INTUITIONISTIC FUZZY PMS-SUBALGEBRA OF A PMS-ALGEBRA

Beza Lamesgin Derseh*, Berhanu Assaye Alaba, and Yohannes Gedamu Wondifraw

Abstract. In this paper, we introduce the notion of intuitionistic fuzzy PMS-subalgebra of a PMS-algebra. The idea of level subsets of an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra is introduced. The relation between intuitionistic fuzzy sets and their level sets in a PMS-algebra is examined, and some interesting results are obtained.

1. Introduction


In this paper, we introduced the notion of intuitionistic fuzzy PMS-subalgebra of PMS-algebras and investigate some of their properties. The idea of level subsets of an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra is introduced. The relation between intuitionistic fuzzy sets and their level sets in a PMS-algebra is examined, and some interesting results are obtained.


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* Corresponding author.


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2. Preliminaries

In this section, we recall some basic definitions and results that are used in the study of this paper.

**Definition 2.1.** [12] A nonempty set $X$ with a constant $0$ and a binary operation $' \ast '$ is called PMS-algebra if it satisfies the following axioms.

1. $0 \ast x = x$
2. $(y \ast x) \ast (z \ast x) = z \ast y$, for all $x, y, z \in X$.

In $X$, we define a binary relation $\leq$ by $x \leq y$ if and only if $x \ast y = 0$.

**Definition 2.2.** [12] Let $S$ be a nonempty subset of a PMS-algebra $X$, then $S$ is called a PMS-sub algebra of $X$ if $x \ast y \in S$, for all $x, y \in S$.

**Example 2.3.** [12] Let $Z$ be the set of all integers, and let $\ast$ be a binary relation on $Z$ defined by $x \ast y = y - x$, for all $x, y \in Z$, where ‘$-$’ the usual subtraction of integers. Then $(Z, \ast, 0)$ is a PMS-algebra since

1. $0 \ast x = x - 0 = x$
2. $(y \ast x) \ast (z \ast x) = (z \ast x) - (y \ast x) = (x - z) - (x - y) = y - z = z \ast y$.

Clearly, the set $E$ of all even integers is a PMS-subalgebra of a PMS-algebra $Z$, since $x \ast y = y - x \in E$ for all $x, y \in E$.

**Proposition 2.4.** [12] In any PMS-algebra $(X, \ast, 0)$ the following properties hold for all $x, y, z \in X$.

1. $x \ast x = 0$
2. $(y \ast x) \ast x = y$
3. $x \ast (y \ast x) = y \ast 0$
4. $(y \ast x) \ast z = (z \ast x) \ast y$
5. $(x \ast y) \ast 0 = y \ast x = (0 \ast y) \ast (0 \ast x)$

**Definition 2.5.** [15] Let $X$ be a nonempty set. A fuzzy subset $A$ of the set $X$ is defined as $A = \{ (x, \mu_A(x)) | x \in X \}$ where the mapping $\mu_A : X \rightarrow [0, 1]$ defines the degree of membership.

**Definition 2.6.** [11] A fuzzy set $A$ in a PMS-algebra $X$ is called fuzzy PMS-subalgebra of $X$ if $\mu_A(x \ast y) \geq \min \{ \mu_A(x), \mu_A(y) \}$ for all $x, y \in X$.

**Definition 2.7.** [2, 4] An intuitionistic fuzzy set (IFS) $A$ in a nonempty set $X$ is an object having the form $A = \{ (x, \mu_A(x), \nu_A(x)) | x \in X \}$, where the functions $\mu_A : X \rightarrow [0, 1]$ and $\nu_A : X \rightarrow [0, 1]$ define the degree of membership and the degree of non membership, respectively, satisfying the condition $0 \leq \mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$.

**Remark 2.8.** Ordinary fuzzy sets over $X$ may be viewed as special intuitionistic fuzzy sets with the non membership function $\nu_A(x) = 1 - \mu_A(x)$. So each Ordinary fuzzy set may be written as $\{ (x, \mu_A(x), 1 - \mu_A(x)) | x \in X \}$ to define an intuitionistic fuzzy set. For the sake of simplicity we write $A = (\mu_A, \nu_A)$ for an intuitionistic fuzzy set $A = \{ (x, \mu_A(x), \nu_A(x)) | x \in X \}$.

**Definition 2.9.** [2–4] Let $A$ and $B$ be two intuitionistic fuzzy subsets of the set $X$, where $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$, then
Let $A$ be an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra $X$ denoted by proposition 2.1(1), we have
\[
\mu_A(x) = \begin{cases} 
1 & \text{if } x = 0 \\
0.5 & \text{if } x = 1, 2 \\
0 & \text{if } x = 3
\end{cases}
\]
and
\[
\nu_A(x) = \begin{cases} 
0 & \text{if } x = 0 \\
0.4 & \text{if } x = 1, 2 \\
1 & \text{if } x = 3
\end{cases}
\]

For intuitionistic fuzzy set $A$ in a PMS-algebra $X$ with membership values $\mu_A(x)$ and non membership values $\nu_A(x)$ as defined above, definition 3.1 is satisfied. Therefore $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS- subalgebra of the PMS-algebra $X$.

**Lemma 3.3.** If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $X$, then $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$ for all $x \in X$.

**Proof.** Suppose $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $X$. Since $x * x = 0$ for every $x \in X$ by proposition 2.1(1), we have
\[
\mu_A(0) = \mu_A(x * x) \geq \min\{\mu_A(x), \mu_A(x)\} = \mu_A(x) \text{ and }
\nu_A(0) = \nu_A(x * x) \leq \max\{\nu_A(x), \nu_A(x)\} = \nu_A(x)
\]

Hence $\mu_A(0) \geq \mu_A(x)$ and $\nu_A(0) \leq \nu_A(x)$ for all $x \in X$.

**Lemma 3.4.** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of $X$, if $x * y \leq z$, then $\mu_A(x) \geq \min\{1, \mu_A(y), \mu_A(z)\}$ and $\nu_A(x) \leq \max\{\nu_A(y), \nu_A(z)\}$.

