ON THE GENERALIZED BOUNDARY
AND THICKNESS

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ABSTRACT. We introduced the concepts of the generalized accumulation points and the generalized density of a subset of the Euclidean space in [1] and [2]. Using those concepts, we introduce the concepts of the generalized closure, the generalized interior, the generalized exterior and the generalized boundary of a subset and investigate some properties of these sets. The generalized boundary of a subset is closely related to the classical boundary. Finally, we also introduce and study a concept of the thickness of a subset.

1. Introduction

In this section, we introduce a concept of the generalized closure of a set and study some properties of the generalized dense subset which we need later. Throughout this paper, \( \epsilon_0 \geq 0 \) denotes any, but fixed, non-negative real number. We denote the open ball, the closed ball and the sphere with radius \( \epsilon \) and center at \( \alpha \) in the space \( R^m \) by \( B(\alpha, \epsilon) = \{ x \in R^m : \|x - \alpha\| < \epsilon \} \), \( \overline{B}(\alpha, \epsilon) = \{ x \in R^m : \|x - \alpha\| \leq \epsilon \} \) and \( S(\alpha, \epsilon) = \{ x \in R^m : \|x - \alpha\| = \epsilon \} \), respectively.

Definition 1.1. Let \( S \) be a subset of \( R^m \). A point \( a \in R^m \) is an \( \epsilon_0 \)–accumulation point of the subset \( S \) if and only if \( B(a, \epsilon) \cap (S - \{a\}) \neq \emptyset \) for all \( \epsilon > \epsilon_0 \). And a point \( a \in S \) is an \( \epsilon_0 \)–isolated point of \( S \) if and only if \( B(a, \epsilon) \cap (S - \{a\}) = \emptyset \) for some positive number \( \epsilon_1 > \epsilon_0 \).
Definition 1.2. For a subset $S$ of $\mathbb{R}^m$, we define the $\epsilon_0$-derived set of $S$ as the set of all the $\epsilon_0$-accumulation points of $S$ and denote it by $S'_{(\epsilon_0)}$.

Definition 1.3. Let $S$ be a subset of $\mathbb{R}^m$. The $\epsilon_0$-closure of $S$ is defined by $\overline{S}_{(\epsilon_0)} = Cl_{\epsilon_0}(S) = S'_{(\epsilon_0)} \cup S$.

Definition 1.4. Let $E$ be any non-empty and open subset of $\mathbb{R}^m$ and $\epsilon_0 \geq 0$. And let a subset $D$ of $E$ be given. We define that $D$ is an $\epsilon_0$-dense subset of $E$ in $E$ if and only if $E \subseteq \overline{D}_{(\epsilon_0)}$. In this case, we say that $D$ is $\epsilon_0$-dense in $E$.

Definition 1.5. Let $E$ be an open non-empty subset of $\mathbb{R}^m$. And let $D$ be an $\epsilon_0$-dense subset of $E$ in $E$. An element $a \in D$ is called a point of the $\epsilon_0$-dense ace of $D$ in $E$ if and only if $D - \{a\}$ is not $\epsilon_0$-dense in $E$.

Lemma 1.6. Let $E$ be an open subset of $\mathbb{R}^m$ and $D$ be a non-empty subset of $E$. Suppose that $E \subseteq \bigcup_{b \in D} B(b, \epsilon_0)$. Then $D$ is $\epsilon_0$-dense in $E$.

Proof. See the proof of the lemma 2.10 in [1].

Lemma 1.7. Let $D$ be a non-empty subset of an open subset $E$ of $\mathbb{R}^m$ and $\overline{D} = D'_{(0)} \cup D$. Then $D$ is $\epsilon_0$-dense in $E$ if and only if $E \subseteq \bigcup_{b \in \overline{D}} B(b, \epsilon_0)$.

Proof. See the proof of the theorem 2.11 in [1].

2. The generalized interior and boundary

In this section, we investigate about the concepts of the $\epsilon_0$-interior, the $\epsilon_0$-exterior and the $\epsilon_0$-boundary of subsets in $\mathbb{R}^m$ and research the shapes of these sets. Throughout this section, $\epsilon_0 \geq 0$ denotes any, but fixed, non-negative real number unless otherwise stated.

Definition 2.1. Let $S$ be a subset of $\mathbb{R}^m$. A point $x$ is called the $\epsilon_0$-interior point of $S$ if and only if there is a positive real number $\epsilon_1 > \epsilon_0$ such that $x \in B(x, \epsilon_1) \subseteq S$. Let’s denote the set of all the $\epsilon_0$-interior points of $S$ in $\mathbb{R}^m$ by $Int_{\epsilon_0}(S)$ or $S'_{(\epsilon_0)}$. 
DEFINITION 2.2. Let $S$ be a subset of $\mathbb{R}^m$. A point $x$ is called the $\epsilon_0$-boundary point of $S$ if and only if $B(x, \epsilon_1) \cap S \neq \emptyset$ and $B(x, \epsilon_1) \cap S^C \neq \emptyset$ for each positive real number $\epsilon_1 > \epsilon_0$. Let’s denote the set of all the $\epsilon_0$-boundary points of $S$ in $\mathbb{R}^m$ by $Bd_{\epsilon_0}(S)$ or $\partial_\epsilon S$.

DEFINITION 2.3. Let $S$ be a subset of $\mathbb{R}^m$. A point $x$ is called the $\epsilon_0$-exterior point of $S$ if and only if $x$ is an $\epsilon_0$-interior point of $S^C = \mathbb{R}^m - S$. Let’s denote the set of all the $\epsilon_0$-exterior points of $S$ in $\mathbb{R}^m$ by $Ext_{\epsilon_0}(S)$.

