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RELATIVE LOGARITHMIC ORDER OF AN ENTIRE FUNCTION

Chinmay Ghosh*, Anirban Bandyopadhyay, and Soumen Mondal

Abstract. In this paper, we extend some results related to the growth rates of entire functions by introducing the relative logarithmic order $\rho_g^l(f)$ of a nonconstant entire function f with respect to another nonconstant entire function g. Next we investigate some theorems related the behavior of $\rho_g^l(f)$. We also define the relative logarithmic proximate order of f with respect to g and give some theorems on it.

1. Introduction

Let f be a nonconstant entire function. Then the maximum modulus function $M_f(r)$ of f, defined by $M_f(r) = \max_{|z|=r} |f(z)|$ is continuous and strictly increasing function of r. In such case the inverse function $M_f^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists and is also continuous, strictly increasing and $\lim_{s\to\infty} M_f^{-1}(s) = \infty$. The growth of an entire function f is generally measured by its order and type.

In 1988, Luis Bernal [1] introduced the order of growth of a nonconstant entire function f relative to another entire function g, which is defined by

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu), \text{ for all } r > r_0\} \\ &= \limsup_{r\to\infty} \frac{\log M_g^{-1}(M_f(r))}{\log r}.\end{aligned}$$

In general, techniques that work well for functions of finite positive order often do not work for functions of order zero. In order to make some progress with functions of order zero, in 2005, P. T. Y. Chern [2] defined the logarithmic order of an entire function f, given by

$$\rho^l = \limsup_{r\to\infty} \frac{\log^+ \log^+ M_f(r)}{\log\log r},$$

where $\log^+ x = \max\{\log x, 0\}$, for all $x \geq 0$.

In this paper we want to sort out the same type of limitations, occurring for the functions of relative order zero, by introducing the relative logarithmic order of f with respect to g, $\rho_g^l(f)$, for two nonconstant entire functions f and g. And then we investigate some theorems related the behavior of $\rho_g^l(f)$.

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* Corresponding author.

Moreover, In 1923, Valiron [7] initiated the terminology and generalized the concept of proximate order and in 1946, S. M. Shah [6] defined it in more justified form and gave a simple proof of its existence. In this paper, we also define the relative logarithmic proximate order of f with respect to g.

2. Basic definitions and preliminary lemmas

In this section we state some definitions and lemmas which will be used to prove our main results.

DEFINITION 2.1. A nonconstant entire function f is said to be satisfy the property (A) if and only if for each $\sigma > 1$,

$$M_f(r)^2 \leq M_f(r^\sigma),$$

exists.

For example $\exp z, \cos z$ etc satisfy the property (A). But no polynomial satisfies property (A). Moreover, there are some transcendental functions which do not satisfy property (A).

LEMMA 2.2. [1] *Let f be a nonconstant entire function, then f satisfies the property (A) if and only if for each $\sigma > 1$ and positive integer n,*

$$M_f(r)^n \leq M_f(r^\sigma), \text{ for all } r > 0.$$

LEMMA 2.3. [1] *Let f be a nonconstant entire function, $\alpha > 1, 0 < \beta < \alpha, s > 1, 0 < \mu < \lambda$ and n be a positive integer. Then*
a) $M_f(\alpha r) > \beta M_f(r)$.
b) There exist, $K = K(s, f) > 0$ such that

$$f(r)^s \leq K M_f(r^s), \text{ for all } r > 0.$$

c) $\lim\limits_{r\to\infty} \frac{M_f(r^s)}{M_f(r)} = \infty = \lim\limits_{r\to\infty} \frac{M_f(r^\lambda)}{M_f(r^\mu)}$.
d) If f is transcendental, then $\lim\limits_{r\to\infty} \frac{M_f(r^s)}{r^n M_f(r)} = \infty = \lim\limits_{r\to\infty} \frac{M_f(r^\lambda)}{r^n M_f(r^\mu)}$.

LEMMA 2.4. [1] *Suppose that f and g are entire functions, $f(0) = 0$ and $h = g \circ f$. Then there exist $c \in (0, 1)$, independent of f and g, such that*

$$M_h(r) > M_g\left(cM_f\left(\frac{r}{2}\right)\right), \text{ for all } r > 0.$$

LEMMA 2.5. [1] *Let $R > 0, \eta \in \left(0, \frac{3e}{2}\right)$ and f be analytic in $|z| \leq 2eR$ with $f(0) = 1$. Then on the disc $|z| \leq R$, excluding a family of discs the sum of whose radii is not greater than $4\eta R$, it is verified that*

$$\log|f(z)| > -T(\eta)\log M_f(2eR),$$

where $T(\eta) = 2 + log\left(\frac{3e}{2\eta}\right)$.

LEMMA 2.6. [1] *Let f be a nonconstant entire function and $A(r) = \max\{\operatorname{Re} f(z) : |z| = r\}$, then*

$$M_f(r) < A(145r).$$

LEMMA 2.7. *[1] Let f be a nonconstant entire function, then*

$$T(r) \leq \log^+ M_f(r) \leq \left(\frac{R+r}{R-r}\right) T(r), \text{ for } 0 < r < R.$$

3. Main Results

In this section we first define the relative logarithmic order of f with respect to g, relative logarithmic lower order of f with respect to g and then establish some theorems related to these. Finally we introduce the relative logarithmic proximate order of f with respect to g.