**Proof.** Suppose $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of $X$. Let $x, y, z \in X$ such that $x * y \leq z$. Then by the binary relation $\leq$ defined in $X$, we have
\((x \ast y) \ast z = 0\). Thus by definition 2.1 and proposition 2.4 (4), we have 
\[ \mu_{A}(x) = \mu_{A}(0 \ast x) = \mu_{A}((x \ast y) \ast z) \ast x) = \mu_{A}((z \ast y) \ast x) \ast x = \mu_{A}(x \ast (z \ast y)) = \mu_{A}(0 \ast (z \ast y)) = \mu_{A}(z \ast y) \geq \min\{\mu_{A}(z), \mu_{A}(y)\} \]

Hence \(\mu_{A}(x) \geq \min\{\mu_{A}(z), \mu_{A}(y)\} \)

Similarly, \(\nu_{A}(x) \leq \max\{\nu_{A}(z), \nu_{A}(y)\} \)

**Theorem 3.5.** Let \(A = (\mu_{A}, \nu_{A})\) be an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra \(X\) and let \(x \in X\), then \(\mu_{A}(x \ast y) = \mu_{A}(y)\) and \(\nu_{A}(x \ast y) = \nu_{A}(y)\) for each \(y \in X\) if and only if \(\mu_{A}(x) = \mu_{A}(0)\) and \(\nu_{A}(x) = \nu_{A}(0)\), where 0 is a constant in \(X\).

**Proof.** Suppose \(\mu_{A}(x \ast y) = \mu_{A}(y)\) and \(\nu_{A}(x \ast y) = \nu_{A}(y)\) for each \(y \in X\). Then we need to show that \(\mu_{A}(x) = \mu_{A}(0)\) and \(\nu_{A}(x) = \nu_{A}(0)\), where 0 is a constant in \(X\). By lemma 3.3, \(\mu_{A}(0) \geq \mu_{A}(x)\) and \(\nu_{A}(0) \leq \nu_{A}(x)\) for each \(x \in X\). By proposition 2.4 (2) \((x \ast 0) \ast 0 = x\). Then \(\mu_{A}(x) = \mu_{A}((x \ast 0) \ast 0) \geq \min\{\mu_{A}(x \ast 0), \mu_{A}(0)\} = \mu_{A}(0)\).

Also, \(\nu_{A}(x) = \nu_{A}((x \ast 0) \ast 0) \leq \max\{\nu_{A}(x \ast 0), \nu_{A}(0)\} = \nu_{A}(0)\).

Hence \(\mu_{A}(x) \geq \mu_{A}(0)\) and \(\nu_{A}(x) \leq \nu_{A}(0)\).

Therefore \(\mu_{A}(x) = \mu_{A}(0)\) and \(\nu_{A}(x) = \nu_{A}(0)\)

Conversely, Suppose \(\mu_{A}(x) = \mu_{A}(0)\) and \(\nu_{A}(x) = \nu_{A}(0)\). Then we need to prove that \(\mu_{A}(x \ast y) = \mu_{A}(y)\) and \(\nu_{A}(x \ast y) = \nu_{A}(y)\), for each \(y \in X\).

By lemma 3.3 \(\mu_{A}(x) \geq \mu_{A}(y)\) and \(\nu_{A}(x) \leq \nu_{A}(y)\) for each \(y \in X\). Since \(A\) is an intuitionistic fuzzy PMS-subalgebra of \(X\), Then \(\mu_{A}(x \ast y) \geq \min\{\mu_{A}(x), \mu_{A}(y)\}\) = \(\mu_{A}(y)\) and \(\nu_{A}(x \ast y) \leq \max\{\nu_{A}(x), \nu_{A}(y)\}\) = \(\nu_{A}(y)\). Thus \(\mu_{A}(x \ast y) \geq \mu_{A}(y)\) and \(\nu_{A}(x \ast y) \leq \nu_{A}(y)\) for each \(y \in X\).

But, using Proposition 2.4 (2) and 2.4 (5) it follows that
\[ \mu_{A}(y) = \mu_{A}((y \ast x) \ast x) \geq \min\{\mu_{A}(y \ast x), \mu_{A}(x)\} \]
\[ = \min\{\mu_{A}((x \ast y) \ast 0), \mu_{A}(x)\} \]
\[ \geq \min\{\min\{\mu_{A}(x \ast y), \mu_{A}(0)\}, \mu_{A}(x)\} \]
\[ = \min\{\mu_{A}(x \ast y), \mu_{A}(x)\} = \mu_{A}(x \ast y) \]

and
\[ \nu_{A}(y) = \nu_{A}((y \ast x) \ast x) \leq \max\{\nu_{A}(y \ast x), \nu_{A}(x)\} \]
\[ = \max\{\nu_{A}((x \ast y) \ast 0), \nu_{A}(x)\} \]
\[ \leq \max\{\max\{\nu_{A}(x \ast y), \nu_{A}(0)\}, \nu_{A}(x)\} \]
\[ = \max\{\nu_{A}(x \ast y), \nu_{A}(x)\} = \nu_{A}(x \ast y) \]

Hence \(\mu_{A}(x \ast y) = \mu_{A}(y)\) and \(\nu_{A}(x \ast y) = \nu_{A}(y)\) for each \(y \in X\). \qed

**Theorem 3.6.** Let \(A = (\mu_{A}, \nu_{A})\) be an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra \(X\). If \(\mu_{A}(x \ast y) = \mu_{A}(0)\) and \(\nu_{A}(x \ast y) = \nu_{A}(0)\) for all \(x, y \in X\), then \(\mu_{A}(x) = \mu_{A}(y)\) and \(\nu_{A}(x) = \nu_{A}(y)\)
Proof. Let \( x, y \in X \) such that \( \mu_A(x * y) = \mu_A(0) \) and \( \nu_A(x * y) = \nu_A(0) \).

Claim \( \mu_A(x) = \mu_A(y) \) and \( \nu_A(x) = \nu_A(y) \)

Now, \( \mu_A(x) = \mu_A((y * y) * x) \)
\[
= \mu_A((x * y) * y) \\
\geq \min\{\mu_A(x * y), \mu_A(y)\} \\
= \min\{\mu_A(0), \mu_A(y)\} = \mu_A(y)
\]

Conversely, \( \mu_A(y) = \mu_A((x * x) * y) \)
\[
= \mu_A((y * x) * x) \\
\geq \min\{\mu_A(y * x), \mu_A(y)\} \\
= \min\{\mu_A((x * y) * 0), \mu_A(y)\} \\
\geq \min\{\min\{\mu_A(x * y), \mu_A(0)\}, \mu_A(x)\} \\
= \min\{\mu_A(0), \mu_A(x)\} = \mu_A(x)
\]

Thus \( \mu_A(x) = \mu_A(y) \)

By similar argument we have \( \nu_A(x) = \nu_A(y) \)

Theorem 3.7. The intersection of any two intuitionistic fuzzy PMS-subalgebras of \( X \) is also an intuitionistic fuzzy PMS-subalgebra of \( X \).