REMARK 2.4. The union $\mathbb{R}^m = Int_{\epsilon_0}(S) \cup Bd_{\epsilon_0}(S) \cup Ext_{\epsilon_0}(S)$ is the mutually disjoint one, $S^o_\circ = S^o$ and $Int_{\epsilon_0}(S) \subseteq Int_0(S) = S^o$ for all $\epsilon_0 \geq 0$.

LEMMA 2.5. Let $S$ be a subset of $\mathbb{R}^m$ and suppose that $\epsilon_0 \geq 0$. Then $Int_{\epsilon_0}(S)$ and $Ext_{\epsilon_0}(S)$ are open subsets of $\mathbb{R}^m$. Hence $Bd_{\epsilon_0}(S)$ is closed in $\mathbb{R}^m$.

Proof. Let any element $x \in Int_{\epsilon_0}(S)$ be given. Then there is a positive real number $\epsilon_1 > \epsilon_0$ such that $x \in B(x, \epsilon_1) \subseteq S$. Consider the set $B(x, \frac{1}{3}(\epsilon_1 - \epsilon_0))$. For any point $y \in B(x, \frac{1}{3}(\epsilon_1 - \epsilon_0))$, we have, for any point $z \in B(y, \epsilon_0 + \frac{1}{3}(\epsilon_1 - \epsilon_0))$,
\[
\|x - z\| \leq \|x - y\| + \|y - z\| \\
< \frac{1}{3}(\epsilon_1 - \epsilon_0) + \epsilon_0 + \frac{1}{3}(\epsilon_1 - \epsilon_0) \\
< \epsilon_0 + \epsilon_1 - \epsilon_0 = \epsilon_1.
\]
Hence we have $B(y, \epsilon_0 + \frac{1}{3}(\epsilon_1 - \epsilon_0)) \subseteq B(x, \epsilon_1) \subseteq S$. Thus we have $y \in Int_{\epsilon_0}(S)$ since $\epsilon_0 + \frac{1}{3}(\epsilon_1 - \epsilon_0) > \epsilon_0$. Therefore, we have
\[
x \in B(x, \frac{1}{3}(\epsilon_1 - \epsilon_0)) \subseteq Int_{\epsilon_0}(S).
\]
This implies that $Int_{\epsilon_0}(S)$ is open. And $Ext_{\epsilon_0}(S)$ is also open since it is the $\epsilon_0$-interior of $S^C$. Since $\mathbb{R}^m = Int_{\epsilon_0}(S) \cup Bd_{\epsilon_0}(S) \cup Ext_{\epsilon_0}(S)$ is the disjoint union, $Bd_{\epsilon_0}(S) = \mathbb{R}^m - \{Int_{\epsilon_0}(S) \cup Ext_{\epsilon_0}(S)\}$ is closed in $\mathbb{R}^m$.

LEMMA 2.6. Let $S$ be a subset of $\mathbb{R}^m$ and suppose that $\epsilon_0 \geq 0$. Then we have $S'_{(\epsilon_0)} \subseteq Int_{\epsilon_0}(S) \cup Bd_{\epsilon_0}(S)$.

Proof. Let any element $x \in S'_{(\epsilon_0)}$ be given. Since $\mathbb{R}^m = Int_{\epsilon_0}(S) \cup Bd_{\epsilon_0}(S) \cup Ext_{\epsilon_0}(S)$ is a disjoint union, we need only to show that $x \notin$
Thus we need only to show that $x \in \text{Ext}_{\epsilon_0}(S)$. Then there is $\epsilon_1 > \epsilon_0$ such that $x \in B(x, \epsilon_1) \subseteq S^C$. Hence $B(x, \epsilon_1) \cap S = \emptyset$. This is a contradiction since $x \in S''_{(\epsilon_0)}$.

**Theorem 2.7.** Let $S$ be a subset of $R^m$ and suppose that $\epsilon_0 \geq 0$. Then $\overline{S}_{(\epsilon_0)} = \text{Int}_{\epsilon_0}(S) \cup \text{Bd}_{\epsilon_0}(S)$.

**Proof.** Since $R^m = \text{Int}_{\epsilon_0}(S) \cup \text{Bd}_{\epsilon_0}(S) \cup \text{Ext}_{\epsilon_0}(S)$ is the disjoint union and $S$ is disjoint from $\text{Ext}_{\epsilon_0}(S)$, we have $S \subseteq \text{Int}_{\epsilon_0}(S) \cup \text{Bd}_{\epsilon_0}(S)$. Hence, by lemma 2.6, we have

$$\overline{S}_{(\epsilon_0)} = S \cup S'_{(\epsilon_0)} \subseteq \text{Int}_{\epsilon_0}(S) \cup \text{Bd}_{\epsilon_0}(S).$$

In order to prove the equality, let any element $x \in \text{Int}_{\epsilon_0}(S) \cup \text{Bd}_{\epsilon_0}(S)$ be given. If $x \in S$ then we are done. Suppose that $x \notin S$. Then $x \notin \text{Int}_{\epsilon_0}(S)$. Thus we have $x \in \text{Bd}_{\epsilon_0}(S)$. Hence we have

$$\forall \epsilon_1 > \epsilon_0, B(x, \epsilon_1) \cap S \neq \emptyset \text{ and } B(x, \epsilon_1) \cap S^C \neq \emptyset.$$ 

Thus we have $\exists y_{\epsilon_1} \in S$ s.t. $y_{\epsilon_1} \in B(x, \epsilon_1)$. Since $y_{\epsilon_1} \neq x$, we have

$$\forall \epsilon_1 > \epsilon_0, y_{\epsilon_1} \in B(x, \epsilon_1) \cap (S - \{x\}) \neq \emptyset.$$ 

This implies that $x \in S'_{(\epsilon_0)}$ which completes the proof.