DEFINITION 3.1 (Relative logarithmic order of f with respect to g). Let f and g be two entire functions. The relative logarithmic order of f with respect to g is given by

$$\begin{aligned} \rho_g^l(f) &= \inf\{\mu > 0 : M_f(r) < M_g((\log r)^\mu), \text{ for all } r > r_0(\mu) > 0\} \\ &= \limsup_{r\to\infty} \frac{\log^+ M_g^{-1}(M_f(r))}{\log\log r}. \end{aligned}$$

DEFINITION 3.2 (Relative logarithmic lower order of f with respect to g). Let f and g be two entire functions. The relative logarithmic lower order of f with respect to g is given by

$$\lambda_g^l(f) = \liminf_{r\to\infty} \frac{\log^+ M_g^{-1}(M_f(r))}{\log\log r}.$$

3.1. Some general properties on relative logarithmic order.

THEOREM 3.3. *Let f, g, h be nonconstant entire functions and $L_i (i = 1, 2, 3, 4)$ are nonconstant linear functions, i.e. $L_i(z) = a_i z + b_i$, for all $z \in \mathbb{C}$, with $a_i, b_i \in \mathbb{C}, a_i \neq 0 (i = 1, 2, 3, 4)$. Then*

a) $\rho_g^l(f) = \limsup_{r\to\infty} \frac{\log^+ M_g^{-1}(M_f(r))}{\log\log r} = \limsup_{r\to\infty} \frac{\log^+ M_g^{-1}(r)}{\log\log^+ M_f^{-1}(r)}$,

b) If g is a polynomial and f is a transcendental, then $\rho_g^l(f) = \infty$,

c) If f and g are polynomials, then $\rho_g^l(f) = \infty$,

d) If $M_f(r) \leq M_g(r)$, then we have $\rho_h^l(f) \leq \rho_h^l(g)$,

e) If $M_g(r) \leq M_h(r)$, then we have $\rho_g^l(f) \geq \rho_h^l(f)$,

f) $\rho_{(L_4\circ g\circ L_3)}^l(L_2 \circ f \circ L_1) = \rho_g^l(f)$.

Proof. a) This follows from the definition.

b) Let the degree of g be n. Then $M_g(r) \leq Kr^n$ and $M_f(r) > Lr^m$, where K, L are constant and $m > 0$ be any real number, for sufficiently large r.

Then,

$$\begin{aligned}\frac{\log^+ M_g^{-1}(M_f(r))}{\log\log r} &> \frac{\log^+ M_g^{-1}(Lr^m)}{\log\log r}\\ &\geq \frac{\log^+\left(\frac{1}{K}(Lr^m)^{\frac{1}{n}}\right)}{\log\log r}\\ &= \frac{\log^+ \frac{L^{\frac{1}{n}}}{K} + \log^+ r^{\frac{m}{n}}}{\log\log r}\\ &= \frac{m}{n}\frac{\log^+ r}{\log\log r} + \frac{\log^+ \frac{L^{\frac{1}{n}}}{K}}{\log\log r},\end{aligned}$$

which tends to ∞ as $r \to \infty$.

Hence,

$$\rho_g^l(f) = \limsup_{r\to\infty} \frac{\log^+ M_g^{-1}(M_f(r))}{\log\log r} = \infty.$$

c) Let

$$f(z) = a_0 z^m + a_1 z^{m-1} + ... + a^m, a_0 \neq 0$$

and

$$g(z) = b_0 z^n + b_1 z^{n-1} + ... + b^n, b_0 \neq 0.$$

Then $M_f(r) \geq \frac{1}{2}|a_0| r^m$ and $M_g(r) \leq Kr^n$, where K is a constants, for sufficiently large r.

Then,

$$\begin{aligned}\frac{\log^+ M_g^{-1}(M_f(r))}{\log\log r} &\geq \frac{\log^+ M_g^{-1}\left(\frac{1}{2}|a_0| r^m\right)}{\log\log r}\\ &\geq \frac{\log^+\left(\frac{1}{K}\left(\frac{1}{2}|a_0| r^m\right)^{\frac{1}{n}}\right)}{\log\log r}\\ &= \frac{m}{n}\frac{\log^+ r}{\log\log r} + \frac{\log^+\left(\frac{|a_0|^{\frac{1}{n}}}{2^{\frac{1}{n}}K}\right)}{\log\log r},\end{aligned}$$

which tends to ∞ as $r \to \infty$.

Hence,

$$\rho_g^l(f) = \limsup_{r\to\infty} \frac{\log^+ M_g^{-1}(M_f(r))}{\log\log r} = \infty.$$

Proofs of d), e) and f) are omitted. □

REMARK 3.4. If f is a polynomial and g is a transcendental, then $\rho_g^l(f)$ may be zero or a positive finite number.

EXAMPLE 3.5. Let $f(z) = z$ and $g(z) = e^z$.

Then, $M_f(r) = r$ and $M_g(r) = e^r$.

Therefore

$$\rho_g^l(f) = \limsup_{r\to\infty} \frac{\log\log r}{\log\log r} = 1.$$

Example 3.6. Let $f(z) = z$ and $g(z) = e^{e^z}$.
Then, $M_f(r) = r$ and $M_g(r) = e^{e^r}$.
Therefore

$$\rho_g^l(f) = \limsup_{r\to\infty} \frac{\log\log\log r}{\log\log r} = 0.$$

3.2. Relative logarithmic order of composition.

Theorem 3.7. *Let f, f_1, f_2, g and m be nonconstant entire functions and $h = g\circ f$, then*
a) $\rho^l_{g\circ f_2}(g\circ f_1) = \rho^l_{f_2}(f_1)$,
b) $\max\{\rho^l_m(f), \rho^l_m(g)\} \le \rho^l_m(h)$,
c) If f is a polynomial, then $\rho^l_m(h) = \rho^l_m(g)$ and $\rho^l_g(h) = \infty$.

Proof. a) Let $h_i = g\circ f_i, (i = 1, 2)$. Then h_i is a nonconstant entire function.
We can suppose that $f_i(0) = 0$, if not, we take $f_i^*(z) = f_i(z) - f_i(0)$ and $g_i^*(z) = g(z + f_i(0))$ and we would have $h_i = g_i^* \circ f_i^*$, and by the first Theorem [**f**) part], we get $\rho^l_{f_2^*}(f_1^*) = \rho^l_{f_2}(f_1)$.
So, without loss of generality we take $f_i(0) = 0$.
We have by Lemma 2.4

$$M_{h_i}(r) \ge M_g\left(cM_{f_i}\left(\frac{r}{2}\right)\right), \text{ for all } r > 0, i = 1, 2.$$

Again using Lemma 2.3we have

$$\begin{aligned} M_{f_i}\left(\frac{1}{d}.\frac{dr}{2}\right) &> \frac{1}{c}.M_{f_i}\left(\frac{dr}{2}\right) \\ &\Rightarrow M_{f_i}\left(\frac{r}{2}\right) > \frac{1}{c}M_{f_i}\left(\frac{dr}{2}\right) \text{ for all } d\in(0,c), \text{ since } M_{h_i} \le M_g\circ M_{f_i}. \end{aligned}$$

