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SUM AND PRODUCT THEOREMS OF (p, q)-φ RELATIVE GOL'DBERG TYPE AND (p, q)-φ RELATIVE GOL'DBERG WEAK TYPE OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Abstract. In this paper, we established sum and product theorems connected to (p, q)-φ relative Gol'dberg type and (p, q)-φ relative Gol'dberg weak type of entire functions of several complex variables with respect to another one under somewhat different conditions.

1. Introduction, Definitions and Notations

The symbols $\mathbb{C}^n$ and R^n will stand for complex and real n-spaces respectively . In addition, let us assume that the points $(z_1, z_2, \cdots, z_n)$, $(m_1, m_2, \cdots, m_n)$ of $\mathbb{C}^n$ or I^n be represented by their corresponding unsuffixed symbols z, m respectively where I denotes the set of non-negative integers. Then the modulus of z, denoted by $|z|$, is defined as $|z| = (|z_1|^2 + \cdots + |z_n|^2)^{\frac{1}{2}}$. If the coordinates of the vector m are non-negative integers, then the expression $z_1^{m_1} \cdots z_n^{m_n}$ will be denoted by z^m where $\|m\| = m_1 + \cdots + m_n$.

Consider $D \subseteq \mathbb{C}^n$ to be an arbitrary bounded complex n-circular domain with center at the origin of coordinates. Then for any entire

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function $f(z)$ of n complex variables and $R > 0$, $M_{f,D}(R)$ may be defined as $M_{f,D}(R) = \sup_{z \in D_R} |f(z)|$ where a point $z \in D_R$ if and only if $\frac{z}{R} \in D$. If $f(z)$ is non-constant, then $M_{f,D}(R)$ is strictly increasing and its inverse $M_{f,D}^{-1} : (|f(0)|, \infty) \to (0, \infty)$ exists such that $\lim_{R\to\infty} M_{f,D}^{-1}(R) = \infty$.

For $k \in \mathbb{N}$, we define $\exp^{[k]} R = \exp(\exp^{[k-1]} R)$ and $\log^{[k]} R = \log(\log^{[k-1]} R)$ where $\mathbb{N}$ is the set of all positive integers. We also denote $\log^{[0]} R = R$, $\log^{[-1]} R = \exp R$, $\exp^{[0]} R = R$ and $\exp^{[-1]} R = \log R$. Further we assume that throughout the present paper p, q and m always denote positive integers. Also throughout the paper an entire function $f(z)$ of n-complex variables will stand for an entire function $f(z)$ for any bounded complete n-circular domain D with center at origin in $\mathbb{C}^n$. Considering this Biswas et al. [3] introduced the definitions of (p,q)-φ Gol'dberg order and (p,q)-φ Gol'dberg lower order of an entire function $f(z)$ of n-complex variables which are as follows:

Definition 1.1. [3] Let $\varphi(R) : [0, +\infty) \to (0, +\infty)$ be a non-decreasing unbounded function. Then the (p,q)-φ Gol'dberg order $\rho_D^{(p,q)}(f,\varphi)$ and (p,q)-φ Gol'dberg lower order $\lambda_D^{(p,q)}(f,\varphi)$ of an entire function $f(z)$ of n-complex variables are defined as

$$\begin{matrix} \rho_D^{(p,q)}(f,\varphi) \\ \lambda_D^{(p,q)}(f,\varphi) \end{matrix} = \lim_{r\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_{f,D}(R)}{\log^{[q]} \varphi(R)}.$$

However, an entire function $f(z)$ for which $\rho_D^{(p,q)}(f,\varphi)$ and $\lambda_D^{(p,q)}(f,\varphi)$ are the same is called a function of regular (p,q)-φ Gol'dberg growth. Otherwise, $f(z)$ is said to be irregular (p,q)-φ Gol'dberg growth.

Remark 1.2. [3] If $\lim_{R\to+\infty} \frac{\log^{[q]} \varphi(\alpha R)}{\log^{[q]} \varphi(R)} = 1$ for all $\alpha > 0$ where $\varphi(R)$: $[0, +\infty) \to (0, +\infty)$ is any non-decreasing unbounded function, then $\rho_D^{(p,q)}(f,\varphi)$ and $\lambda_D^{(p,q)}(f,\varphi)$ are independent of the choice of the domain D.