**Corollary 2.8.** Let $S$ be a subset of $R^m$ and suppose that $\epsilon_0 \geq 0$. Then

$$\overline{S}_{(\epsilon_0)} = \left\{ \left( S^C \right)_{(\epsilon_0)} \right\}^C.$$ 

**Proof.** Since $R^m = \text{Int}_{\epsilon_0}(S) \cup \text{Bd}_{\epsilon_0}(S) \cup \text{Ext}_{\epsilon_0}(S)$ is the disjoint union and $\overline{S}_{(\epsilon_0)} = \text{Int}_{\epsilon_0}(S) \cup \text{Bd}_{\epsilon_0}(S)$, the union of the equation $R^m = \overline{S}_{(\epsilon_0)} \cup \text{Ext}_{\epsilon_0}(S)$ is the disjoint one. Hence we have $\{ \overline{S}_{(\epsilon_0)} \}^C = \text{Ext}_{\epsilon_0}(S) = \{ S^C \}''_{(\epsilon_0)}$. Thus we have $\overline{S}_{(\epsilon_0)} = \left\{ \left( S^C \right)_{(\epsilon_0)} \right\}^C$.

**Theorem 2.9.** Let $S$ be a subset of $R^m$ and suppose that $\epsilon_0 \geq 0$. Then $R^m - \overline{S}_{(\epsilon_0)} = \text{Int}_{\epsilon_0}(S)$, i.e., $\overline{S^C}_{(\epsilon_0)} = \text{Int}_{\epsilon_0}(S)$.

**Proof.** By the definition of the $\epsilon_0$–closure of the set $S^C$, we have $\overline{S^C}_{(\epsilon_0)} = [S^C]''_{(\epsilon_0)} \cup S^C$. Hence we have $R^m - \overline{S^C}_{(\epsilon_0)} = \{ [S^C]''_{(\epsilon_0)} \}^C \cap S$. Thus we need only to show that $\text{Int}_{\epsilon_0}(S) = \{ [S^C]''_{(\epsilon_0)} \}^C \cap S$. Let any
element \( x \in \text{Int}_{\epsilon_0}(S) \) be given. Then we have

\[ \exists \epsilon_1 > \epsilon_0 \text{ s.t. } x \in B(x, \epsilon_1) \subseteq S \]

\[ \Rightarrow B(x, \epsilon_1) \cap S^C = \emptyset \text{ and } x \in S \]

\[ \Rightarrow x \notin [S^C]^{(\epsilon_0)} \text{ and } x \in S \]

\[ \Rightarrow x \in \{[S^C]^{(\epsilon_0)} \}^C \cap S. \]

Conversely, let any element \( x \in \{[S^C]^{(\epsilon_0)} \}^C \cap S \) be given. Since \( x \in S \) is not a member of \([S^C]^{(\epsilon_0)}\), we have

\[ \exists \epsilon_1 > \epsilon_0 \text{ s.t. } B(x, \epsilon_1) \cap (S^C - \{x\}) = \emptyset. \]

Since \( x \in S \) and \( S^C - \{x\} = S^C \), we also have \( B(x, \epsilon_1) \cap S^C = \emptyset \). Thus we have \( x \in B(x, \epsilon_1) \subseteq S \). Therefore, we have \( x \in \text{Int}_{\epsilon_0}(S) \) which completes the proof.

**Theorem 2.10.** (Representation) Let \( S \) be a subset of \( \mathbb{R}^m \) and suppose that \( \epsilon_0 \geq 0 \). Then we have

\[ \text{Bd}_{\epsilon_0}(S) = \bigcup_{x \in \partial S} \overline{B}(x, \epsilon_0). \]

Moreover, if \( \epsilon_0 > 0 \) then \( \partial S \) is an \( \epsilon_0 \)-dense subset of the interior of the subset \( \text{Bd}_{\epsilon_0}(S) \).

**Proof.** Let \( x \in \partial S \) and any element \( y \in \overline{B}(x, \epsilon_0) \) be given. For each positive real number \( \epsilon > \epsilon_0 \), we have \( x \in B(y, \epsilon) \). Hence \( x \in B(x, \epsilon - \epsilon_0) \subseteq B(y, \epsilon) \). Since \( x \in \partial S \),

\[ B(x, \epsilon - \epsilon_0) \cap S \neq \emptyset \text{ and } B(x, \epsilon - \epsilon_0) \cap S^C \neq \emptyset. \]

Thus we have

\[ B(y, \epsilon) \cap S \neq \emptyset \text{ and } B(y, \epsilon) \cap S^C \neq \emptyset. \]

Hence we have \( y \in \text{Bd}_{\epsilon_0}(S) \). Thus we have \( \overline{B}(x, \epsilon_0) \subseteq \text{Bd}_{\epsilon_0}(S) \) for all elements \( x \in \partial S \). Therefore, we have \( \bigcup_{x \in \partial S} \overline{B}(x, \epsilon_0) \subseteq \text{Bd}_{\epsilon_0}(S) \). Conversely, let any element \( y \in \text{Bd}_{\epsilon_0}(S) \) be given. For each natural number \( n \), we have

\[ B(y, \epsilon_0 + \frac{1}{n}) \cap S \neq \emptyset \text{ and } B(y, \epsilon_0 + \frac{1}{n}) \cap S^C \neq \emptyset. \]