Then

$$M_{h_i}(r) > M_g\left(M_{f_i}\left(\frac{dr}{2}\right)\right) \ge M_{h_i}\left(\frac{dr}{2}\right), \text{ for } i = 1, 2. \tag{1}$$

Again from (1)

$$\begin{aligned} M_{h_1}(r) &> M_g\left(M_{f_1}\left(\frac{dr}{2}\right)\right) \\ &\Rightarrow M_{h_2}^{-1}(M_{h_1}(r)) > M_{h_2}^{-1}\left(M_g\left(M_{f_1}\left(\frac{dr}{2}\right)\right)\right) \end{aligned}$$

Again since, $M_{h_2}^{-1}\circ M_g(t) \ge M_{f_2}^{-1}(t)$,

$$M_{h_2}^{-1}(M_{h_1}(r)) > M_{h_2}^{-1}\left(M_g\left(M_{f_1}\left(\frac{dr}{2}\right)\right)\right) > M_{f2}^{-1}\left(M_{f_1}\left(\frac{dr}{2}\right)\right). \tag{2}$$

In (1) , for $i = 2$, we put $M_{h_2}(r) = t$. i.e., $r = M_{h_2}^{-1}(t)$ and we get

$$\begin{aligned} t &> M_g\left(M_{f_2}\left(\frac{d}{2}M_{h_2}^{-1}(t)\right)\right) \\ M_{f_2}^{-1}(M_g^{-1}(t)) &> \frac{d}{2}M_{h_2}^{-1}(t) \Rightarrow M_{h_2}^{-1}(t) < \frac{2}{d}M_{f_2}^{-1}(M_g^{-1}(t)). \end{aligned}$$

Putting $t = M_{h_1}(r)$, we have

$$\begin{aligned} M_{h_2}^{-1}(M_{h_1}(r)) &< \frac{2}{d} M_{f_2}^{-1}(M_g^{-1}(M_{h_1}(r))) \\ &\leq \frac{2}{d} M_{f_2}^{-1}(M_{f_1}(r)). \end{aligned}$$

Combining (2) and (3) we have,

$$M_{f_2}^{-1}\left(M_{f_1}\left(\frac{dr}{2}\right)\right) < M_{h_2}^{-1}(M_{h_1}(r)) < \frac{2}{d} M_{f_2}^{-1}(M_{f_1}(r)).$$

Taking logarithm and dividing by $\log\log r$ and then taking $\limsup$ as $r \to \infty$, we get

$$\rho^l_{g\circ f_2}(g \circ f_1) = \rho^l_{f_2}(f_1).$$

b) As in part (a), we can assume that $f(0) = 0$.

Since f and g are nonconstant, there exist $\alpha > 0$ such that $M_f(r) > \alpha r$ and $M_g(r) > \alpha r$.

Applying the Lemma 2.4$c \in (0, 1)$ such that

$$\begin{aligned} M_h(r) &\geq M_g\left(cM_f\left(\frac{r}{2}\right)\right) > \alpha.c.M_f\left(\frac{r}{2}\right) > M_f(r^\sigma), \text{ for sufficiently large } r, \\ &\Rightarrow M_m^{-1}(M_h(r)) > M_m^{-1}(M_f(r^\sigma)), \qquad (3) \end{aligned}$$

and also

$$M_h(r) > M_g\left(cM_f\left(\frac{r}{2}\right)\right) > M_g\left(c\alpha\frac{r}{2}\right) > M_g(r^\sigma), \text{ for sufficiently large } r.$$

Taking logarithm and dividing by $\log\log r$ and using (3), we get

$$\begin{aligned} \frac{\log^+ M_m^{-1}(M_h(r))}{\log\log r} &> \frac{\log^+ M_m^{-1}(M_f(r^\sigma))}{\log\log r} \\ &= \frac{\log^+ M_m^{-1}(M_f(s))}{\log\log s^{\frac{1}{\sigma}}}, \text{ [taking } r^\sigma = s] \\ &= \frac{\log^+ M_m^{-1}(M_f(s))}{\log\left(\frac{1}{\sigma}\log s\right)} \\ &= \frac{\log^+ M_m^{-1}(M_f(s))}{\log\frac{1}{\sigma} + \log\log s} \\ &= \frac{\frac{\log^+ M_m^{-1}(M_f(s))}{\log\log s}}{\frac{\log\frac{1}{\sigma}}{\log\log s} + 1} \end{aligned}$$

Now taking $\limsup$ as $r \to \infty$, we get

$$\begin{aligned} \rho^l_m(h) &\geq \limsup_{s\to\infty} \frac{\log^+ M_m^{-1}(M_f(s))}{\log\log s}, \text{ [since } s \to \infty \text{ as } r = \infty] \\ &\Rightarrow \rho^l_m(h) \geq \rho^l_m(f). \end{aligned}$$

Similarly from (3.7) , we get

$$\rho^l_m(h) \geq \rho^l_m(g).$$

From the above two results (b) follows.

c) Let f be a polynomial of degree $n \geq 1$, then there exist $\alpha > 0, \beta > 0$ such that $\alpha r^n < M_f(r) < \beta r^n$.

So, using Lemma 2.4

$$\begin{aligned} M_g(\gamma r^n) &< M_g\left(cM_f\left(\frac{r}{2}\right)\right) \leq M_h(r) \leq M_g\left(M_f\left(r\right)\right) < M_g(\beta r^n) \\ &\Rightarrow M_m^{-1}\left(M_g(\gamma r^n)\right) < M_m^{-1}\left(M_h(r)\right) < M_m^{-1}\left(M_g(\beta r^n)\right), \end{aligned}$$

where $\gamma = c\left(\frac{\alpha}{2}\right)^n$.

