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ON 2-INNER PRODUCT SPACES AND REPRODUCING PROPERTY

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Abstract. This paper is devoted to study the reproducing property on 2-inner product Hilbert spaces. We focus on a new structure to produce reproducing kernel Hilbert and Banach spaces. According to multi variable computing, this structures play the key role in probability, mathematical finance and machine learning.

1. Introduction

It is well known by functional analysis that point evaluations, which can be considered as a functional or operator, due to the algebraic construction of range space, are not always bounded. There are some special cases. For example, a point evaluation from the normed space $(\mathsf{B}(\Omega), ||\cdot||_\infty)$ of all bounded from Ω to a field $\mathbb{K}$ is always bounded. A reproducing kernel Hilbert space (RKHS) is a Hilbert space H of functions defined on a determined set Ω, such that point-evaluations for all $x \in \Omega$ are continuous linear functionals. It is clear that continuity should be held in the sense of the norm in H. These kind of Hilbert spaces (RKHS) are a proper hosting of applications in pure approach to complex analysis, harmonic analysis, quantum mechanics and applied approach to machine learning and applied statistics [2, 6–9, 14–18].

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The most important theorem on this approach is a fundamental theorem of Aronszajn which establishes an explicit correspondence between positive definite kernels and RKHSs in two sides of a seesaw. On the other hand, every positive definite kernel is also the covariance kernel of a Gaussian process.By the continuity of the point-evaluations, it is obvious that every finite dimension Hilbert space is naturally a RKHS. This causes us to restrict our focus on the infinite Hilbert spaces. So for a given positive kernel, we should explore to find an application of infinite spaces.

On this view side, there are many studies which are working on varied features of the application and all of them are constructing an ordinary inner product, using the properties of positive kernels, to reach the results. Our research is established on another property, 2-inner product.

Studying the 2-inner product reproducing kernel spaces has found diverse applications in different approaches. As we may touch a RKHS as an extension of a Hilbert space, a 2-inner product is an extension of an inner product, endowed with a property which modifies some fundamental properties of an ordinary one.

Since 1963, S. Gahler published two papers entitled *2-metrische Räume und ihr topologische Strukhur* and *Lineare 2-normierte Raume*, a number of authors have done considerable works on geometric structures of 2-metric spaces and linear 2-normed spaces, and have applied these spaces to several fields of mathematics in many ways. In 1969, S. Gahler introduced also the concept of n metric spaces in a series of his papers entitled *Untersuchungen uber verallemeinerte n-metrischke Raume*, which extend the concept of 2-metric spaces to the general case, and provided many properties of topological and geometrical structures. Recently, A. Misiak introduced the concept of n-inner product spaces and extended many results in 2 inner product spaces, which in turn were introduced and studied by C. Diminnie, S. Gahler and A. White, to n-inner product spaces in his doctoral dissertation.

In the following, we prepare some preliminaries of 2-inner product spaces and review some important and useful theorems and lemmas.Then we defined a 2-inner product reproducing kernel Hilbert spaces and proved some theorems.

2. Preliminaries and Theorems

This section is to review the required concepts and definitions. we also, recall some important lemmas and theorems. Among these content, Riesz representation theorem on 2-inner product spaces plays a key roll in the next sections. This theorem is a wise extension of basic type in [20] to 2-inner product spaces.

Definition 2.1. [1] The Hilbert space H of functions defined on a fixed set Ω such that for each $x \in \Omega$ the evaluation functional at x, i.e., $\delta_x(f) := f(x), f \in \mathsf{H}$, is bounded on H is called a reproducing kernel Hilbert space.

By the Riesz representation theorem, corresponding to each $x \in \Omega$ there exist a unique function $k_x : \Omega \times \Omega \to \mathbb{C}$ such that

$$\begin{cases} \{k_x\}_{x\in\Omega} \subset \mathsf{H}, \\ f(x) = \langle f, k_x\rangle_{\mathsf{H}}, \quad x \in \Omega, f \in \mathsf{H}. \end{cases}$$

Due to the existence of such k_x, there exist a function $K : \Omega \times \Omega \to \mathbb{C}$ such that

$$K(x, y) = \langle k_x, k_y\rangle_{\mathsf{H}}.$$

The above function K is called the reproducing kernel of H. It can be easily shown that the space generated by $\{k_x\}_{x\in\Omega}$ is dense in H and every element of H can be estimated by linear combinations of functions k_x.

Similar structure can be defined for Banach spaces with a quick difference. More details about these kind of spaces can be found in [1, 19, 23]. In following, we extend these kind of spaces to multi linear forms, based on 2-inner products and 2-norms.