Hence there are two sequences \( \{w_n\}, \{z_n\} \) in \( \mathbb{R}^m \) such that \( \{w_n\} \subseteq S \), \( \{z_n\} \subseteq S^C \) and \( w_n, z_n \in B(y, \epsilon_0 + \frac{1}{n}) \) for each natural number \( n \). Since they are bounded, we may assume by using their subsequences that \( \lim_{n \to \infty} w_n = w_0 \) and \( \lim_{n \to \infty} z_n = z_0 \) for some elements \( w_0 \in S \) and \( z_0 \in S^C \).
Note that $\partial S = \partial S^C$. If $w_0 \in \partial S$ or $z_0 \in \partial S$ then we are done since $y \in \overline{B}(w_0, \epsilon_0)$ with $w_0 \in \partial S$ or $y \in \overline{B}(z_0, \epsilon_0)$ with $z_0 \in \partial S$. Now suppose that $w_0 \notin \partial S$ and $z_0 \notin \partial S$. Then we must have $w_0 \in \text{Int}(S)$ and $z_0 \in \text{Ext}(S)$. Now consider the line segment $\overline{w_0z_0}$ joining the points $w_0$ and $z_0$. We have $\overline{w_0z_0} \cap \partial S \neq \emptyset$ since $\overline{w_0z_0}$ is connected. Choosing an element $x_0 \in \overline{w_0z_0} \cap \partial S$, we have $x_0 = t_0w_0 + (1 - t_0)z_0$ for some real number $0 < t_0 < 1$. Thus we have

$$
\|y - x_0\| = \|t_0y + (1 - t_0)y - \{t_0w_0 + (1 - t_0)z_0\}\|
\leq t_0\|y - w_0\| + (1 - t_0)\|y - z_0\| 
\leq t_0\epsilon_0 + (1 - t_0)\epsilon_0 = \epsilon_0.
$$

Hence $y \in \overline{B}(x_0, \epsilon_0) \subseteq \bigcup_{x \in \partial S} \overline{B}(x, \epsilon_0)$. Moreover, if $\epsilon_0 > 0$ then $\partial S$ is a subset of the interior of $\text{Bd}_{\epsilon_0}(S)$. Thus $\partial S$ is an $\epsilon_0$-dense subset of the interior of the subset $\text{Bd}_{\epsilon_0}(S)$ by the lemma 1.6. \qed

**THEOREM 2.11.** (Core) Let $S$ be a subset of $\mathbb{R}^m$ and suppose that $\epsilon_0 \geq 0$. Then

$$
\text{Int}_{\epsilon_0}(S) = S - \bigcup_{x \in \partial S} \overline{B}(x, \epsilon_0).
$$

**Proof.** By the theorem just above, we need only to show that $\text{Int}_{\epsilon_0}(S) = S - \text{Bd}_{\epsilon_0}(S)$. Let any element $x \in S - \text{Bd}_{\epsilon_0}(S)$ be given. Then $x \in S$ and $x \notin \text{Bd}_{\epsilon_0}(S)$. Since $\mathbb{R}^m = \text{Int}_{\epsilon_0}(S) \cup \text{Bd}_{\epsilon_0}(S) \cup \text{Ext}_{\epsilon_0}(S)$ is the disjoint union, we must have $x \in \text{Int}_{\epsilon_0}(S)$. Conversely, let any element $x \in \text{Int}_{\epsilon_0}(S)$ be given. Then we clearly have $x \in S$, $x \notin \text{Ext}_{\epsilon_0}(S)$ and $x \notin \text{Bd}_{\epsilon_0}(S)$. Thus we have $x \in S - \text{Bd}_{\epsilon_0}(S)$. \qed

**LEMMA 2.12.** A subset $F$ of $\mathbb{R}^m$ is the boundary of some open subset in $\mathbb{R}^m$ if and only if $F$ is closed and nowhere dense.

**Proof.** First, suppose that $F$ is the boundary of some open subset $S$ in $\mathbb{R}^m$. Then it is clear that $F$ is closed. Since the interior $S$ of the set $S$ is disjoint from the boundary $F$ of $S$, we have $S \cap F = \emptyset$. If some point $x \in F$ is an interior point of $F$ then there is a positive real number $\epsilon_1 > 0$ such that $x \in B(x, \epsilon_1) \subseteq F$. Since $S \cap F = \emptyset$, this implies that $B(x, \epsilon_1) \cap S = \emptyset$. Thus we have $x \in B(x, \epsilon_1) \subseteq S^C$. This implies that $x \in \text{Ext}(S)$. This is a contradiction since the boundary is disjoint from the exterior. This contradiction implies that $F$ is nowhere dense. Now suppose that $F$ is closed and nowhere dense. Take $S = F^C$. Then $S$ is an open subset of $\mathbb{R}^m$. We need only to prove that $F = \partial F^C$. First, we have $\partial F^C \cap F^C = \emptyset$ since $F^C$ is open. Hence we have $\partial F^C \subseteq F$. Next,
let any element \( x \in F \) be given. Then \( B(x, \epsilon) \cap F \neq \emptyset \) for all positive real number \( \epsilon > 0 \) since this intersection contains the element \( x \). Moreover, the open ball \( B(x, \epsilon) \) cannot be a subset of \( F \) for all positive real number \( \epsilon > 0 \) since \( F \) is nowhere dense. Thus we also have \( B(x, \epsilon) \cap F^c \neq \emptyset \) for all positive real number \( \epsilon > 0 \). Hence we have \( x \in \partial F^c = \partial S \). Thus \( F = \partial S \).

**Corollary 2.13.** Let \( S \) be any subset of \( R^m \). Then \( \partial S \) is the boundary of some open subset of \( R^m \).

**Proof.** Let \( S \) be any subset of \( R^m \). Since \( S^c \) is open, \( \partial S^c \) is nowhere dense by the lemma just above. But we have \( \partial S = \partial S^c \). Hence we have \( \{ \partial S \}^o = \{ \partial S^c \}^o = \emptyset \).

**Theorem 2.14.** Let \( F \) be a non-empty subset of \( R^m \) and \( \epsilon_0 \geq 0 \). Then \( F \) is the \( \epsilon_0 \)-boundary of some open subset of \( R^m \) if and only if \( F = \bigcup_{x \in S} B(x, \epsilon_0) \) for some closed and nowhere dense subset \( S \) of \( R^m \).