Taking logarithms and dividing by $\log\log r$, we get

$$\frac{\log^+ M_m^{-1}\left(M_g(\gamma r^n)\right)}{\log\log r} < \frac{\log^+ M_m^{-1}\left(M_h(r)\right)}{\log\log r} < \frac{\log^+ M_m^{-1}\left(M_g(\beta r^n)\right)}{\log\log r}$$

$$\begin{aligned} &\Rightarrow \frac{\log^+ M_m^{-1}\left(M_g(s)\right)}{\log\log\left(\frac{s}{\gamma}\right)^{\frac{1}{n}}} < \frac{\log^+ M_m^{-1}\left(M_h(r)\right)}{\log\log r} \\ &\qquad < \frac{\log^+ M_m^{-1}\left(M_g(t)\right)}{\log\log\left(\frac{t}{\beta}\right)^{\frac{1}{n}}} \ [\text{taking } \gamma r^n = s \text{ and } \beta r^n = t] \\ &\Rightarrow \frac{\log^+ M_m^{-1}\left(M_g(s)\right)}{\log\frac{1}{n} + \log\log\left(\frac{s}{\gamma}\right)} < \frac{\log^+ M_m^{-1}\left(M_h(r)\right)}{\log\log r} < \frac{\log^+ M_m^{-1}\left(M_g(t)\right)}{\log\frac{1}{n} + \log\log\left(\frac{t}{\beta}\right)} \\ &\Rightarrow \frac{\frac{\log^+ M_m^{-1}(M_g(s))}{\log\log s}}{\frac{\log\frac{1}{n}}{\log\log s} + \frac{\log\log\left(\frac{s}{\gamma}\right)}{\log\log s}} < \frac{\log^+ M_m^{-1}\left(M_h(r)\right)}{\log\log r} < \frac{\frac{\log^+ M_m^{-1}(M_g(t))}{\log\log s}}{\frac{\log\frac{1}{n}}{\log\log t} + \frac{\log\log\left(\frac{t}{\beta}\right)}{\log\log t}} \end{aligned}$$

Now taking lim sup as $r \to \infty$, we get

$$\rho_m^l(h) = \rho_m^l(g)$$

Again from (4), we get

$$\begin{aligned} M_g^{-1}\left(M_h(r)\right) &> M_g^{-1}\left(M_g(\gamma r^n)\right) = \gamma r^n \\ &\Rightarrow \frac{\log^+ M_g^{-1}\left(M_h(r)\right)}{\log\log r} > \frac{\log^+ \gamma + n\log^+ r}{\log\log r}, \end{aligned}$$

which tends to ∞ as $r \to \infty$.

Hence,

$$\rho_g^l(h) = \limsup_{r\to\infty} \frac{\log^+ M_g^{-1}\left(M_h(r)\right)}{\log\log r} = \infty.$$

□

3.3. Relative logarithmic order of sum and product.

THEOREM 3.8. *Let f, g, f_1, f_2 be nonconstant entire functions and P be a polynomial not identically zero. Then*

a) $\rho_g^l(f_1 + f_2) \leq \max\{\rho_g^l(f_1), \rho_g^l(f_2)\}$, giving equality if $\rho_g^l(f_1) \neq \rho_g^l(f_2)$,

b) If f is transcendent, then $\rho_g^l(Pf) = \rho_g^l(f)$, and if g is transcendent, then $\rho_{Pg}^l(f) = \rho_g^l(f)$,

c) $\rho_g^l(f) = \rho_g^l(f^n)$, where n is a positive integer.

d) if g satisfies property (A), then $\rho_g^l(f_1 f_2) \leq \max\{\rho_g^l(f_1), \rho_g^l(f_2)\}$, giving equality if $\rho_g^l(f_1) \neq \rho_g^l(f_2)$.

The same result is true for $\frac{f_1}{f_2}$, assuming it is an entire function.

Proof. a) Let $h = f_1 + f_2$, $\rho^l = \rho_g^l(h), \rho_i^l = \rho_g^l(f_i)$ for $i = 1, 2$.

If h is constant, then it is trivial.

Suppose h is not a constant and $\rho_1^l \leq \rho_2^l$.

Given $\varepsilon > 0$,

$$M_{f_1}(r) \leq M_g\left((\log r)^{\rho_1^l+\varepsilon}\right) \leq M_g\left((\log r)^{\rho_2^l+\varepsilon}\right)$$

and

$$M_{f_2}(r) \leq M_g\left((\log r)^{\rho_2^l+\varepsilon}\right),$$

for $r > r_0$.

Therefore,

$$M_h(r) \leq M_{f_1}(r) + M_{f_2}(r) \leq 2M_g\left((\log r)^{\rho_2^l+\varepsilon}\right) < M_g\left(3(\log r)^{\rho_2^l+\varepsilon}\right).$$

Taking logarithm and dividing by $\log\log r$, we get

$$\begin{aligned}
\frac{\log^+ M_g^{-1} M_h(r)}{\log\log r} &< \frac{\log^+ 3 + \left(\rho_2^l + \varepsilon\right)\log^+\log r}{\log\log r} \\
&\Rightarrow \limsup_{r\to\infty} \frac{\log^+ M_g^{-1} M_h(r)}{\log\log r} \\
&< \limsup_{r\to\infty} \frac{\log^+ 3 + \left(\rho_2^l + \varepsilon\right)\log^+\log r}{\log\log r} \\
&= \rho_2^l + \varepsilon \\
&\Rightarrow \rho_g^l(h) \leq \rho_2^l + \varepsilon, \text{ for each } \varepsilon > 0,
\end{aligned}$$

and consequently,

$$\rho^l \leq \rho_2^l = \max\{\rho_1^l, \rho_2^l\}.$$

Now suppose that, $\rho_1^l < \rho_2^l$ and take $\lambda \in \left(\rho_1^l, \rho_2^l\right)$ and $\mu \in \left(\rho_1^l, \lambda\right)$.

Then, $M_{f_1}(r) < M_g\left((\log r)^\mu\right)$ and there is a sequence $\{r_n\} \to \infty$ with $M_g\left((\log r_n)^\lambda\right) < M_{f_2}(r)$, for all n.

Again by Lemma 2.3¿$2M_g\left((\log r)^\mu\right)$.