Proof. Let \( A = (\mu_A, \nu_A) \) and \( B = (\mu_B, \nu_B) \) be any two intuitionistic fuzzy PMS-subalgebras of a PMS-algebra \( X \).

Claim: \( A \cap B \) is an intuitionistic fuzzy PMS-subalgebra of \( X \). Then for \( x, y \in X \), we have
\[
\mu_{A \cap B}(x * y) = \min\{\mu_A(x * y), \mu_B(x * y)\} \\
\geq \min\{\min\{\mu_A(x), \mu_B(x)\}, \min\{\mu_A(y), \mu_B(y)\}\} \\
= \min\{\min\{\mu_A(x), \mu_B(x)\}, \min\{\mu_A(y), \mu_B(y)\}\} \\
= \min\{\mu_{A \cap B}(x), \mu_{A \cap B}(y)\}
\]

and
\[
\nu_{A \cap B}(x * y) = \max\{\nu_A(x * y), \nu_B(x * y)\} \\
\leq \max\{\max\{\nu_A(x), \nu_B(x)\}, \max\{\nu_A(y), \nu_B(y)\}\} \\
= \max\{\max\{\nu_A(x), \nu_B(x)\}, \max\{\nu_A(y), \nu_B(y)\}\} \\
= \max\{\nu_{A \cap B}(x), \nu_{A \cap B}(y)\}
\]

Hence \( A \cap B \) is an intuitionistic fuzzy PMS-subalgebra of \( X \)

The above theorem proves that the intersection of any two intuitionistic fuzzy PMS-subalgebras of \( X \) is again an intuitionistic fuzzy subalgebra of \( X \). It can also be generalized to any family of intuitionistic fuzzy PMS-subalgebra of \( X \) as follows:

Corollary 3.8. If \( \{A_i : i \in I\} \) be a family of intuitionistic fuzzy PMS-subalgebra of \( X \), then \( \bigcap_{i \in I} A_i \) is also an intuitionistic fuzzy PMS-subalgebra of \( X \), where \( \bigcap_{i \in I} \mu_{A_i}(x) = \inf_{i \in I} \mu_{A_i}(x) \) and \( \bigcap_{i \in I} \nu_{A_i}(x) = \sup_{i \in I} \nu_{A_i}(x) \)

Remark 3.9. The union of any two intuitionistic fuzzy PMS-subalgebras of a PMS-algebra \( X \) is not necessarily an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra \( X \).
EXAMPLE 3.10. Let $X = \{0, 1, 2, 3\}$ be a set with the table as in example 3.2 and $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy set in $X$ as defined in example 3.2. Let $B = (\mu_B, \nu_B)$ be an intuitionistic fuzzy set in $X$ defined by

$$
\mu_B(x) = \begin{cases} 
1 & \text{if } x = 0 \\
0.6 & \text{if } x = 1, 3 \\
0 & \text{if } x = 2 
\end{cases}
$$

and

$$
\nu_B(x) = \begin{cases} 
0.2 & \text{if } x = 1, 3 \\
1 & \text{if } x = 2 
\end{cases}
$$

Now, \(\mu_{A \cup B}(1*0) = \max\{\mu_A(1), \mu_B(0)\} = \max\{0.5, 0\} = 0.5\) (i)

\(\min\{\mu_{A \cup B}(0), \mu_{A \cup B}(0)\} = \min\{\max\{\mu_A(0), \mu_B(1)\}, \max\{\mu_A(0), \mu_B(0)\}\}

= \min\{\max\{0.5, 0.6\}, \max\{1, 1\}\}

= \min\{0.6, 1\} = 0.6\) (ii)

and

\(\nu_{A \cup B}(1*0) = \nu_{A \cup B}(2) = \min\{\nu_A(2), \nu_B(2)\} = \min\{0.4, 1\} = 0.4\) (iii)

\(\max\{\nu_{A \cup B}(1), \nu_{A \cup B}(0)\} = \max\{\min\{\nu_A(1), \nu_B(1)\}, \min\{\nu_A(0), \nu_B(0)\}\}

= \max\{\min\{0.4, 0.2\}, \min\{0, 0\}\}

= \max\{0.2, 0\} = 0.2\) (iv)

From (i) and (ii) we see that \(\mu_{A \cup B}(1*0) = 0.5 < 0.6 = \min\{\mu_{A \cup B}(1), \mu_{A \cup B}(0)\}\)

and from (iii) and (iv) we see that \(\nu_{A \cup B}(1*0) = 0.4 > 0.2 = \max\{\nu_{A \cup B}(1), \nu_{A \cup B}(0)\}\)

which is a contradiction. This shows that the union of any two intuitionistic fuzzy PMS-subalgebras of a PMS-algebra $X$ may not be an intuitionistic fuzzy PMS-subalgebra.

**Lemma 3.11.** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in $X$. Then the following statements hold for any $x, y \in X$.

1. $1 - \max\{\mu_A(x), \mu_A(y)\} = \min\{1 - \mu_A(x), 1 - \mu_A(y)\}$
2. $1 - \min\{\mu_A(x), \mu_A(y)\} = \max\{1 - \mu_A(x), 1 - \mu_A(y)\}$
3. $1 - \max\{\nu_A(x), \nu_A(y)\} = \min\{1 - \nu_A(x), 1 - \nu_A(y)\}$
4. $1 - \min\{\nu_A(x), \nu_A(y)\} = \max\{1 - \nu_A(x), 1 - \nu_A(y)\}$

Now, we can prove the next two theorems using the above Lemma.

**Theorem 3.12.** An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of a PMS-algebra $X$ is an intuitionistic fuzzy PMS-subalgebra of $X$ if and only if the fuzzy subsets $\mu_A$ and $\nu_A$ are fuzzy subalgebras of $X$.

**Proof.** Suppose $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $X$. Clearly, $\mu_A$ is a fuzzy PMS-subalgebra of $X$ and $\nu_A$ is an intuitionistic fuzzy PMS-subalgebra of $X$. Now, for all $x, y \in X$,

$$
\nu_A(x \ast y) = 1 - \nu_A(x \ast y) \geq 1 - \max\{\nu_A(x), \nu_A(y)\}

= \min\{1 - \nu_A(x), 1 - \nu_A(y)\} \quad \text{(By Lemma 3.11(3))}

= \min\{\nu_A(x), \nu_A(y)\}

$$

Therefore $\nu_A$ is a fuzzy PMS-subalgebra of $X$.