**Proof.** First, suppose that \( F \) is the \( \epsilon_0 \)-boundary of some open subset \( G \) of \( R^m \). Then the boundary \( S = \partial G \) of \( G \) is closed and nowhere dense subset of \( R^m \) by the lemma just above. Moreover, we have \( F = \bigcup_{x \in S} B(x, \epsilon_0) \) by the theorem 2.10. Hence we have \( F = \bigcup_{x \in S} B(x, \epsilon_0) \). Conversely, suppose that \( F = \bigcup_{x \in S} B(x, \epsilon_0) \) for some closed and nowhere dense subset \( S \) of \( R^m \). Then \( S \) is the boundary \( \partial G \) of some open subset \( G \) of \( R^m \) by the lemma just above. The \( \epsilon_0 \)-boundary of this open subset \( G \) is given by \( \partial S = \bigcup_{x \in S} B(x, \epsilon_0) \) by the theorem 2.10. Thus \( F \) is the \( \epsilon_0 \)-boundary of the open subset \( G \).

**Lemma 2.15.** Let \( S, T \) be any subsets of \( R^m \) and suppose that \( \epsilon_0 \geq 0 \). Then

(1) \( \text{Int}_{\epsilon_0}(S \cap T) = \text{Int}_{\epsilon_0}(S) \cap \text{Int}_{\epsilon_0}(T) \).

(2) \( \text{Ext}_{\epsilon_0}(S \cup T) = \text{Ext}_{\epsilon_0}(S) \cap \text{Ext}_{\epsilon_0}(T) \).

**Proof.** (1) Since \( S \cap T \) is a subset of \( S \) and \( T \), we have \( \text{Int}_{\epsilon_0}(S \cap T) \subseteq \text{Int}_{\epsilon_0}(S) \cap \text{Int}_{\epsilon_0}(T) \). Conversely, if \( x \in \text{Int}_{\epsilon_0}(S) \cap \text{Int}_{\epsilon_0}(T) \) is any element then we have \( \exists \epsilon_1 > \epsilon_0 \text{ s.t. } x \in B(x, \epsilon_1) \subseteq S \) and \( \exists \epsilon_2 > \epsilon_0 \text{ s.t. } x \in B(x, \epsilon_2) \subseteq T \).
Hence we have the statement
\[ \exists \epsilon_3 = \min\{\epsilon_1, \epsilon_2\} > \epsilon_0 \text{ s.t. } x \in B(x, \epsilon_3) \subseteq S \cap T \]
which implies that \( x \in \text{Int}_{\epsilon_0}(S \cap T) \). (2) By (1), we have
\[ \text{Int}_{\epsilon_0}(S^C \cap T^C) = \text{Int}_{\epsilon_0}(S^C) \cap \text{Int}_{\epsilon_0}(T^C). \]
Since \( \text{Int}_{\epsilon_0}(S^C) = \text{Ext}_{\epsilon_0}(S) \), we have the desired result \( \text{Ext}_{\epsilon_0}(S \cup T) = \text{Ext}_{\epsilon_0}(S) \cap \text{Ext}_{\epsilon_0}(T) \).

Note that \( \text{Int}_{\epsilon_0}(S) \cup \text{Int}_{\epsilon_0}(T) \subseteq \text{Int}_{\epsilon_0}(S \cup T) \) in general.

**Theorem 2.16.** Let \( S, T \) be any subsets of \( \mathbb{R}^m \) and suppose that \( \epsilon_0 \geq 0 \). Then \( \overline{\text{Cl}_{\epsilon_0}(S \cup T)} = \overline{\text{Cl}_{\epsilon_0}(S)} \cup \overline{\text{Cl}_{\epsilon_0}(T)} \).

**Proof.** By the corollary 2.8 and the lemma 2.15, we have
\[ (S \cup T)_{(\epsilon_0)} = [((S \cup T)^C)_{(\epsilon_0)}]^C \]
\[ = [((S^C \cap T^C))_{(\epsilon_0)}]^C \]
\[ = ((S^C)_{(\epsilon_0)} \cap (T^C)_{(\epsilon_0)})^C \]
\[ = ((S^C)_{(\epsilon_0)})^C \cup ((T^C)_{(\epsilon_0)})^C \]
\[ = \overline{S(\epsilon_0)} \cup \overline{T(\epsilon_0)} \]
which completes the proof. \( \square \)

### 3. Thickness

By the corollary 2.13, we have \( \{\partial S\}_o = \emptyset \) for all subsets of \( \mathbb{R}^m \). But the similar relation \( \{\partial_{\epsilon_0} S\}_{(\epsilon_0)} = \emptyset \) is not true in general if \( \epsilon_0 \neq 0 \). For if \( S = \{A, B, C\} \) is the vertices of the equilateral triangle in \( \mathbb{R}^2 \), then we have \( \frac{A+B+C}{3} \in \{\partial_{\epsilon_0} S\}_{(\epsilon_0)} \) with \( \epsilon_0 = \|A - B\| \). This leads us to the following concept of the thickness.

**Definition 3.1.** Let \( S \) be a non-empty subset of \( \mathbb{R}^m \) and \( \epsilon_0 \geq 0 \). Then \( S \) is said to be \( \epsilon_0 \)--thick at a point \( p \in S \) if and only if \( p \in \text{Int}_{\epsilon_0}(S) \). In this case, we call that \( p \) is an \( \epsilon_0 \)--thick point or spot of \( S \).

Note that \( \text{Int}_{\epsilon_0}(S) \) is the set of all the \( \epsilon_0 \)--thick points of \( S \). We call the closure \( \overline{\text{Int}_{\epsilon_0}(S)} \) the \( \epsilon_0 \)--core of \( S \). In accordance to the theorem 2.11, the \( \epsilon_0 \)--core of \( S \) is the closure of the set \( \text{Int}_{\epsilon_0}(S) = S - \bigcup_{x \in \partial S} B(x, \epsilon_0) \).