Therefore

$$2M_{f_1}(r_n) < 2M_g\left((\log r_n)^\mu\right) < M_g\left((\log r_n)^\lambda\right) < M_{f_2}(r_n) \text{ for sufficiently large } n.$$

Which implies

$$\begin{aligned}
M_h(r_n) &\geq M_{f_2}(r_n) - M_{f_1}(r_n) \\
&> \frac{1}{2} M_{f_2}(r_n) \\
&> \frac{1}{2} M_g\left((\log r_n)^\lambda\right) \\
&> M_g\left(\frac{1}{3}(\log r_n)^\lambda\right), \text{ for } n \text{ sufficiently large } n, \text{ by Lemma 2.3 } a).
\end{aligned}$$

Therefore

$$\begin{aligned}\rho^l &\geq \limsup_{r\to\infty} \frac{\log^+ M_g^{-1} M_h(r_n)}{\log\log r_n}\\ &> \limsup_{r\to\infty} \frac{\log^+ \frac{1}{3} + \lambda \log^+ \log r_n}{\log\log r_n} = \lambda, \text{ for each } \lambda \in (\rho_1^l, \rho_2^l).\end{aligned}$$

So, $\rho^l \geq \rho_2^l = \max\{\rho_1^l, \rho_2^l\}$.
Hence

$$\rho^l = \max\{\rho_1^l, \rho_2^l\}.$$

b) Since $P(z)$ is a polynomial there exists a real number $\alpha > 0$ and a positive integer n such that

$$2\alpha < |P(z)| < r^n \qquad (|z| = r)$$

Since f is transcendental, $h = Pf$ and $s > 1$, then

$$\begin{aligned}M_f(\alpha r) &< 2\alpha M_f(r), \text{ using Lemma 2.3 } a)\\ &< |P(z)| M_f(r), \text{ on } |z| = r\\ &= M_h(r)\\ &< r^n M_f(r)\\ &< M_f(r^s), \text{ using Lemma 2.3 } d), \text{ for sufficiently large } r.\end{aligned}$$

Therefore

$$M_g^{-1} M_f(\alpha r) < M_g^{-1} M_h(r) < M_g^{-1} M_f(r^s)$$

$$\Rightarrow \frac{\log^+ M_g^{-1} M_f(\alpha r)}{\log\log(\alpha r)} . \frac{\log^+ \log(\alpha r)}{\log\log(r)} < \frac{\log^+ M_g^{-1} M_h(r)}{\log\log r} < \frac{\log^+ M_g^{-1} M_f(r^s)}{\log\log r^s} . \frac{\log\log r^s}{\log\log r}$$

Taking $\limsup$ as $r \to \infty$, we have

$$\begin{aligned}\rho_g^l(f).1 &\leq \rho_g^l(h) \leq \rho_g^l(f).1\\ &\Rightarrow \rho_g^l(h) = \rho_g^l(f)\end{aligned}$$

c) It is obvious that $\rho_g^l(f^n) \geq \rho_g^l(f)$.
Let $M_{f^n}(r) = \max\{|f^n(z)| : |z| = r\}$.
Therefore

$$\begin{aligned}M_{f^n}(r) &\leq K M_f(r^n) < M_f((K+1)r^n), \text{ by Lemma 2.3 } a), 2.3\ b)\\ &\Rightarrow M_g^{-1} M_{f^n}(r) < M_g^{-1} M_f((K+1)r^n)\\ &\Rightarrow \frac{\log^+ M_g^{-1} M_{f^n}(r)}{\log\log r} < \frac{\log^+ M_g^{-1} M_f((K+1)r^n)}{\log\log r}\end{aligned}$$

Taking $\limsup$ as $r \to \infty$, we get

$$\rho_g^l(f^n) \leq \rho_g^l(f).$$

Therefore

$$\rho_g^l(f^n) = \rho_g^l(f).$$

d) Let us assume f_1, f_2 be transcendental, otherwise it would be trivial.
Denote $h = f_1 f_2, \rho^l = \rho_g^l(h), \rho_i^l = \rho_g^l(f_i), (i = 1, 2)$.
First we assume $\rho_1 \leq \rho_2 < \infty$(If $\rho_2 = \infty$, it is trivial)
Now given $\varepsilon > 0$,

$$M_{f_i}(r) < M_g\left((\log r)^{\rho_2 + \frac{\varepsilon}{2}}\right), i = 1, 2.$$

Then

$$M_h(r) \leq M_{f_1}(r)M_{f_2}(r) < \left(M_g\left((\log r)^{\rho_2+\frac{\varepsilon}{2}}\right)\right)^2 < M_g\left((\log r)^{\rho_2+\varepsilon}\right),$$

applying Property (A), with $\sigma = \frac{\rho_2^l+\varepsilon}{\rho_2^l+\frac{\varepsilon}{2}}$.

Then

$$\rho^l \leq \rho_2^l = \max\{\rho_1^l, \rho_2^l\}.$$

Next suppose that, $\rho_1^l < \rho_2^l$.

From part b) we have, the product of f by a factor $\frac{c}{z^n}$ does not alter its order, so we can assume without loss of generality that $f_1(0) = 1$.

Take λ, μ with $\rho_1^l < \mu < \lambda < \rho_2^l$.

Then there is a succession $R_n \to \infty$ such that

$$M_{f_2}(R_n) > M_g((\log R_n)^\lambda), \text{ for all } n \text{ and } M_{f_1}(r) < M_g((\log r)^\mu).$$

Let us apply the Lemma $2.5\frac{1}{16}$, we get

$$\log|f_1(z)| > -(2+\log(24e))\log M_{f_1}(4eR_n)$$

on the disc $|z| \leq 2R_n$, excluding a family of discs, the sum of whose radii exceeds $\frac{R_n}{2}$.