Conversely, Suppose $\mu_A$ and $\nu_A$ are fuzzy PMS-subalgebras of $X$. So, we need to show that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $X$. Since $\mu_A$ and $\nu_A$ are fuzzy PMS-subalgebras of $X$, we have that $\mu_A(x \ast y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x \ast y) \geq \min\{\nu_A(x), \nu_A(y)\}$, for all $x, y \in X$. Now it suffices to show that
\( \nu_A(x \ast y) \leq \max\{\nu_A(x), \nu_A(y)\} \) for all \( x, y \in X \).
\[
1 - \nu_A(x \ast y) = \nu_A(x \ast y) \geq \min\{\nu_A(x), \nu_A(y)\} \\
= \min\{1 - \nu_A(x), 1 - \nu_A(y)\} \\
= 1 - \max\{\nu_A(x), \nu_A(y)\} \quad \text{(By Lemma 3.11(3))}
\]
\[\Rightarrow \nu_A(x \ast y) \leq \max\{\nu_A(x), \nu_A(y)\}, \text{ for all } x, y \in X.\]

Hence \( A = (\mu_A, \nu_A) \) is an intuitionistic fuzzy PMS-subalgebra of \( X \).

**Corollary 3.13.** If \( \mu_A \) is a fuzzy PMS-subalgebra of \( X \), then \( A = (\mu_A, \bar{\mu}_A) \) is an intuitionistic fuzzy PMS-subalgebra of \( X \).

**Proof.** Suppose \( \mu_A \) is a fuzzy PMS-subalgebra of \( X \). Then we want to show that \( A = (\mu_A, \bar{\mu}_A) \) is an intuitionistic fuzzy PMS-subalgebra of \( X \). Since \( \mu_A \) is a fuzzy PMS-subalgebra of \( X \), it follows that \( \mu_A(x \ast y) \geq \min\{\mu_A(x), \mu_A(y)\} \). Then it suffices to show that \( \bar{\mu}_A(x \ast y) \leq \max\{\bar{\mu}_A(x), \bar{\mu}_A(y)\} \).
\[
\bar{\mu}_A(x \ast y) = 1 - \mu_A(x \ast y) \leq 1 - \min\{\mu_A(x), \mu_A(x)\} \\
= \max\{1 - \mu_A(x), 1 - \mu_A(x)\} \\
= \max\{\mu_A(x), \mu_A(x)\}
\]
\[\Rightarrow \bar{\mu}_A(x \ast y) \leq \max\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}
\]
Hence \( A = (\mu_A, \bar{\mu}_A) \) is an intuitionistic fuzzy PMS-subalgebra of \( X \).

**Corollary 3.14.** If \( \bar{\nu}_A \) is a fuzzy PMS-subalgebra of \( X \), then \( A = (\bar{\nu}_A, \nu_A) \) is an intuitionistic fuzzy PMS-subalgebra of \( X \).

**Proof.** Similar to corollary 3.13

**Theorem 3.15.** An intuitionistic fuzzy subset \( A = (\mu_A, \nu_A) \) of \( X \) is an intuitionistic fuzzy PMS-subalgebra of \( X \) if and only if \( \Box A = (\mu_A, \bar{\mu}_A) \) and \( \Diamond A = (\bar{\nu}_A, \nu_A) \) are intuitionistic fuzzy PMS-subalgebras of \( X \).

**Proof.** Assume that an intuitionistic fuzzy subset \( A = (\mu_A, \nu_A) \) of \( X \) is an intuitionistic fuzzy PMS-subalgebra of \( X \), then \( \mu_A(x \ast y) \geq \min\{\mu_A(x), \mu_A(y)\} \) and \( \nu_A(x \ast y) \leq \max\{\nu_A(x), \nu_A(y)\} \).

Claim: \( \Box A = (\mu_A, \bar{\mu}_A) \) and \( \Diamond A = (\bar{\nu}_A, \nu_A) \) are intuitionistic fuzzy PMS-subalgebras of \( X \).

(i) To show that \( \Box A \) is an intuitionistic fuzzy PMS-subalgebra of \( X \), it suffices to show that \( \bar{\mu}_A(x \ast y) \leq \max\{\bar{\mu}_A(x), \bar{\mu}_A(y)\} \), for all \( x, y \in X \). Let \( x, y \in X \), then
\[
\bar{\mu}_A(x \ast y) = 1 - \mu_A(x \ast y) \leq 1 - \min\{\mu_A(x), \mu_A(y)\} \\
= \max\{1 - \mu_A(x), 1 - \mu_A(y)\} \\
= \max\{\mu_A(x), \mu_A(x)\}
\]
\[\Rightarrow \bar{\mu}_A(x \ast y) \leq \max\{\bar{\mu}_A(x), \bar{\mu}_A(y)\}, \forall x, y \in X.
\]
Hence \( \Box A \) is an intuitionistic fuzzy PMS-subalgebra of \( X \).

(ii) To show that \( \Diamond A \) is an intuitionistic fuzzy PMS-subalgebra of \( X \), it suffices to show that \( \bar{\nu}_A(x \ast y) \geq \min\{\bar{\nu}_A(x), \bar{\nu}_A(y)\} \), for all \( x, y \in X \). Let \( x, y \in X \), then
\[
\bar{\nu}_A(x \ast y) = 1 - \nu_A(x \ast y) \geq 1 - \max\{\bar{\nu}_A(x), \bar{\nu}_A(y)\} \\
= \min\{1 - \nu_A(x), 1 - \nu_A(y)\} \\
= \min\{\nu_A(x), \nu_A(y)\}
\]
\[\Rightarrow \bar{\nu}_A(x \ast y) \geq \min\{\bar{\nu}_A(x), \bar{\nu}_A(y)\}, \forall x, y \in X.
\]
Hence $\diamondsuit A$ is an intuitionistic fuzzy PMS-subalgebra of $X$.

The proof of the converse of this theorem is trivial. \hfill $\Box$

4. Level Subsets of Intuitionistic Fuzzy PMS-subalgebras

In this section, the idea of level subsets of an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra is introduced. Characterizations of level subsets of a fuzzy PMS-subalgebra of a PMS-algebra are given.

**Theorem 4.1.** If $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of $X$, then the sets $X_{\mu_A} = \{x \in X | \mu_A(x) = \mu_A(0)\}$ and $X_{\nu_A} = \{x \in X | \nu_A(x) = \nu_A(0)\}$ are PMS-subalgebra of $X$.