Definition 2.2. [3, 12] Let V be a real or complex vector space of dimension greater than 1. The functional $\langle\cdot, \cdot; \cdot\rangle$ defined on V^3 is called a 2-inner product on V provided the following conditions hold,

(i) $\langle x, x; z\rangle \geq 0$ and $\langle x, x; z\rangle = 0$ iff x and z are linearly dependent.
(ii) $\langle x, x; z\rangle = \overline{\langle z, z; x\rangle}$.
(iii) $\langle y, x; z\rangle = \overline{\langle x, y; z\rangle}$.
(iv) $\langle \alpha x, y; z\rangle = \alpha\langle x, y; z\rangle$, for all scalars α.
(v) $\langle x_1 + x_2, y; z\rangle = \langle x_1, y; z\rangle + \langle x_2, y; z\rangle$.

In this case, the pair $(\mathsf{V}, \langle ., .; .\rangle)$ is called a 2-inner product space.

For a 2-inner product space $(\mathsf{V}, \langle ., .; . \rangle)$ and $x, y, b \in \mathsf{V}$

$$x \perp^b y \Leftrightarrow \langle x, y; b \rangle = 0.$$

A 2-norm on $\mathsf{V} \times \mathsf{V}$ can be defined naturally, corresponding to 2-inner product as

$$\|x, y\|_2^2 = \langle x, x; y \rangle.$$

The normed space $(\mathsf{V}^2, \|\cdot, \cdot\|_2)$ and its relation to corresponding ℓ^2 space may be studied deeply for the convergence properties, but our plan is different in this research and we use $\|\cdot, \cdot\|$ just referring to $\|\cdot, \cdot\|_2$.

By the above definition, this norm satisfies the following conditions:

1. $\|x, z\| \geq 0$ and $\|x, z\| = 0$ if and only if x and z are linearly dependent.
2. $\|x, z\| = \|z, x\|$.
3. $\|\alpha x, z\| = |\alpha| \|x, z\|$ for all scalar α.
4. $\|x_1 + x_2, z\| \leq \|x_1, z\| + \|x_2, z\|$.

2-norms have several interesting properties. More details can be found in [11].

Example 2.3. [10] For a ℓ^2 space V, the standard 2-inner product is defined on V by

$$\langle x, y; z \rangle = \begin{vmatrix} \langle x, y \rangle & \langle x, z \rangle \\ \langle z, y \rangle & \langle z, z \rangle \end{vmatrix} = \langle x, y \rangle \langle z, z \rangle - \langle x, z \rangle \langle z, y \rangle.$$

This norm can be extended to a ℓ^p space for $p > 2$ as a subspace of ℓ^2. (See [10])

For a 2-inner product space V, let $\{e_i\}_{i=1}^n$ be a linearly independent subset of V, then $\{e_i\}_{i=1}^n$ is a b-orthonormal system if for $b \in \mathsf{V}$, $\langle e_i, e_j; b \rangle = 0$ for $i \neq j$ and $\langle e_i, e_1; b \rangle = 1$ where $1 \leq i \leq n$.

Lemma 2.4. [11] *For a 2-inner product space* V, *the subspace spanned by a b-orthonormal system* $\{e_i\}_{i=1}^n$ *is closed.*

Definition 2.5. [11] Let V be a 2-inner product space and $b \in \mathsf{V}$, then

1. A sequence $\{x_n\}$ in V is said to be a b-Cauchy sequence if for every $\epsilon > 0$ there exists $N > 0$ such that for every $m, n \geq N$, $0 < \|x_n - x_m, b\| < \epsilon$.
2. V is said to be b-Hilbert if every b-Cauchy sequence is convergent in the semi-normed space $(\mathsf{V}, \|., b\|)$.

For a 2-normed space $(\mathsf{V}, \|.,.\|)$ and a subspace W of V, let $b \in \mathsf{V}$ be fixed. A functional $T : \mathsf{W} \times \langle b \rangle \to \mathbb{R}(\mathbb{C})$ is called a b-linear functional on $\mathsf{W} \times \langle b \rangle$ whenever it is linear corresponding to the first variable. In this case, we can looking at T as an operator and speak about its boundedness. A b-linear functional $T : \mathsf{W} \times \langle b \rangle \to \mathbb{R}(\mathbb{C})$ is said to be bounded if there exists a real number $M > 0$ such that $|T(x,b)| \leq M\|x,b\|$ for every $x \in W$.