Therefore there exists $r_n \in (R_n, 2R_n)$ such that $|z| = r_n$ does not intersect any of the excluded discs, then

$$\log|f_1(z)| > -7\log M_{f_1}(4eR_n) \text{ in } |z| = r_n.$$

Also

$$M_{f_2}(r_n) > M_{f_2}(R_n) > M_g\left((\log R_n)^\lambda\right) > M_g\left(\left(\log\frac{r_n}{2}\right)^\lambda\right)$$

If z_r is a point in $|z| = r$, with $M_{f_2}(r) = |f_2(z_r)|$, we have

$$M_h(r) \geq |f_1(z_r)|\,|f_2(z_r)| = |f_1(z_r)|\,M_{f_2}(r).$$

Therefore

$$\begin{aligned} M_h(r) &> (M_{f_1}(4eR_n))^{-7} M_g\left(\left(\log\frac{r_n}{2}\right)^\lambda\right) \\ &> (M_g((\log 4eR_n)^\mu))^{-7} M_g\left(\left(\log\frac{r_n}{2}\right)^\lambda\right) \\ &> (M_g((\log 4er_n)^\mu))^{-7} M_g\left(\left(\log\frac{r_n}{2}\right)^\lambda\right), \text{ for sufficiently large } n. \end{aligned}$$

Take $\nu \in (\mu, \lambda), \sigma = \frac{v}{\mu} > 1$, we obtain

$$\begin{aligned} M_h(r_n) &> M_g((\log 4er_n)^\nu)(M_g((\log 4er_n)^\mu))^{-7} \\ &> (M_g((\log 4er_n)^\mu))^8(M_g((\log 4er_n)^\mu))^{-7} \\ &= M_g((\log 4er_n)^\mu), \text{ applying Lemma 2.2 for } n = 8 \text{ and } r = (\log 4er_n)^{\mu.} \\ &> M_g((\log r_n)^\mu), \text{ for sufficiently large } n. \end{aligned}$$

Consequently

$$\begin{aligned} \rho^l &\geq \mu, \text{ for each } \mu < \rho_2^l \\ &\Rightarrow \rho^l \geq \rho_2^l \end{aligned}$$

Hence

$$\rho^l = \rho_2^l.$$

For the last part of d), let $h = \frac{f_1}{f_2}$, i.e. $f_1 = hf_2$.

We keep the same notations and without loss of generality let us suppose $\rho_1^l \leq \rho_2^l$.

If possible let, $\rho^l > \rho_2^l = \max\{\rho_1^l, \rho_2^l\}$. Then from the previous part equality occurs, i.e. $\rho_1^l = \max\{\rho^l, \rho_2^l\} = \rho^l$. Therefore $\rho_1^l > \rho_2^l$ and we came to a contradiction.

Therefore,

$$\rho^l \leq \rho_2^l.$$

Next we suppose that $\rho_1^l < \rho_2^l$. We are to show in this case equality holds. If possible assume that $\rho^l < \rho_2^l$, then $\max\{\rho^l, \rho_2^l\} = \rho_1^l$. From the previous part, then $\rho_2^l = \rho_1^l$ and we come back to a contradiction again.

So,

$$\rho^l = \rho_2^l.$$

□

3.4. Relative logarithmic order of derivative.

THEOREM 3.9. *Let f and g be both transcendental entire functions. Then*

$$\rho_g^l(f) = \rho_g^l(f') = \rho_{g'}^l(f) = \rho_{g'}^l(f').$$

Proof. Without loss of generality we can assume that $f(0) = 0$.

Then

$$f(z) = \int_0^z f'(t)dt$$

where we have taken the integral over the segment that joins the origin with z. Then

$$M_f(r) \leq rM_{f'}(r).$$

Using Cauchy's formula, we get

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(t)}{(t-z)^2}dt,$$

where $C = \{t : |t - z| = r\}$; then

$$M_{f'}(r) \leq \frac{1}{2\pi}\frac{M_f(r)}{r^2}.2\pi r = \frac{M_f(r)}{r} \leq \frac{M_f(2r)}{r}.$$

Summarizing we get,

$$\frac{M_f(r)}{r} \leq M_{f'}(r) \leq \frac{M_f(2r)}{r}, \text{ for each } r > 0.$$

Next let $\sigma \in (0,1)$, then from Lemma 2.3 $d)$ and taking $\lambda = 1, \mu = \sigma$, we get

$$\begin{aligned} \lim_{r\to\infty} \frac{M_f(r)}{rM_f(r^\sigma)} &= \infty \\ &\Rightarrow M_f(r) > rM_f(r^\sigma), \text{ for sufficiently large } r. \end{aligned}$$

Therefore

$$M_f(r^\sigma) \leq \frac{M_f(r)}{r} \leq M_{f'}(r) \leq \frac{M_f(2r)}{r} \leq M_f(2r)$$

$$\Rightarrow\ M_f(r^\sigma) \le M_{f'}(r) \le M_f(2r)$$
$$\Rightarrow\ M_g^{-1}M_f(r^\sigma) \le M_g^{-1}M_{f'}(r) \le M_g^{-1}M_f(2r)$$
$$\Rightarrow\ \frac{\log^+ M_g^{-1}M_f(r^\sigma)}{\log\log r} \le \frac{\log^+ M_g^{-1}M_{f'}(r)}{\log\log r} \le \frac{\log^+ M_g^{-1}M_f(2r)}{\log\log r}$$
$$\Rightarrow\ \frac{\log^+ M_g^{-1}M_f(r^\sigma)}{\log\log r^\sigma}.\frac{\log\log r^\sigma}{\log\log r} \le \frac{\log^+ M_g^{-1}M_{f'}(r)}{\log\log r} \le \frac{\log^+ M_g^{-1}M_f(2r)}{\log\log 2r}.\frac{\log\log 2r}{\log\log r},$$

taking $\limsup$ as $r \to \infty$, we get

$$\begin{aligned}\rho_g^l(f).1 &\le \rho_g^l(f') \le \rho_g^l(f).1\\ &\Rightarrow \rho_g^l(f) = \rho_g^l(f')\end{aligned}$$