**Proof.** Suppose $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $X$ and let $x, y \in X_{\mu_A}$. Then $\mu_A(x) = \mu_A(0) = \mu_A(y)$. So, $\mu_A(x * y) \geq \min \{\mu_A(x), \mu_A(y)\}$ implies $\mu_A(0) = \mu_A(0)$. Hence, $\nu_A(x * y) = \nu_A(0)$ which imply that $x * y \in X_{\mu_A}$. Also, let $x, y \in X_{\nu_A}$. Then $\nu_A(x) = \nu_A(0) = \nu_A(y)$ and so $\nu_A(x * y) \leq \max \{\nu_A(x), \nu_A(y)\} = \max \{\nu_A(0), \nu_A(0)\} = \nu_A(0)$. Hence, $\nu_A(x * y) = \nu_A(0)$ which imply that $x * y \in X_{\nu_A}$.

Hence, the sets $X_{\mu_A}$ and $X_{\nu_A}$ are PMS-subalgebras of $X$. \hfill $\Box$

**Theorem 4.2.** Let $S$ be a nonempty subset of a PMS-algebra $X$ and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in $X$ defined by

$$
\mu_A(x) = \begin{cases} p & \text{if } x \in S \\ q & \text{if } x \notin S \end{cases} \quad \text{and} \quad \nu_A(x) = \begin{cases} r & \text{if } x \in S \\ s & \text{if } x \notin S \end{cases}
$$

for all $p, q, r, s \in [0, 1]$ with $p \geq q, r \leq s$ and $0 \leq p + r \leq 1, 0 \leq q + s \leq 1$. Then $A$ is an intuitionistic fuzzy PMS-subalgebra of $X$ if and only if $S$ is a PMS-subalgebra of $X$. Furthermore, in this situation, $X_{\mu_A} = S = X_{\nu_A}$.

**Proof.** Let $A$ be an intuitionistic fuzzy PMS-subalgebra of $X$. Then we want to show that $S$ is a PMS-subalgebra of $X$. Let $x, y \in X$ such that $x, y \in S$.

Since $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $X$, we have

$$
\mu_A(x * y) \geq \min \{\mu_A(x), \mu_A(y)\} = \min \{p, p\} = p \quad \text{and} \quad \nu_A(x * y) \leq \max \{\nu_A(x), \nu_A(y)\} = \max \{r, r\} = r.
$$

Hence $x * y \in S$. So, $S$ is a PMS-subalgebra of $X$.

Conversely, suppose that $S$ is a PMS-subalgebra of $X$. We claim to show that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $X$.

Let $x, y \in X$. Now consider the following cases:

- **Case (i).** If $x, y \in S$, then $x * y \in S$, since $S$ is a PMS-subalgebra of $X$. Thus, $\mu_A(x * y) = p = \min \{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) = r = \max \{\nu_A(x), \nu_A(y)\}$.

- **Case (ii).** If $x \in S, y \notin S$, then $\mu_A(x) = p, \mu_A(y) = q$ and $\nu_A(x) = r, \nu_A(y) = s$.

Thus, $\mu_A(x * y) \geq \min \{p, q\} = \min \{\mu_A(x), \mu_A(y)\}$ implies $\mu_A(x * y) \geq \min \{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) \leq s = \max \{r, s\} = \max \{\nu_A(x), \nu_A(y)\}$.

- **Case (iii).** If $x \notin S, y \in S$, then interchanging the roles of $x$ and $y$ in Case (ii), yields similar results $\mu_A(x * y) \geq \min \{\mu_A(x), \mu(y)\}$ and $\nu_A(x * y) \leq \max \{\nu_A(x), \nu_A(y)\}$.
case (iv). If \( x, y \notin S \), then \( \mu_A(x) = q = \mu_A(y) \) and \( \nu_A(x) = s = \nu_A \), this implies that 
\[ \mu_A(x * y) \geq q = \min \{ \mu_A(x), \mu_A(y) \} \] 
and 
\[ \nu_A(x * y) \leq s = \max \{ \nu_A(x), \nu_A(y) \} \]
Hence \( A = (\mu_A, \nu_A) \) is an intuitionistic fuzzy PMS-subalgebra of \( X \).

Furthermore, we have 
\[ X_{\mu_A} = \{ x \in X | \mu_A(x) = \mu_A(0) \} = \{ x \in X | \mu_A(x) = p \} = S \] 
and 
\[ X_{\nu_A} = \{ x \in X | \nu_A(x) = \nu_A(0) \} = \{ x \in X | \nu_A(x) = r \} = S. \]
Hence \( X_{\mu_A} = S = X_{\nu_A} \).

Definition 4.3. Let \( A = (\mu_A, \nu_A) \) be any intuitionistic fuzzy subset of a PMS-algebra \( X \) such that \( t, s \in [0, 1] \), then the set \( U(\mu_A, t) = \{ x \in X : \mu_A(x) \geq t \} \) is called an upper \( t \)-level set of an intuitionistic fuzzy subset \( A \) of \( X \) and the set 
\[ L(\mu_A, s) = \{ x \in X : \nu_A(x) \leq s \} \] 
is called a lower \( s \)-level set of an intuitionistic fuzzy subset \( A \) of \( X \).

Theorem 4.4. An intuitionistic fuzzy subset \( A = (\mu_A, \nu_A) \) of a PMS-algebra \( X \) is an intuitionistic fuzzy PMS-subalgebra of \( X \) if and only if the nonempty level subsets \( U(\mu_A, t) \) and \( L(\nu_A, s) \) of \( A \) are PMS-subalgebras of \( X \) for all \( t, s \in [0, 1] \) with \( 0 \leq t + s \leq 1 \).

Proof. Assume that \( A = (\mu_A, \nu_A) \) is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra \( X \) such that \( U(\mu_A, t) \neq \emptyset \) and \( L(\nu_A, s) \neq \emptyset \). Now we claim that \( U(\mu_A, t) \) and \( L(\nu_A, t) \) are PMS-subalgebras of \( X \) for all \( t, s \in [0, 1] \) with \( 0 \leq t + s \leq 1 \). Let \( x, y \in U(\mu_A, t) \), then we have \( \mu_A(x) \geq t \) and \( \mu_A(y) \geq t \). Thus 
\[ \mu_A(x * y) \geq \min \{ \mu_A(x), \mu_A(y) \} \geq \min \{ t, t \} = t \]
\[ \Rightarrow x * y \in U(\mu_A, t) \]
Hence \( U(\mu_A, t) \) is a PMS-subalgebra of \( X \).

Also, let \( x, y \in L(\nu_A, s) \), then \( \nu_A(x) \leq s \) and \( \nu_A(y) \leq s \)
So, 
\[ \nu_A(x * y) \leq \max \{ \nu_A(x), \nu_A(y) \} \leq \max \{ t, t \} = t \Rightarrow x * y \in L(\nu_A, s) \]
Hence \( L(\nu_A, s) \) is a PMS-subalgebra of \( X \).