Note also that if \( S \) is \( \epsilon_0 \)--thick at a point \( p \in S \) then \( S \) is \( \epsilon \)--thick at a point \( p \in S \) for all \( 0 < \epsilon < \epsilon_1 \) for some \( \epsilon_1 \) with \( \epsilon_0 < \epsilon_1 \).
DEFINITION 3.2. Let $S$ be a non-empty subset of $R^m$ and $\epsilon_0 \geq 0$. Then $S$ is said to be not $\epsilon_0$—thick anywhere or nowhere $\epsilon_0$—thick if and only if $Int_{\epsilon_0}(S) = \emptyset$.

THEOREM 3.3. Let $S$ be any subsets of $R^m$ and suppose that $\epsilon_0 \geq 0$. If $Bd_{\epsilon_0}(S)$ is nowhere $\epsilon_0$—thick then $Bd(S)$ is closed and nowhere dense, but not conversely.

Proof. The boundary $Bd(S)$ is clearly closed in $R^m$. Suppose that $Bd(S)$ is not nowhere dense. Then $Int(Bd(S)) \neq \emptyset$. Hence there is a point $x_0 \in Bd(S)$ such that $B(x_0, \epsilon_1) \subseteq Bd(S)$ for some positive real number $\epsilon_1 > 0$. Then we have

$$x_0 \in B(x_0, \epsilon_0 + \frac{\epsilon_1}{2}) \subseteq \bigcup_{x \in Bd(S)} \overline{B}(x, \epsilon_0) = Bd_{\epsilon_0}(S).$$

Thus we have $x_0 \in Int_{\epsilon_0}(Bd_{\epsilon_0}(S))$. Hence $Bd_{\epsilon_0}(S)$ is $\epsilon_0$—thick at $x_0$. In order to show that the converse is not true in general, choose the open set $S = B(0, \epsilon_0)$ with $\epsilon_0 > 0$. Then we have $Bd(S) = \{x \in R^m : \|x - 0\| = \epsilon_0\} = S(0, \epsilon_0)$. The sphere $S(0, \epsilon_0)$ is closed and nowhere dense. But we have

$$0 \in B(0, \frac{3}{2} \epsilon_0) \subseteq \bigcup_{x \in Bd(S)} \overline{B}(x, \epsilon_0) = Bd_{\epsilon_0}(S).$$

Hence $Bd_{\epsilon_0}(S)$ is $\epsilon_0$—thick at the origin $0$.

Let $u$ be any non-zero vector in $R^m$. Let’s denote the orthogonal space by $u^\perp = \{z \in R^m : z \cdot u = 0\}$. Recall that the projection of a vector $x \in R^m$ along the vector $u$ is given by $proj_u(x) = \frac{u \cdot x}{u \cdot u} u$. Let’s denote the parallel projection from $R^m$ to $u^\perp$ by $\Pi_{(u^\perp)}(x) = x - proj_u(x)$.

THEOREM 3.4. Let $S$ be any subsets of $R^m$ and suppose that $\epsilon_0 \geq 0$. If $S$ is $\epsilon_0$—thick at a point $p \in S$ in $R^m$ then for any non-zero vector $u \in R^m$ the set $\Pi_{(u^\perp)}(S) = \{\Pi_{(u^\perp)}(x) : x \in S\}$ is $\epsilon_0$—thick at the point $\Pi_{(u^\perp)}(p)$ in the $m-1$ dimensional space $\Pi_{(u^\perp)}(R^m)$, but not conversely.

Proof. Suppose that $S$ is $\epsilon_0$—thick at a point $p \in S$ in $R^m$ and let $u$ be any non-zero vector in $R^m$. Then there is a positive real number $\epsilon_1 > \epsilon_0$ such that $p \in B(p, \epsilon_1) \subseteq S$. Hence we have

$$\Pi_{(u^\perp)}(p) \in \Pi_{(u^\perp)}(B(p, \epsilon_1)) \subseteq \Pi_{(u^\perp)}(S).$$

This completes the proof of the first part since $\Pi_{(u^\perp)}(B(p, \epsilon_1))$ is an open ball in $\Pi_{(u^\perp)}(R^m)$ with the same radius $\epsilon_1$. Now let $\{A, B, C\}$ be the vertices of the equilateral triangle in $R^2$ with $\|A - B\| = 2\epsilon_0$. Then the
set \( S = B(A, \epsilon_0) \cup B(B, \epsilon_0) \cup B(C, \epsilon_0) \) is not \( \epsilon_0 \)--thick at any point. But the set \( \Pi_{(u^+)}(S) \) is obviously \( \epsilon_0 \)--thick at some point for any direction \( u \) in \( \mathbb{R}^n \).

**Lemma 3.5.** Let \( \epsilon_0 > 0 \) be given. If \( P, Q \in \mathbb{R}^2 \) are distinct points with \( \| P - Q \| < 2 \epsilon_0 \), then there are two points \( U, V \in \mathbb{R}^2 \) such that \( \| U - P \| = \| U - Q \| = \epsilon_0 = \| V - P \| = \| V - Q \| \).

**Proof.** We clearly have \( S(P, \epsilon_0) \cap S(Q, \epsilon_0) = \{ U, V \} \).

**Remark 3.6.** It is obvious that \( \text{Int}_{\epsilon_0} [\overline{B}(P, \epsilon_0) \cup \overline{B}(Q, \epsilon_0)] = \emptyset \) for any two points \( P, Q \) in \( \mathbb{R}^2 \).