Again from (4)

$$\frac{\log\log^+ M_g^{-1}M_f(r^\sigma)}{\log r} \le \frac{\log\log^+ M_g^{-1}M_{f'}(r)}{\log r} \le \frac{\log\log^+ M_g^{-1}M_f(2r)}{\log r}$$
$$\Rightarrow \frac{\log\log^+ M_g^{-1}(s)}{\log^+(\frac{1}{\sigma}M_f^{-1}(s))} \le \frac{\log\log^+ M_g^{-1}(s)}{\log^+ M_{f'}^{-1}(s)} \le \frac{\log\log^+ M_g^{-1}(s)}{\log^+(\frac{1}{2}M_f^{-1}(s))},$$

taking $\liminf$ as $s \to \infty$, we get

$$\begin{aligned}\liminf_{s\to\infty} \frac{\log\log^+ M_g^{-1}(s)}{\log^+ M_f^{-1}(s)} &\le \liminf_{s\to\infty} \frac{\log\log^+ M_g^{-1}(s)}{\log^+ M_{f'}^{-1}(s)} \le \liminf_{s\to\infty} \frac{\log\log^+ M_g^{-1}(s)}{\log^+ M_f^{-1}(s)}\\ &\Rightarrow \liminf_{s\to\infty} \frac{\log\log^+ M_g^{-1}(s)}{\log^+ M_f^{-1}(s)} = \liminf_{s\to\infty} \frac{\log\log^+ M_g^{-1}(s)}{\log^+ M_{f'}^{-1}(s)}.\end{aligned}$$

Interchanging the role of f and g, we get

$$\begin{aligned}\liminf_{s\to\infty} \frac{\log\log^+ M_f^{-1}(s)}{\log^+ M_g^{-1}(s)} &= \liminf_{s\to\infty} \frac{\log\log^+ M_f^{-1}(s)}{\log^+ M_{g'}^{-1}(s)}\\ &\Rightarrow \frac{1}{\limsup_{s\to\infty} \frac{\log^+ M_g^{-1}(s)}{\log\log^+ M_f^{-1}(s)}} = \frac{1}{\limsup_{s\to\infty} \frac{\log^+ M_{f'}^{-1}(s)}{\log\log^+ M_g^{-1}(s)}}\\ &\Rightarrow \frac{1}{\rho_g^l(f)} = \frac{1}{\rho_{g'}^l(f)}\\ &\Rightarrow \rho_g^l(f) = \rho_{g'}^l(f).\end{aligned}$$

Consequently from (4) and (4) , we get

$$\rho_g^l(f) = \rho_g^l(f') = \rho_{g'}^l(f) = \rho_{g'}^l(f')$$

□

Note that, it is trivial when either f and g both are polynomials, or f is transcendent and g is polynomial. But the theorem does not hold for f is polynomial and g is transcendental, as shown in the following example.

Example 3.10. let $f(z) = z, g(z) = \exp z$.
Then $f'(z) = 1, M_f(r) = r, M_{f'}(r) = 1$ and $M_g(r) = \exp r$.

Therefore

$$\begin{aligned}\rho_g^l(f) &= \frac{\log^+ \log r}{\log\log r}\\ &= 1\end{aligned}$$

whereas,

$$\begin{aligned}\rho_g^l(f') &= \frac{\log^+ \log 1}{\log\log r}\\ &= 0.\end{aligned}$$

3.5. Relative logarithmic order of real and imaginary parts.

Theorem 3.11. *Let f and g are nonconstant entire functions.*
Let

$$\begin{aligned}A(r) &= \max\{\operatorname{Re} f(z) : |z| = r\},\\ B(r) &= \max\{\operatorname{Im} f(z) : |z| = r\},\\ C(r) &= \max\{\operatorname{Re} g(z) : |z| = r\},\\ D(r) &= \max\{\operatorname{Im} g(z) : |z| = r\}.\end{aligned}$$

Then

$$\begin{aligned}\rho_g^l(f) &= \inf\{\mu > 0 : M(r) < N\left((\log r)^\mu\right)\}\\ &= \limsup_{r\to\infty} \frac{\log N^{-1}(M(r))}{\log\log r}.\end{aligned}$$

where M is any of the functions $A, B \circ F$ and N is any of the functions $C, D \circ G$.

Proof. It is clear that A, B, C and D are continuous strictly increasing functions of r, then A^{-1}, B^{-1}, C^{-1} and D^{-1} exist. From Lemma 2.5 we obtain the existence of a constant $\alpha > 0$ with

$$M(r) \le F(r) \le M(\alpha r)$$

and

$$N(r) \le G(r) \le N(\alpha r).$$

Let $\rho^l = \rho_g^l(f)$ and $\beta = \inf\{\mu > 0 : M(r) < N((\log r)^\mu)\}$.
We first prove that $\beta \le \rho^l$.
If $\rho^l = \infty$, it is trivial.
So assume that ρ^l is finite, choose λ, μ with $\rho^l < \lambda < \mu < \infty$.
Then $M_f(r) < M_g((\log r)^\lambda)$ and

$$M(r) \le M_f(r) < M_g((\log r)^\lambda) < N((\log \alpha r)^\lambda) < N((\log r)^\mu), \text{ for sufficiently large } r.$$

Which implies

$$\begin{aligned}\beta &\le \mu, \text{ for all } \mu > \rho^l.\\ &\Rightarrow \beta \le \rho^l\end{aligned}$$

Finally let us prove, $\beta \ge \rho^l$.
If $\rho^l = 0$, the case is trivial.
So let $\rho^l > 0$, choose λ, μ such that $0 < \mu < \lambda < \rho^l$.
Then there is a sequence $\{r_n\} \to \infty$ such that

$$M_f(r_n) > M_g((\log r)^\lambda), \text{ for all } n.$$

Therefore
$$M(\alpha r_n) > M_f(r_n) > M_g((\log r_n)^{\lambda}) > M_g((\log \alpha r_n)^{\mu}) > N((\log \alpha r_n)^{\mu}),$$
for sufficiently large n.
Which implies
$$\begin{aligned} \beta &\geq \mu \text{ for each } \mu < \rho^l. \\ &\Rightarrow \beta \geq \rho^l. \end{aligned}$$
Therefore we have $\beta = \rho^l$. □

3.6. Relative logarithmic order of Nevanlinna's characteristic function.

THEOREM 3.12. *Let f and g are nonconstant entire functions. Then*
$$\begin{aligned} \rho_g^l(f) &= \inf\{\mu > 0 : T_f(r) < T_g((\log r)^{\mu})\} \\ &= \limsup_{r\to\infty} \frac{\log^+ T_g^{-1}(T_f(r))}{\log\log r}. \end{aligned}$$