The norm of the b-linear functional T is naturally defined as an extension of operator norm by

$$\begin{aligned}\|T\| &= \inf\{M > 0; |T(x,b)| \leq M\|x,b\|, \forall x \in \mathsf{W}\}\\ &= \sup\{|T(x,b)|; \|x,b\| = 1\},\end{aligned}$$

and $|T(x,b)| \leq \|T\|\|x,b\|$.

Suppose V is a vector space and $b \in \mathsf{V}$. let $y_1, y_2 \in V$, then y_1 is said to be b-congruent to y_2 iff $(y_1 - y_2) \in \langle b \rangle$, the subspace generated by b.

THEOREM 2.6. *(Riesz Representation Theorem on 2-Inner Product Spaces) [19] Let H be a b-Hilbert space and $T \in H_b^*$ then there exists a unique $y \in H$ up to b-congruence such that $T(x,b) = \langle x, y; b\rangle$ and $\|T\| = \|y,b\|$.*

In the following, we extend the concept of *reproducing kernel* to 2-inner product spaces. This extension is along with the previews concepts. The main goal is to construct a new space with more applicable properties.

First of all, let us to consider the conditional covariance. For a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let S be the space of all independent and identically distributed (iid) random variables X and Y but not independent to Z. For every three random variables X, Y and Z, conditional covariance is defined as:

$$\mathrm{Cov}(X, Y|Z) = \mathrm{E}(XY|Z) - \mathrm{E}(X|Z)\mathrm{E}(Y|Z),$$

where $\mathrm{E}(X)$ is the expectation value of X. Define a bilinear form on S as:

$$\langle X, Y|Z\rangle_S = \begin{cases}\mathrm{Cov}(X,Y|Z) & X \neq Y,\\ 0 & X = Y.\end{cases}$$

It is easy to check the conditions of Definition 2.2:

(i) By the properties of covariance, the first part holds. For the second part, there are several examples that X and Z are dependent and $\mathrm{Var}(X|Z) = 0$. Since $\mathrm{Cov}(X, X|Z) = \mathrm{Var}(X|Z)$. This shows that it may be possible that $\mathrm{Cov}(X, X|Z) = 0$ while $X \neq 0$.

(ii) The second property is about the symmetric property of equal first and second entry with the third one. It doesn't holds in our manner in general.

(iii) The third property holds naturally, because the bilinear form is real valued.

(iv) For the fourth property we have

$$\begin{aligned}\langle \alpha X, Y|Z\rangle_S &= \mathrm{Cov}(\alpha X, Y|Z)\\ &= \mathrm{E}(\alpha XY|Z) - \mathrm{E}(\alpha X|Z)\mathrm{E}(Y|Z)\\ &= \alpha\mathrm{E}(XY|Z) - \alpha\mathrm{E}(X|Z)\mathrm{E}(Y|Z)\\ &= \alpha\mathrm{Cov}(X, Y|Z).\end{aligned}$$

(v) For this property we have

$$\begin{aligned}&\langle X_1 + X_2, Y|Z\rangle_S\\ &= \mathrm{Cov}(X_1 + X_2, Y|Z)\\ &= \mathrm{E}((X_1 + X_2)Y|Z) - \mathrm{E}((X_1 + X_2|Z)\mathrm{E}(Y|Z)\\ &= \mathrm{E}(X_1Y + X_2Y|Z) - \mathrm{E}(X_1 + X_2|Z)\mathrm{E}(Y|Z)\\ &= \mathrm{E}(X_1Y|Z) + \mathrm{E}(X_2Y|Z) - [\mathrm{E}(X_1|Z) + \mathrm{E}(X_2)]\mathrm{E}(Y|Z)\\ &= \mathrm{E}(X_1Y|Z) + \mathrm{E}(X_2Y|Z) - \mathrm{E}(X_1|Z)\mathrm{E}(Y|Z) - \mathrm{E}(X_2)\mathrm{E}(Y|Z)\\ &= \mathrm{E}(X_1Y|Z) - \mathrm{E}(X_1|Z)\mathrm{E}(Y|Z) + \mathrm{E}(X_2Y|Z) - \mathrm{E}(X_2)\mathrm{E}(Y|Z)\\ &= \mathrm{Cov}(X_1, Y|Z) + \mathrm{Cov}(X_2, Y|Z)\\ &= \langle X_1, Y, Z\rangle_S + \langle X_2, Y, Z\rangle_S.\end{aligned}$$

We used the independency of X_1, X_2 and Y in the above equations.

A bilinear form with properties (i,iii,iv,v) of Definition 2.2 is called a partial symmetric semi 2-inner product. In this case, conditional covariance is a partial symmetric semi 2-inner product.