Conversely, Suppose that \( U(\mu_A, t) \) and \( L(\nu_A, s) \) are PMS-subalgebras of \( X \) for all \( t, s \in [0, 1] \) with \( 0 \leq t + s \leq 1 \).

Claim: \( A \) is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra \( X \).

Let \( x, y \in X \) such that \( \mu_A(x) = t_1 \) and \( \mu_A(y) = t_2 \) for \( t_1, t_2 \in [0, 1] \). Then \( x \in U(\mu_A, t_1) \) and \( y \in U(\mu_A, t_2) \).
Choose \( t = \min \{ t_1, t_2 \} \), then \( t \leq t_1 \) and \( t \leq t_2 \)
\[ \Rightarrow U(\mu_A, t_1) \subseteq U(\mu_A, t) \] 
and 
\[ U(\mu_A, t_2) \subseteq U(\mu_A, t). \]
Since \( U(\mu_A, t) \) is a PMS-Subalgebra of \( X \), it follows that \( x * y \in U(\mu_A, t) \).
Thus \( \mu_A(x * y) \geq t = \min \{ t_1, t_2 \} = t = \min \{ \mu_A(x), \mu_A(y) \} \).
Hence \( \mu_A(x * y) \geq \min \{ \mu_A(x), \mu_A(y) \} \) for all \( x, y \in X \).
And also, let \( x, y \in X \) such that \( \nu_A(x) = s_1 \) and \( \nu_A(y) = s_2 \) for \( s_1, s_2 \in [0, 1] \).
Then \( x \in L(\nu_A, s_1) \) and \( y \in L(\nu_A, s_2) \).
Choose \( s = \max \{ s_1, s_2 \} \), then \( s_1 \leq s \) and \( s_2 \leq s \)
\[ \Rightarrow L(\nu_A, s_1) \subseteq L(\nu_A, s) \] 
and 
\[ L(\nu_A, s_2) \subseteq L(\nu_A, s). \]
Since \( L(\nu_A, s) \) is a PMS-subalgebra of \( X \), it follows that \( x * y \in L(\nu_A, s) \).
Thus \( \nu_A(x * y) \leq s = \max \{ s_1, s_2 \} = \max \{ \nu_A(x), \nu_A(y) \} \).
Hence \( \nu_A(x * y) \leq \max \{ \nu_A(x), \nu_A(y) \} \) for all \( x, y \in X \).
Hence \( A \) is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra \( X \).
Remark 4.5. The PMS-subalgebras $U(\mu_A, t)$ and $L(\nu_A, s)$ of $X$ for all $t, s \in [0, 1]$ obtained in the above theorem are called level PMS-subalgebras of $X$.

Corollary 4.6. An intuitionistic fuzzy subset $A = (\mu_A, \nu_A)$ of a PMS-algebra $X$ is an intuitionistic fuzzy PMS-subalgebra of $X$ if and only if the level subsets $U(\mu_A, t)$ and $L(\nu_A, s)$ of $A$ are PMS-subalgebras of $X$ for all $t \in \text{Im}(\mu_A)$ and $s \in \text{Im}(\nu_A)$ with $0 \leq t + s \leq 1$.

Theorem 4.7. Let $S$ be a subset of $X$ and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in $X$ defined by

$$
\mu_A(x) = \begin{cases} 
t & \text{if } x \in S \\
0 & \text{if } x \notin S
\end{cases} \quad \text{and} \quad 
\nu_A(x) = \begin{cases} 
s & \text{if } x \in S \\
1 & \text{if } x \notin S
\end{cases}
$$

for all $t, s \in [0, 1]$ such that $0 \leq t + s \leq 1$. If $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $X$, then $S$ is a level PMS-subalgebra of $X$.

Proof. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of $X$. Then we need to show that $S$ is a level PMS-subalgebra of $X$. Let $x, y \in S$, then $\mu_A(x) = t = \mu_A(y)$ and $\nu_A(x) = s = \nu_A(y)$. So, $\mu_A(x * y) \geq \min \{\mu_A(x), \mu_A(y)\} = \min \{t, t\} = t$ and $\nu_A(x * y) \leq \max \{\mu_A(x), \nu_A(y)\} = \max \{s, s\}$ which implies that $x * y \in S$. Hence $S$ is a PMS-subalgebra of $X$. Also, by Theorem 4.4, $U(\mu_A, t)$ is a level subalgebra of $X$, and $U(\mu_A, t) = \{x \in X : \mu_A(x) \geq t\} = S = \{x \in X : \nu_A(x) \leq s\}$.

Thus, $S$ is a level PMS-Subalgebra of $X$ corresponding to the intuitionistic fuzzy PMS-subalgebra $A = (\mu_A, \nu_A)$ of $X$.

Theorem 4.8. If $S$ is any PMS-subalgebra of $X$, then there exists an intuitionistic fuzzy PMS-subalgebra $A$ of $X$, in which $S$ satisfies both the upper level and lower level PMS-subalgebra of $A$ in $X$.

Proof. Let $S$ be a PMS-subalgebra of a PMS-algebra $X$ and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set in $X$ defined by

$$
\mu_A(x) = \begin{cases} 
t & \text{if } x \in S \\
0 & \text{if } x \notin S
\end{cases} \quad \text{and} \quad 
\nu_A(x) = \begin{cases} 
s & \text{if } x \in S \\
1 & \text{if } x \notin S
\end{cases}
$$

for all $t, s \in [0, 1]$ such that $0 \leq t + s \leq 1$.

Clearly, $U(\mu_A, t) = \{x \in X : \mu_A(x) \geq t\} = S$. Let $x, y \in X$. To prove that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra $X$, we consider the following cases:

case(i). If $x, y \in S$, then $x * y \in S$. Since $S$ is a PMS-subalgebra of a PMS-algebra $X$.

$$
\mu_A(x) = \mu_A(y) = \mu_A(x * y) = t \quad \text{and} \quad \nu_A(x) = \nu_A(y) = \nu_A(x * y) = s.
$$

Therefore $\mu_A(x * y) = \min \{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) = \max \{\nu_A(x), \nu_A(y)\}$

case(ii). If $x \in S, y \notin S$, then we have $\mu_A(x) = t, \mu_A(y) = 0$ and $\nu_A(x) = s, \nu_A(y) = 1$. Thus, $\mu_A(x * y) \geq 0 = \min \{t, 0\} = \min \{\mu_A(x), \mu_A(y)\}$ which implies that $\mu_A(x * y) \geq \min \{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) \leq 1 = \max \{s, 1\} = \max \{\nu_A(x), \nu_A(y)\}$ implies $\nu_A(x * y) \leq \max \{\nu_A(x), \nu_A(y)\}$

case(iii). If $x \notin S, y \in S$, then interchanging the roles of $x$ and $y$ in Case (ii), yields similar results $\mu_A(x * y) \geq \min \{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) \leq \max \{\nu_A(x), \nu_A(y)\}$

case(iv). If $x, y \notin S$ then $\mu_A(x) = 0 = \mu_A(y)$ and $\nu_A(x) = 1 = \nu_A(y)$. Then $\mu_A(x * y) \geq 0 = \min \{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x * y) \leq 1 = \max \{\nu_A(x), \nu_A(y)\}$.
So, in all cases we get $\mu_A(x \ast y) \geq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x \ast y) \leq \max\{\nu_A(x), \nu_A(y)\}$, for all $x, y \in X$.