Proof. Let $\rho^l = \rho_g^l(f)$ and $\alpha = \inf\{\mu > 0 : T_f(r) < T_g((\log r)^{\mu})\}$
Let us prove that $\alpha \leq \rho^l$.
If $\rho^l = \infty$, the case is trivial.
So, we take ρ^l be finite and let's take $\gamma, \delta, \lambda, \mu$ such that $\rho^l < \gamma < \delta < \lambda < \mu < \infty$.
Now for sufficiently large r, it is clear that
$$\frac{\gamma}{\delta} < \frac{(\log r)^{\mu} - (\log r)^{\lambda}}{(\log r)^{\mu} + (\log r)^{\lambda}}.$$
By Lemma 2.3
$$M_g(r^{\gamma})^s = M_g(r^{\gamma})^{\frac{\delta}{\gamma}} \leq KM_g(r^{\delta}) < M_g(r^{\lambda}).$$
Hence
$$\begin{aligned} M_g((\log r)^{\gamma})^s &= M_g\left((\log r)^{\gamma}\right)^{\frac{\delta}{\gamma}} \\ &\leq KM_g((\log r)^{\delta}), \text{ for sufficiently large } r \\ &< M_g((\log r)^{\lambda}). \end{aligned}$$
Therefore,
$$\frac{\delta}{\gamma} \log M_g((\log r)^{\gamma}) < \log M_g((\log r)^{\lambda}).$$
Which implies
$$\begin{aligned} \log^+ M_g((\log r)^{\gamma}) &< \frac{\gamma}{\delta} \log^+ M_g((\log r)^{\lambda}) \\ &< \frac{(\log r)^{\mu} - (\log r)^{\lambda}}{(\log r)^{\mu} + (\log r)^{\lambda}} \log^+ M_g((\log r)^{\lambda}) \\ &\leq T_g((\log r)^{\mu}). \end{aligned}$$
Again from Lemma 2.7
$$\begin{aligned} T_f(r) &\leq \log^+ M_f(r) < \log^+ M_g((\log r)^{\lambda}) \\ &\Rightarrow T_f(r) < T_g((\log r)^{\mu}) \\ &\Rightarrow \mu \geq \alpha, \text{ for all } \mu > \rho^l \\ &\Rightarrow \rho^l \geq \alpha. \end{aligned}$$

Next we prove, $\alpha \geq \rho^l$.
If $\rho^l = 0$, the case is trivial.
So let $\rho^l > 0$, and take γ, δ, μ with $0 < \mu < \lambda < \gamma < \rho^l$.
Then there exist $\{r_n\} \to \infty$ such that

$$M_f(r_n) > M_g((\log r_n)^\gamma), \text{ for all } n.$$

Let $c \in \left(\frac{\lambda}{\gamma}, 1\right)$ and $d > \frac{1+c}{1-c}$.
Then

$$\begin{aligned} T_f(dr_n) &> \frac{dr_n - r_n}{dr_n + r_n} \log^+ M_f(r_n) \\ &= \frac{d-1}{d+1} \log^+ M_f(r_n) \\ &> c \log^+ M_f(r_n) \\ &> \log^+ M_g((\log r_n)^\gamma)^c \\ &> \log^+ \frac{M_g((\log r_n)^{\gamma c})}{K}, \text{ using Lemma 2.3 } b) \text{ for } c < 1 \\ &> \log^+ M_g((\log r_n)^\lambda), \text{ as } c > \frac{\lambda}{\gamma} \\ &\geq \log^+ M_g((d \log r_n)^\mu), \text{ for sufficiently large } n \\ &\geq T_g((d \log r_n)^\mu). \end{aligned}$$

Therefore,

$$\begin{aligned} T_f(dr_n) &> T_g((d \log r_n)^\mu), \text{ for sufficiently large } n \\ &\Rightarrow \alpha \geq \mu, \text{ for all } \mu < \rho^l \\ &\Rightarrow \alpha \geq \rho^l. \end{aligned}$$

Hence,

$$\rho^l = \alpha = \{\mu > 0 : T_f(r) < T_g((\log r)^\mu)\}.$$

□

3.7. Relative logarithmic proximate order.

Definition 3.13 (Relative logarithmic proximate order of f with respect to g). Let f and g be two entire functions with finite logarithmic order of growth of f relative to g (i.e. $\rho_g^l(f)$ is finite). A non-negative real valued continuous function $\rho_g^l(f)(r)$, defined in $(0, +\infty)$, is said to be a logarithmic proximate order of growth of f relative to g if the following properties holds:

i) $\rho_g^l(f)(r)$ is differentiable for $r > r_0$ except at isolated points at which $\left[\rho_g^l(f)\right]'(r-0)$ and $\left[\rho_g^l(f)\right]'(r+0)$ exist,

ii) $\lim\limits_{r\to\infty} \rho_g^l(f)(r) = \rho_g^l(f)$,

iii) $\lim\limits_{r\to\infty} r.[\rho_g^l(f)]'(r). \log\log r = 0$,

iv) $\limsup\limits_{r\to\infty} \frac{M_g^{-1}(M_f(r))}{(\log r)^{\rho_g^l(f)(r)}} = 1$.

Theorem 3.14. *Let f and g be two entire functions with finite logarithmic order of f with respect to g. Then there exist a logarithmic proximate order $\rho_g^l(f)(r)$.*

The proof of this theorem is omitted because it can be carried out in the same line of S. M. Shah [6].

3.8. Future aspects. Keeping in mind the results already established, one may explore the analogous theorems using more generalized order such as iterated order [5], $(p, q)-$order [4], $\phi-$order [3] etc.

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Chinmay Ghosh
Department of Mathematics, Kazi Nazrul University,
Nazrul Road, P.O.- Kalla C.H. Asansol-713340, West Bengal, India
E-mail: chinmayarp@gmail.com

Anirban Bandyopadhyay
53, Gopalpur Primary School, Raninagar-I, Murshidabad,
Pin-742304, West Bengal, India
E-mail: anirbanbanerjee010@gmail.com

Soumen Mondal
28, Dolua Dakshinpara Haridas Primary School,
Beldanga, Murshidabad, Pin-742133, West Bengal, India
E-mail: mondalsoumen79@gmail.com