**Theorem 3.7.** Let \( P, Q, U, V \in \mathbb{R}^2 \) be the four points in the above lemma with \( P \) on the left, \( Q \) on the right, \( U \) at the top and \( V \) at the bottom. If a point \( T \in \mathbb{R}^2 \) is an element of the intersection \( \overline{B}(U, \epsilon_0) \cap \overline{B}(V, \epsilon_0) \) then we have

\[
\text{Int}_{\epsilon_0} [\overline{B}(P, \epsilon_0) \cup \overline{B}(Q, \epsilon_0) \cup \overline{B}(T, \epsilon_0)] = \emptyset.
\]

**Proof.** Put \( Z = \overline{B}(P, \epsilon_0) \cup \overline{B}(Q, \epsilon_0) \cup \overline{B}(T, \epsilon_0) \). If \( T \) is a boundary point of the intersection \( \overline{B}(U, \epsilon_0) \cap \overline{B}(V, \epsilon_0) \) then the three spheres \( S(T, \epsilon_0), S(P, \epsilon_0) \) and \( S(Q, \epsilon_0) \) meet at the point \( U \) or \( V \). Suppose that they meet at the point \( V \). Then for any point \( x \in \overline{B}(V, \epsilon_0) \) we have \( \| x - V \| \leq \epsilon_0 \). Since \( V \) is a boundary point of the union \( Z \), this implies that any point \( x \) in the set \( \overline{B}(V, \epsilon_0) \cap Z \) is not an \( \epsilon_0 \)--interior point of \( Z \). Since the sphere \( S(V, \epsilon_0) \) passes through the center points \( P, Q, T \) of the three spheres \( S(P, \epsilon_0), S(Q, \epsilon_0) \) and \( S(T, \epsilon_0) \), we also have \( \text{dist}(x, \partial(Z - \overline{B}(V, \epsilon_0))) \leq \epsilon_0 \) for all the points \( x \in Z - \overline{B}(V, \epsilon_0) \). Thus we have \( \text{Int}_{\epsilon_0}(Z) = \emptyset \). The proof of the case where they meet at the point \( U \) is similarly handled. On the other hand, suppose that the point \( T \) is in the interior of the intersection \( \overline{B}(U, \epsilon_0) \cap \overline{B}(V, \epsilon_0) \). Then the center points \( U, V \) are in the open ball \( \overline{B}(T, \epsilon_0) \) and the sphere \( S(T, \epsilon_0) \) meets the boundary of the union \( \overline{B}(P, \epsilon_0) \cup \overline{B}(Q, \epsilon_0) \) at the four points, say \( A, B, C \) and \( D \). Let’s call the point on the upper left \( A \), the point on the lower left \( B \), the point on the upper right \( C \) and the point on the lower right \( D \). Then, for any point \( x \) of the union of the rhombi \( \diamond APBT \) and \( \diamond CTDQ \), we have \( \text{dist}(x, \partial(Z)) \leq \epsilon_0 \) since the points \( A, B, C \) and \( D \) are in the boundary of \( Z \). And, for any point \( x \) in the union of the four circular sectors \( \diamond APB, \diamond ATC, \diamond BTD \) and \( \diamond CQD \), we also have \( \text{dist}(x, \partial(Z)) \leq \epsilon_0 \) since all of the circular arcs of these four
circular sectors are parts of the boundary of $Z$. Therefore, we have $\text{dist}(x, \partial(Z)) \leq \epsilon_0$ for all the points $x \in Z$. Consequently, we have $\text{Int}_{\epsilon_0}(Z) = \emptyset$. □

**Corollary 3.8.** Let $P_1, P_2, P_3$ be three points in $R^2$. Suppose that

$$\text{Int}_{\epsilon_0} \left[ \overline{B}(P_1, \epsilon_0) \cup \overline{B}(P_2, \epsilon_0) \cup \overline{B}(P_3, \epsilon_0) \right] \neq \emptyset.$$  

Then we have

1. $S(P_1, \epsilon_0) \cap S(P_2, \epsilon_0) = \{U_1, V_2\}$ and $P_3 \notin \overline{B}(U_1, \epsilon_0) \cap \overline{B}(V_2, \epsilon_0)$
2. $S(P_2, \epsilon_0) \cap S(P_3, \epsilon_0) = \{U_2, V_3\}$ and $P_1 \notin \overline{B}(U_2, \epsilon_0) \cap \overline{B}(V_3, \epsilon_0)$
3. $S(P_3, \epsilon_0) \cap S(P_1, \epsilon_0) = \{U_3, V_1\}$ and $P_2 \notin \overline{B}(U_3, \epsilon_0) \cap \overline{B}(V_1, \epsilon_0)$.

**Proof.** (1) From the theorem just above, if $S(P_1, \epsilon_0) \cap S(P_2, \epsilon_0) = \{U_1, V_2\}$ and $P_3 \in \overline{B}(U_1, \epsilon_0) \cap \overline{B}(V_2, \epsilon_0)$ then

$$\text{Int}_{\epsilon_0} \left[ \overline{B}(P_1, \epsilon_0) \cup \overline{B}(P_2, \epsilon_0) \cup \overline{B}(P_3, \epsilon_0) \right] = \emptyset.$$  

The proofs of (2) and (3) are quite similar to the proof of (1) and we omit them. □

**Theorem 3.9.** Let $P, Q, U, V$ be the four mutually distinct points in $R^2$ such that $S(P, \epsilon_0) \cap S(Q, \epsilon_0) = \{U, V\}$ with $P$ on the left, $Q$ on the right, $U$ at the top and $V$ at the bottom. If a point $T \in R^2$ is an element of the union

$$[B(U, \epsilon_0) - B(V, \epsilon_0)] \cup [B(V, \epsilon_0) - B(U, \epsilon_0)]$$

then $Z = B(P, \epsilon_0) \cup B(Q, \epsilon_0) \cup B(T, \epsilon_0)$ is $\epsilon_0$-thick at some point.