However for any two entire functions $f(z)$ and $g(z)$ of n-complex variables, Mondal et al. [6] introduced the concept of relative Gol'dberg order of $f(z)$ with respect to $g(z)$. In the case of relative Gol'dberg order, it therefore seems reasonable to define suitably the (p,q)-φ relative Gol'dberg order. With this in view Biswas et al. [3] introduced the following definitions:

Definition 1.3. [3] Let $\varphi(R) : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. Also let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables. The (p,q)-φ relative Gol'dberg order and the (p,q)-φ relative Gol'dberg lower order of of $f(z)$ with respect to $g(z)$ are defined as

$$\begin{matrix} \rho_{g,D}^{(p,q)}(f,\varphi) \\ \lambda_{g,D}^{(p,q)}(f,\varphi) \end{matrix} = \lim_{R\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p]} M_{g,D}^{-1}(M_{f,D}(R))}{\log^{[q]} \varphi(R)}.$$

Further an entire function $f(z)$ of n-complex variables for which $\rho_{g,D}^{(p,q)}(f,\varphi)$ and $\lambda_{g,D}^{(p,q)}(f,\varphi)$ are the same is called a function of regular (p,q)-φ relative Gol'dberg growth with respect to an entire function $g(z)$ of n-complex variables. Otherwise, $f(z)$ is said to be irregular (p,q)-φ relative Gol'dberg growth.with respect to $g(z)$.

However in the present paper, we assume that the nondecreasing unbounded function $\varphi(R) : [0,+\infty) \to (0,+\infty)$ always satisfies $\lim\limits_{R\to+\infty} \frac{\log^{[q]}\varphi(\alpha R)}{\log^{[q]}\varphi(R)} = 1$ for all $\alpha > 0$. Since, Biswas et al. [3] have already shown that $\rho_{g,D}^{(p,q)}(f,\varphi)$ and $\lambda_{g,D}^{(p,q)}(f,\varphi)$ are independent of the choice of the domain D when $\varphi(R) : [0,+\infty) \to (0,+\infty)$ is a nondecreasing unbounded function and satisfies $\lim\limits_{R\to+\infty} \frac{\log^{[q]}\varphi(\alpha R)}{\log^{[q]}\varphi(R)} = 1$ for all $\alpha > 0$, so here we shall always use the notations $\rho_g^{(p,q)}(f,\varphi)$ and $\lambda_g^{(p,q)}(f,\varphi)$ instead of $\rho_{g,D}^{(p,q)}(f,\varphi)$ and $\lambda_{g,D}^{(p,q)}(f,\varphi)$ respectively.

Now, for the development of such growth indicators, one may introduce (p,q)-φ relative Gol'dberg type $\sigma_{g,D}^{(p,q)}(f,\varphi)$, (p,q)-φ relative Gol'dberg lower type $\overline{\sigma}_{g,D}^{(p,q)}(f,\varphi)$, (p,q)-φ relative Gol'dberg weak type $\tau_{g,D}^{(p,q)}(f,\varphi)$ and another growth indicator $\overline{\tau}_{g,D}^{(p,q)}(f,\varphi)$ in the following way:

Definition 1.4. Let $\varphi(R) : [0,+\infty) \to (0,+\infty)$ be a non-decreasing unbounded function. Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables such that $0 < \rho_g^{(p,q)}(f,\varphi) < +\infty$. Then the (p,q)-φ relative Gol'dberg type $\sigma_{g,D}^{(p,q)}(f,\varphi)$ and the (p,q)-φ relative Gol'dberg lower type $\overline{\sigma}_{g,D}^{(p,q)}(f,\varphi)$ of $f(z)$ with respect to $g(z)$ are defined as:

$$\begin{matrix} \sigma_{g,D}^{(p,q)}(f,\varphi) \\ \overline{\sigma}_{g,D}^{(p,q)}(f,\varphi) \end{matrix} = \lim_{R\to+\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log^{[p-1]} M_{g,D}^{-1}(M_{f,D}(R))}{(\log^{[q-1]} \varphi(R))^{\rho_g^{(p,q)}(f,\varphi)}}.$$

Definition 1.5. Let $\varphi(R):[0,+\infty)\to(0,+\infty)$ be a non-decreasing unbounded function. Let $f(z)$ and $g(z)$ be any two entire functions of n-complex variables such that $0<\lambda_g^{(p,q)}(f,\varphi)<+\infty$. Then the (p,q)-φ relative Gol'dberg weak type $\tau_{g,D}^{(p,q)}(f,\varphi)$ and the growth indicator $\overline{\tau}_{g,D}^{(p,q)}(f,\varphi)$ of $f(z)$ with respect to $g(z)$ are defined as:

$$\begin{matrix}\overline{\tau}_{g,D}^{(p,q)}(f,\varphi)\\ \tau_{g,D}^{(p,q)}(f,\varphi)\end{matrix}=\lim_{R\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log^{[p-1]}M_{g,D}^{-1}(M_{f,D}(R))}{(\log^{[q-1]}\varphi(R))^{\lambda_g^{(p,q)}(f,\varphi)}}.$$

As Gol'dberg has shown that (see [4,5]) Gol'dberg type depends on the domain D, all the growth indicators defined in Definition 1.4 and Definition 1.5 also depend on D.