Definition 2.7. Let Ω be a set and H be a linear space of two variable functions $f : \Omega \times \Omega \to \mathbb{R}$ endowed with a 2-inner product. Fix

$g \in H$. So we can speak about g-Hilbert space H_g. Let H_g be a subspace of H_g such that all 2-variable evaluation

$$\delta_{(x,y)}(f) = f(x,y),$$

be bounded functions. So by theorem (2.6), there exist a function $k_{(x,y)}$ such that

$$\delta_{(x,y)}(f) = f(x,y) = \langle f, k_{(x,y)}; g\rangle.$$

Now we can define a kernel function $K : \Omega \times \Omega \times \Omega \to \mathbb{R}$ by

$$K(x,y,z) = \langle k_{(x,y)}, k_{(y,z)}; g\rangle.$$

In this case

$$K(x,x,x) = \langle k_{(x,x)}, k_{(x,x)}; g\rangle = \|k_{(x,x)}, g\|^2.$$

The pair (H_g, K) is called a 2-inner product reproducing kernel Hilbert space (2IPRKHS in short).

An examples of 2-inner product reproducing kernel Hilbert space is as follows:

Example 2.8. Let H_2^2 be the two variable Hardy-Hilbert space, consists of all analytic functions having power series representations with square-summable complex coefficients and H_I be a subspace of H_2^2 such that

$$\mathsf{H}_I = \left\{ f : f(z_1,z_2) = \sum_{n=0}^{\infty} a_n z_1^n z_2^n; \quad a_1 = 0 \quad \textit{and} \quad \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\},$$

Where I denotes the identity function and

$$f(z_1,z_2) = \sum_{n=0}^{\infty} a_n z_1^n z_2^n, \quad \text{and} \quad h(z_1,z_2) = \sum_{n=0}^{\infty} b_n z_1^n z_2^n.$$

The 2-inner product on H_I is defined by

$$\langle f, h; I\rangle = a_0\overline{b_0} + \sum_{n=2}^{\infty} a_n \overline{b_n}.$$

In this case, let $k_{(z_1,z_2)}(t_1,t_2) = 1 + \sum_{n=2}^{\infty} \overline{z_1^n z_2^n} t_1^n t_2^n$, then

$$f(z_1,z_2) = \langle f, k_{(z_1,z_2)}; I\rangle,$$

and

$$K(z_1,z_2,z_3) = \langle k_{(z_1,z_2)}, k_{(z_2,z_3)}; I\rangle = 1 + \sum_{n=2}^{\infty} |z_2|^{2n} z_1^n z_3^n.$$

We follow by studying some properties of 2-inner product reproducing kernels.

PROPOSITION 2.9. *If* H_g *is a 2IPRKHS on* X *with reproducing kernel* $K(x,y,z)$*, then* $K(x,y,x) = \overline{K(y,x,y)}$.

Proof. By properties of 2-inner product we have

$$K(x,y,x) = \langle k_{(x,y)}, k_{(y,x)}; g\rangle = \overline{\langle k_{(y,x)}, k_{(x,y)}; g\rangle} = \overline{K(y,x,y)}.$$

□

Let H_g be a 2-inner product reproducing kernel Hilbert space and $\{e_i\}_i \in I$ be a basis for this space. Note that each e_i is an element of H_g and so is a function. We can define matrix $K = (\langle e_i, e_j; g\rangle)$. This matrix is positive. To see that, let $x \in \mathsf{H}_g$. Then

$$\langle Kx, x\rangle = x^*Kx = (x^*\langle e_i, e_j, g\rangle x) = (\langle e_i, e_j; g\rangle \|x\|^2) \geq 0.$$

In general form and similar to ordinary reproducing kernel Hilbert spaces, 2-inner product reproducing kernel Hilbert space corresponding to a 2-inner product kernel, is unique. Convergence of a sequence of elements of H_g is similar to RKHS.

LEMMA 2.10. *Let* H_g *be a 2IPRKHS and let* $\{f_n\} \subseteq \mathsf{H}_g$. *If* $\lim_n \|f_n - f\|_g = 0$*, then* $f(x) = lim_n f_n(x)$ *for every* x.

Uniqueness of kernel function corresponding to a 2IPRKHS is an other important problems. Naturally, it seems that the kernel function should be unique.