Thus, $A$ is an intuitionistic fuzzy PMS-subalgebra of $X$. \hfill \Box

We can also prove the following theorem as a generalization of theorem 4.8.

**Theorem 4.9.** Let $\{S_i\}$ be any family of a PMS-subalgebra of a PMS-algebra $X$ such that $S_0 \subset S_1 \subset S_2 \subset \ldots \subset S_n = X$, then there exists an intuitionistic fuzzy PMS-subalgebra $A = (\mu_A, \nu_A)$ of $X$ whose level PMS-subalgebras are exactly the PMS-subalgebras $\{S_i\}$.

**Proof.** Suppose $t_0 > t_1 > t_2 > \ldots > t_n$ and $s_0 < s_1 < s_2 \ldots < s_n$ where each $t_i, s_i \in [0,1]$ with $0 \leq t_i + s_i \leq 1$. Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy set defined by

$$
\mu_A(x) = \begin{cases} 
    t_0 & \text{if } x \in S_0 \\
    t_i & \text{if } x \in S_i - S_{i-1}, 0 < i \leq n
\end{cases}
$$

and

$$
\nu_A(x) = \begin{cases} 
    s_0 & \text{if } x \in S_0 \\
    s_i & \text{if } x \in S_i - S_{i-1}, 0 < i \leq n
\end{cases}
$$

Now, we claim that $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $X$ and $U(\mu_A, t_i) = S_i = L(\nu_A, s_i)$ for $0 \leq i \leq n$.

Let $x, y \in X$. Then, we consider the following two cases.

Case (i): Let $x, y \in S_i - S_{i-1}$. Therefore by the definition of $A = (\mu_A, \nu_A)$, we have

$$
\mu_A(x) = t_i = \mu_A(y) \text{ and } \nu_A(x) = s_i = \nu_A(y).
$$

Since $S_i$ is a PMS-subalgebra of $X$, it follows that $x \ast y \in S_i$, and so either $x \ast y \in S_i - S_{i-1}$ or $x \ast y \in S_i$ or $x \ast y \in S_i - S_{i-2}$.

Then, $\mu_A(x) = t_i$ or $\mu_A(x) = t_{i-1} > t_i$ and $\nu_A(x) = s_i$ or $\nu_A(x) = s_{i-1} > s_i$.

In any case we conclude that

$$
\mu_A(x \ast y) \geq t_i = \min\{\mu_A(x), \mu_A(y)\} \text{ and } \nu_A(x \ast y) \leq s_i = \max\{\nu_A(x), \nu_A(y)\}.
$$

Case (ii): For $i > j$, $t_j > t_i, s_j < s_i$ and $S_j \subset S_i$. Let $x \in S_i - S_{i-1}$ and $y \in S_j - S_{j-1}$. Then, $\mu_A(x) = t_i, \mu_A(y) = t_j > t_i, \nu_A(x) = s_i$ and $\nu_A(y) = s_j < s_i$. Then $x \ast y \in S_i$ since $S_i$ is a PMS-subalgebra of $X$ and $S_j \subset S_i$.

Hence $\mu_A(x \ast y) \geq t_i = \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_A(x \ast y) \leq s_i = \max\{\nu_A(x), \nu_A(y)\}$ by case (i). Thus $A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy PMS-subalgebra of $X$.

Also, from the definition of $A = (\mu_A, \nu_A)$, it follows that $Im(\mu_A) = \{t_0, t_1, \ldots, t_n\}$ and $Im(\nu_A) = \{s_0, s_1, \ldots, s_n\}$. So, $U(\mu_A, t_i)$ and $L(\nu_A, s_i)$ are the level subalgebras of $A$ for $0 \leq i \leq n$, and form the chains,

$$
U(\mu_A, t_0) \subset \ldots \subset U(\mu_A, t_n) = X \text{ and } L(\nu_A, s_0) \subset \ldots \subset L(\nu_A, s_n) = X.
$$

Now, $U(\mu_A, t_0) = \{x \in X : \mu_A(x) \geq t_0\} = S_0 = \{x \in X : \nu_A(x) \leq s_0\} = L(\nu_A, s_0)$.

Finally, we prove that $U(\mu_A, t_i) = S_i = L(\nu_A, s_i)$ for $0 < i \leq n$.

Note that the number of PMS-subalgebras of a finite PMS-algebra $X$ is finite whereas the number of level PMS-subalgebras of an intuitionistic fuzzy PMS-subalgebra $A$ appears to be infinite. However, every level PMS-subalgebra of $X$ is a PMS-subalgebra.
of $X$, not all of these PMS-subalgebras are unique. The next theorem illustrates this situation.

**Theorem 4.10.** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of $X$, then

(i). The upper level PMS-subalgebras $U(\mu_A, t_1)$ and $U(\mu_A, t_2)$ (with $t_1 < t_2$) of an
intuitionistic fuzzy PMS-subalgebra $A$ are equal if and only if there is no $x \in X$
such that $t_1 \leq \mu_A(x) < t_2$.

(ii). The lower level PMS-subalgebras $L(\nu_A, s_1)$ and $L(\nu_A, s_2)$ (with $s_1 > s_2$) of an
intuitionistic fuzzy PMS-subalgebra $A$ are equal if and only if there is no $x \in X$
such that $s_1 \geq \nu_A(x) > s_2$.

**Proof.** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of $X$. Since the
proofs for both (i) and (ii) are similar, here we prove for only (ii).

Suppose that $L(\nu_A, s_1) = L(\nu_A, s_2)$, for $s_1 > s_2$. Then we claim that there is no $x \in X$
such that $s_1 \geq \nu_A(x) > s_2$.