In this connection, we finally remind the following definitions from which are needed in the sequel.

Definition 1.6. [7] A non-constant entire function $f(z)$ of n-complex variables is said to have Property (G), if for any $\delta>1$, $(M_{f,D}(R))^2\le M_{f,D}(R^\delta)$.

Definition 1.7. A pair of entire functions $f(z)$ and $g(z)$ of n-complex variables are mutually said to have Property (X) if for all sufficiently large values of R, both $M_{f\cdot g,D}(R)>M_{f,D}(R)$ and $M_{f\cdot g,D}(R)>M_{g,D}(R)$ hold simultaneously.

During the past decades, several authors {cf. [1] to [7]} made closed investigations on the growth properties of entire functions of several complex variables using different growth indicators such as relative gol'dberg order, relative (p,q)-th Gol'dberg order, (p,q)-φ relative Gol'dberg order, (p,q)-φ relative Gol'dberg lower order, (p,q)-φ relative Gol'dberg type etc. In the present paper our aim is to investigate several basic properties of (p,q)-φ relative Gol'dberg type and (p,q)-φ relative Gol'dberg weak type of entire functions of several complex variables with respect to another one under somewhat different conditions.

2. Existing Results

Theorem 2.1. [3] *Let us consider $f_1(z)$, $f_2(z)$ and $g_1(z)$ are any three entire functions of n-complex variables. Also let at least $f_1(z)$ or $f_2(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_1(z)$. Then* $\lambda_{g_1}^{(p,q)}(f_1\pm f_2,\varphi)\le\max\{\lambda_{g_1}^{(p,q)}(f_1,\varphi),\lambda_{g_1}^{(p,q)}(f_2,\varphi)\}$. *The equality holds* *when any one of* $\lambda_{g_1}^{(p,q)}(f_i,\varphi) > \lambda_{g_1}^{(p,q)}(f_j,\varphi)$ *hold with at least* $f_j(z)$ *is of regular* (p,q)-φ *relative Gol'dberg growth with respect to* $g_1(z)$ *where* $i,j = 1,2$ *and* $i \neq j$.

THEOREM 2.2. [3] *Let us consider* $f_1(z)$, $f_2(z)$ *and* $g_1(z)$ *are any three entire functions of* n*-complex variables such that* $\rho_{g_1}^{(p,q)}(f_1,\varphi)$ *and* $\rho_{g_1}^{(p,q)}(f_2,\varphi)$ *exists. Then* $\rho_{g_1}^{(p,q)}(f_1 \pm f_2,\varphi) \leq \max\{\rho_{g_1}^{(p,q)}(f_1,\varphi), \rho_{g_1}^{(p,q)}(f_2,\varphi)\}$. *The equality holds when* $\rho_{g_1}^{(p,q)}(f_1,\varphi) \neq \rho_{g_1}^{(p,q)}(f_2,\varphi)$.

THEOREM 2.3. [3] *Let* $f_1(z)$, $g_1(z)$ *and* $g_2(z)$ *be any three entire functions of* n*-complex variables such that* $\lambda_{g_1}^{(p,q)}(f_1,\varphi)$ *and* $\lambda_{g_2}^{(p,q)}(f_1,\varphi)$ *exists. Then* $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) \geq \min\{\lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_2}^{(p,q)}(f_1,\varphi)\}$. *The equality holds when* $\lambda_{g_1}^{(p,q)}(f_1,\varphi) \neq \lambda_{g_2}^{(p,q)}(f_1,\varphi)$.

THEOREM 2.4. [3] *Let* $f_1(z)$, $g_1(z)$ *and* $g_2(z)$ *be any three entire functions of* n*-complex variables. Also let* $f_1(z)$ *is of regular* (p,q)-φ *relative Gol'dberg growth with respect to at least any one of* $g_1(z)$ *or* $g_2(z)$. *Then* $\rho_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) \geq \min\{\rho_{g_1}^{(p,q)}(f_1,\varphi), \rho_{g_2}^{(p,q)}(f_1,\varphi)\}$. *The equality holds when any one of* $\rho_{g_i}^{(p,q)}(f_1,\varphi) < \rho_{g_j}^{(p,q)}(f_1,\varphi)$ *hold with at least* $f_1(z)$ *is of regular* (p,q)-φ *relative Gol'dberg growth with respect to* $g_j(z)$ *where* $i,j = 1,2$ *and* $i \neq j$.