PROPOSITION 2.11. *Let* H_{gi}*,* $i = 1, 2$ *be 2IPRKHS's with kernels,* $K_i(x,y,z)$*,* $i = 1, 2$*. If* $K_1(x,y,z) = K_2(x,y,z)$ *for all* x, y, z *then* $\mathsf{H}_{g1} = \mathsf{H}_{g2}$ *and* $\|f\|_{g1} = \|f\|_{g2}$ *for every* f *where* $\|.\|_{gi}$ *is the norm corresponding to* H_{gi}.

Proof. It is easy to see that H_g is the closure of a set contains all of functions $k_{(x,y)}$ and also it is a linear space. So let $\{k_{(x_i,y_j)}\}_{i,j}$ be a basis of kernels for H_{g_1} and $\{k'_{(x_i,y_j)}\}_{i,j}$ be a basis of kernels for H_{g2}. By properties of 2-inner product kernels and equality of kernel functions we have

$$\begin{aligned}\|f\|_{g1}^2 &= \sum_{i,j,t} \alpha_{ij}\overline{\alpha_{jt}}\langle k_{(x_i,y_j)}, k_{(y_j,z_t)}; g\rangle \\ &= \sum_{i,j,t} \alpha_{ij}\overline{\alpha_{jt}} K_1(x,y,z) \\ &= \sum_{i,j,t} \alpha_{ij}\overline{\alpha_{jt}} K_2(x,y,z) \\ &= \sum_{i,j,t} \alpha_{ij}\overline{\alpha_{jt}}\langle k'_{(x_i,y_j)}, k'_{(y_j,z_t)}; g\rangle \\ &= \|f\|_{g2}^2.\end{aligned}$$

Moreover, by the previews lemma, boundary elements are the same. Equality of norms on H_{g1} and H_{g2} is a direct consequence of equality on the mentioned dense subspaces. □

Motivated to extend these concepts to wavelets, we try to define Parseval frames in 2IPRKHS's.

Definition 2.12. Let H_g be a 2IPRKHS with 2 inner product, $\langle ., .; .\rangle$. A set of vectors $\{f_s : s \in S\} \subseteq \mathsf{H}_g$ is called a Parseval frame for H_g provided that

$$\|h\|_g^2 = \sum_{s\in S} |\langle h, f_s; g\rangle|^2.$$

for every $h \in \mathsf{H}_g$.

Theorem 2.13. *Let H_g be a 2IPRKHS on X with reproducing kernel $K(x,y,z)$. Then $\{f_s : s \in S\} \subseteq \mathsf{H}_g$ is a Parseval frame for H_g if and only if*

$$K(x,y,z) = \sum_{s\in S} f_s(x,y)\overline{f_s(y,z)}.$$

Where the series converges point-wise.

Proof. Let $\{f_s : s \in S\}$ be Parseval frame. then we have

$$\begin{aligned} K(x,y,z) &= \langle k_{(x,y)}, k_{(y,z)}; g\rangle \\ &= \sum_{s\in S} \langle k_{(x,y)}, f_s; g\rangle \langle f_s, k_{(y,z)}; g\rangle \\ &= \sum_{s\in S} f_s(x,y)\overline{f_s(y,z)}. \end{aligned}$$

Conversely, let α_{ij} are scalars and $h = \sum_{ij} \alpha_{ij} k_{(x_i,y_j)}$ is any finite linear combination of kernel functions, then

$$\begin{aligned} \|h\|_g^2 &= \sum_{i,j,t} \alpha_{ij}\overline{\alpha_{jt}} \langle k_{(x_i,y_j)}, k_{(y_j,z_t)}\rangle \\ &= \sum_{i,j,t} \alpha_{ij}\overline{\alpha_{jt}} K(x_i, y_j, z_t) \\ &= \sum_{i,j,t} \alpha_{ij}\overline{\alpha_{jt}} \sum_{s\in S} \overline{f_s(y_j)} f_s(y_i) \\ &= \sum_{i,j,t} \alpha_{ij}\overline{\alpha_{jt}} \sum_{s\in S} \langle k_{(x_i,y_j)}, f_s; g\rangle \langle f_s, k_{(y_j,z_t)}; g\rangle \\ &= \sum_{s\in S} \langle \sum_{i,j} \alpha_{ij} k_{(x_i,y_j)}, f_s; g\rangle \langle f_s, \sum_{j,t} \alpha_{jt} k_{(y_j,z_t)}; g\rangle \\ &= \sum_{s\in S} |\langle h, f_s; g\rangle|^2. \end{aligned}$$

Now it is easy to see that if we take a limit of a norm convergent sequence of vectors on both sides of this identity, then we obtain the identity for the limit vector, too, and the proof is complete. □

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