**Proof.** We need only to prove the case where $T \in [B(U, \epsilon_0) - B(V, \epsilon_0)]$ since the other case is similarly handled. Then we have $U \in B(T, \epsilon_0)$ and $V \notin B(T, \epsilon_0)$. And the sphere $S(T, \epsilon_0)$ meets the boundary of the set $\overline{B}(P, \epsilon_0) \cup \overline{B}(Q, \epsilon_0)$ at two points, say $L$ on the left, $R$ on the right. Consider the triangle $\triangle LVR$. Let’s denote by $V'$ the point at which the line segment connecting the midpoint $L + \overline{LR}$ and the vertex $V$ intersects the sphere $S(T, \epsilon_0)$. Now if $\angle LV'R \leq \frac{\pi}{2}$ then the radius of the circumscribed circle of the triangles $\triangle LV'R$ is $\epsilon_0$ and $0 < \angle LV'R < \angle LV'R \leq \frac{\pi}{2}$. Hence if $r$ is the radius of the circumscribed circle of the triangle $\triangle LV'R$ then we have

$$2\epsilon_0 = \frac{\overline{TR}}{\sin(\angle LV'R)} < \frac{\overline{TR}}{\sin(\angle LV'R)} = 2r,$$

i.e., $\epsilon_0 < r$. 


On the other hand, if $\angle LV'R > \frac{\pi}{2}$ then the point $T$ is positioned higher than the line segment $LR$. In this case, let $C$ be the image of the reflection of the circle $S(T, \epsilon_0)$ with respect to the line segment $LR$. Let’s denote by $V''$ the point at which the line segment connecting the midpoint $\frac{L+R}{2}$ and the vertex $V$ intersects this circle $C$. Then the point $V''$ lies inside the triangle $\triangle LVR$ and we have $\angle LV''R \leq \frac{\pi}{2}$. Hence the radius $r$ of the circumscribed circle of the triangle $\triangle LVR$ still satisfies the relation $\epsilon_0 < r$ since the radius of the circumscribed circle of the triangles $\angle LV''R$ is $\epsilon_0$ and $0 < \angle LV'R < \angle LV''R \leq \frac{\pi}{2}$. Since the three sides $LV$, $RV$ and $LR$ of the triangle $\triangle LVR$ are parts of the closed balls $B(P, \epsilon_0)$, $B(Q, \epsilon_0)$ and $B(T, \epsilon_0)$, respectively, the circumscribed circle and its interior of the triangle $\triangle LVR$ is a subset of the union $Z$. Thus $Z$ contains an open ball with radius $\frac{\epsilon_0 + r}{2}$ which implies that $Int_{\epsilon_0}(Z) \neq \emptyset$.

**Theorem 3.10.** (Three points thickness) Let $P, Q$ be the two distinct points in $R^2$ with $\|P - Q\| < 2\epsilon_0$ such that $S(P, \epsilon_0) \cap S(Q, \epsilon_0) = \{U, V\}$ with $P$ on the left, $Q$ on the right, $U$ at the top and $V$ at the bottom. For a point $T \in R^2$, the union $Z = B(P, \epsilon_0) \cup B(Q, \epsilon_0) \cup B(T, \epsilon_0)$ is $\epsilon_0$-thick at some point of $Z$ if and only if

$$T \in \{B(U, \epsilon_0) - B(V, \epsilon_0)\} \cup \{B(V, \epsilon_0) - B(U, \epsilon_0)\}.$$

**Proof.** By means of the theorems 3.7 and 3.9, we need only to prove that if $T \notin B(U, \epsilon_0) \cup B(V, \epsilon_0)$ then $Z$ is nowhere $\epsilon_0$-thick. Suppose that $T \notin B(U, \epsilon_0) \cup B(V, \epsilon_0)$. Then we have $U, V \notin B(T, \epsilon_0)$. Now there are three cases depending on the relative position of the two points $U, V$ with respect to the sphere $S(T, \epsilon_0)$.

Case I. $U, V \notin S(T, \epsilon_0)$. In this case, the intersection of the sphere $S(T, \epsilon_0)$ and the boundary of the union $B(P, \epsilon_0) \cup B(Q, \epsilon_0)$ is a subset $A$ of $R^2$ consisting of no point, one point, two points, three points or four points. But all the points of the union $A \cup \{U, V\}$ are the boundary point of the union $Z$. Hence we have $Int_{\epsilon_0}(Z) = \emptyset$.

Case II. $U$ or $V \in S(T, \epsilon_0)$ and $S(T, \epsilon_0) \cap \partial [B(P, \epsilon_0) \cup B(Q, \epsilon_0)]$ is consisting of the two elements. In this case, we may assume that this intersection contains the point $V$ since the case where it contains $U$ is similarly handled. Then we have $\|x - V\| \leq 2\epsilon_0$ for all the points $x \in Z$. Since $V$ is a boundary point of $Z$, this implies that $Int_{\epsilon_0}(Z) = \emptyset$.

Case III. $U$ or $V \in S(T, \epsilon_0)$ and $S(T, \epsilon_0) \cap \partial [B(P, \epsilon_0) \cup B(Q, \epsilon_0)]$ is consisting of the three elements. In this case, we may also assume that
the set of the last intersection is \( \{ E, V, F \} \) with \( E \in S(T, \epsilon_0) \cap S(P, \epsilon_0) \).
Since the quadrilaterals \( \square PETV \) and \( \square QVTF \) are the rhombi, we have \( PQ = EF \). Similarly, we have \( EU = TQ \) and \( PT = UF \) by using the appropriate rhombi. Thus the triangles \( \triangle UEF \) and \( \triangle PQT \) are congruent. Since \( PV = TV = QV = \epsilon_0 \), the point \( V \) is the circumcenter of the triangle \( \triangle PQT \). Hence the radius of the circumscribed circle of \( \triangle UEF \) is \( \epsilon_0 \). Since all the three points \( U, E, F \) are the boundary points of \( Z \), this implies that \( \text{Int}_{\epsilon_0}(Z) = \emptyset \). \( \square \)

References

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