q^n,$$

where $\zeta = \exp(2\pi i/3)$ is the primitive third roof of unity. Now we derive that

$$\begin{aligned}
6R_{\mathcal{T}}(\zeta,q) &= \frac{1}{(q,\zeta q,\zeta^{-1}q)_\infty} + \frac{3}{(q,\zeta q^2,\zeta^{-1}q^2;q^2)_\infty} + \frac{2}{(q^3;q^3)_\infty}\\
&= \frac{3}{(q^3;q^3)_\infty} + \frac{3(q^2;q^2)_\infty}{(q;q^2)_\infty(q^6;q^6)_\infty}\\
&= \frac{3}{(q^3;q^3)_\infty} + \frac{3}{(q^6;q^6)_\infty}\sum_{n\geq 0} q^{n(n+1)/2},
\end{aligned}$$

where we use Gauss identity and $1-x^3 = (1-x)(1-\zeta x)(1-\zeta^{-1}x)$. Thus, we find that the coefficients of q^{3n+2} of $6R_{\mathcal{T}}(\zeta,q)$ is zero, i.e.

$$r_t(0,3,3n+2) + r_t(1,3,3n+2)\zeta + r_t(2,3,3n+2)\zeta^2 = 0.$$

Since $1 + \zeta + \zeta^2$ is a minimal polynomial in $\mathbb{Z}[\zeta]$ we can conclude that the desired result. $\square$

3. Concluding Remark

It is nice if one can define a rank function from $\mathcal{T}$ to $\mathbb{Z}$ as the ordinary partition rank. Like a vector crank of Garvan [6] played a fundamental role in the development of Andrews-Garvan crank [2], it would be great if one can deduce a more natural rank function on triangular partitions from polynomial ranks.

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Byungchan Kim
School of Liberal Arts
Seoul National University of Science and Technology
Seoul 01811, Korea
E-mail: `bkim4@seoultech.ac.kr`