Assume that there exists $x \in X$ such that $s_1 \geq \mu_A(x) < s_2$.

$$\Rightarrow x \in L(\nu_A, s_1) \text{ but } x \notin L(\nu_A, s_2)$$

$$\Rightarrow L(\mu_A, s_2) \text{ is a proper subset of } L(\nu_A, s_1).$$

This contradicts to the assumption that $L(\nu_A, s_1) = L(\nu_A, s_2)$.

Hence there is no $x \in X$ such that $s_1 \geq \nu_A(x) > s_2$.

Conversely, suppose that there is no $x \in X$ such that $s_1 \geq \nu_A(x) > s_2$. Then we
prove that $L(\nu_A, s_1) = L(\nu_A, s_2)$.

Since $s_1 > s_2$, we get $L(\nu_A, s_2) \subseteq L(\nu_A, s_1)$.

Now, $x \in L(\nu_A, s_1) \Rightarrow \nu_A(x) \leq s_1$. \hspace{1cm} (1)

$$\Rightarrow x \in L(\nu_A, s_2).$$

Hence $L(\nu_A, s_1) \subseteq L(\nu_A, s_2)$ \hspace{1cm} (2)

From (1) and (2) we get $L(\nu_A, s_1) = L(\nu_A, s_2)$. \hfill \Box

**Remark 4.11.** As the consequence of Theorem 4.10, the level subalgebras of an
intuitionistic fuzzy PMS-algebra $A = (\mu_A, \nu_A)$ of a finite PMS-algebra $X$ form a
chain,

$U(\mu_A, t_0) \subset U(\mu_A, t_1) \subset \ldots \subset U(\mu_A, t_n) = X$ and $L(\nu_A, s_0) \subset L(\nu_A, s_1) \subset \ldots \subset L(\nu_A, s_n) = X$, where $t_0 > t_1 > \ldots > t_n$ and $s_0 < s_1 < \ldots < s_n$.

**Corollary 4.12.** Let $X$ be a finite PMS-algebra and $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of $X$.

(i). If $Im(\mu_A) = \{v_1, \ldots, v_n\}$, then the family of PMS-subalgebras $\{U(\mu_A, t_i)| 1 \leq i \leq n\}$, constitutes all the upper level PMS-subalgebras of $A$ in $X$.

(ii). If $Im(\nu_A) = \{s_1, \ldots, s_n\}$, then the family of PMS-subalgebras $\{L(\nu_A, s_i)| 1 \leq i \leq n\}$, constitutes all the lower level PMS-subalgebras of $A$ in $X$.

**Proof.** Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy PMS-subalgebra of $X$ such that
$Im(\mu_A) = \{t_1, t_2, \ldots, t_n\}$ with $t_1 < t_2 < \ldots < t_n$ and $Im(\nu_A) = \{s_1, s_2, \ldots, s_n\}$ with $s_1 > s_2 > \ldots > s_n$.

(i). Let $t \in [0, 1]$ and $t \notin Im(\mu_A)$. Now, we can consider the following cases.

case (1). If $t \leq t_1$, then $U(\mu_A, t_1) = X = U(\mu_A, t)$.

case (2). If $t > t_n$, then $U(\mu_A, t) = \{x \in X| \mu_A(x) \geq t\} = \{x \in X| \mu_A(x) > t_n\} = \emptyset$

case (3). If $t_{i-1} < t < t_i$, then $U(\mu_A, t) = U(\mu_A, t_i)$ by theorem 4.10(i), since
there is no \( x \in X \) such that \( t \leq \mu_A(x) < t_i \). Thus for any \( t \in [0,1] \), the level PMS-subalgebra is one of \( \{ U(\mu_A, t_i) \mid i = 1, \ldots, n \} \).

(ii). proof of (ii) is similar to (i) \( \square \)

**Corollary 4.13.** Let \( A = (\mu_A, \nu_A) \) be an intuitionistic fuzzy PMS-subalgebra of \( X \) with finite images.

(i). If \( U(\mu_A, t_i) = U(\mu_A, t_j) \) for any \( t_i, t_j \in \text{Im}(\mu_A) \), then \( t_i = t_j \). 

(ii). If \( L(\nu_A, s_i) = L(\nu_A, s_j) \) for any \( s_i, s_j \in \text{Im}(\nu_A) \), then \( s_i = s_j \).

**Proof.** Let \( A = (\mu_A, \nu_A) \) be an intuitionistic fuzzy PMS-subalgebra of \( X \) with finite images. Here we only prove (ii). the prove of (i) can be done similarly. Assume \( L(\nu_A, s_i) = L(\nu_A, t_j) \) for \( s_i, s_j \in \text{Im}(\nu_A) \). So to show that \( s_i = s_j \) assume on contrary, that is, \( s_i \neq s_j \). Without loss of generality assume \( s_i > s_j \).

Let \( x \in L(\nu_A, s_j) \), then \( \nu_A(x) \leq s_j < s_i \).
\[
\begin{align*}
\nu_A(x) & \leq s_i \\
x & \in L(\nu_A, s_i)
\end{align*}
\]

Let \( x \in X \) such that \( s_i > \nu_A(x) > s_j \). Then \( x \in L(\nu_A, s_i) \) but \( x \notin L(\nu_A, s_j) \).
\[
\begin{align*}
\nu_A(x) & \geq s_j \\
L(\nu_A, s_j) & \subset L(\nu_A, s_i) \\
L(\nu_A, t_i) & \neq L(\nu_A, t_j)
\end{align*}
\]

which contradics the hypothesis that \( L(\nu_A, s_i) = L(\nu_A, s_j) \). Therefore, \( s_i = s_j \). \( \square \)

5. Conclusion

In this paper, we introduced the notion of intuitionistic fuzzy PMS-subalgebras of PMS-algebras and some results are obtained. The idea of level subsets of an intuitionistic fuzzy PMS-subalgebra of a PMS-algebra is introduced. The relation between an intuitionistic fuzzy sets in a PMS-algebra and their level sets is discussed and some interesting results are obtained. The concepts can further be extended to intuitionistic fuzzy ideals of a PMS-algebra for new results in our future work.

**References**

Beza Lamesgin Derseh, Berhanu Assaye Alaba, and Yohannes Gedamu Wondifraw


Beza Lamesgin Derseh
Department of Mathematics, Debre Markos University, Debre Markos, Ethiopia
E-mail: dbezalem@gmail.com

Berhanu Assaye Alaba
Department of Mathematics, Bahir Dar University, Bahir Dar, Ethiopia
E-mail: birhanu.assaye290113@gmail.com

Yohannes Gedamu Wondifraw
Department of Mathematics, Bahir Dar University, Bahir Dar, Ethiopia
E-mail: yohannesg27@gmail.com