THEOREM 2.5. [3] *Let* $f_1(z)$, $g_1(z)$ *and* $g_2(z)$ *be any three entire functions of* n*-complex variables. Then*

$$\begin{aligned}\rho_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2,\varphi) &\leq \max\left[\min\{\rho_{g_1}^{(p,q)}(f_1,\varphi), \rho_{g_2}^{(p,q)}(f_1,\varphi)\}\right. \\ &\qquad \left., \min\{\rho_{g_1}^{(p,q)}(f_2,\varphi), \rho_{g_2}^{(p,q)}(f_2,\varphi)\}\right]\end{aligned}$$

when the following two conditions holds:
(i) $\rho_{g_i}^{(p,q)}(f_1,\varphi) < \rho_{g_j}^{(p,q)}(f_1,\varphi)$ *with at least* $f_1(z)$ *is of regular* (p,q)-φ *relative Gol'dberg growth with respect to* $g_j(z)$ *for* $i = 1, 2$, $j = 1,2$ *and* $i \neq j$; *and*
(ii) $\rho_{g_i}^{(p,q)}(f_2,\varphi) < \rho_{g_j}^{(p,q)}(f_2,\varphi)$ *with at least* $f_2(z)$ *is of regular* (p,q)-φ *relative Gol'dberg growth with respect to* $g_j(z)$ *for* $i = 1, 2$, $j = 1,2$ *and* $i \neq j$.
The equality holds when any one of $\rho_{g_1}^{(p,q)}(f_i,\varphi) < \rho_{g_1}^{(p,q)}(f_j,\varphi)$ *and any one of* $\rho_{g_2}^{(p,q)}(f_i,\varphi) < \rho_{g_2}^{(p,q)}(f_j,\varphi)$ *hold simultaneously for* $i = 1,2$; $j = 1,2$ *and* $i \neq j$.

Theorem 2.6. [3] *Let $f_1(z)$, $g_1(z)$ and $g_2(z)$ be any three entire functions of n-complex variables. Then*

$$\begin{aligned}\lambda_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2, \varphi) \geq \min\big[&\max\{\lambda_{g_1}^{(p,q)}(f_1, \varphi), \lambda_{g_1}^{(p,q)}(f_2, \varphi)\}, \\ &\max\{\lambda_{g_2}^{(p,q)}(f_1, \varphi), \lambda_{g_2}^{(p,q)}(f_2, \varphi)\}\big]\end{aligned}$$

when the following two conditions holds:
(i) $\lambda_{g_1}^{(p,q)}(f_i, \varphi) > \lambda_{g_1}^{(p,q)}(f_j, \varphi)$ with at least $f_j(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_1(z)$ for $i = 1, 2$, $j = 1, 2$ and $i \neq j$; and
(ii) $\lambda_{g_2}^{(p,q)}(f_i, \varphi) > \lambda_{g_2}^{(p,q)}(f_j, \varphi)$ with at least $f_j(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_2(z)$ for $i = 1, 2$, $j = 1, 2$ and $i \neq j$.
The equality holds when any one of $\lambda_{g_i}^{(p,q)}(f_1, \varphi) < \lambda_{g_j}^{(p,q)}(f_1, \varphi)$ and any one of $\lambda_{g_i}^{(p,q)}(f_2, \varphi) < \lambda_{g_j}^{(p,q)}(f_2, \varphi)$ hold simultaneously for $i = 1, 2$; $j = 1, 2$ and $i \neq j$.

Theorem 2.7. [3] *Let us consider $f_1(z)$, $f_2(z)$ and $g_1(z)$ are any three entire functions of n-complex variables. Also let at least $f_1(z)$ or $f_2(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_1(z)$ and $g_1(z)$ satisfy the Property (G). Then $\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2, \varphi) \leq$ $\max\{\lambda_{g_1}^{(p,q)}(f_1, \varphi), \lambda_{g_1}^{(p,q)}(f_2, \varphi)\}$. The equality holds when $f_1(z)$ and $f_2(z)$ satisfy Property (X).*

Theorem 2.8. [3] *Let us consider $f_1(z)$, $f_2(z)$ and $g_1(z)$ are any three entire functions of n-complex variables such that $\rho_{g_1}^{(p,q)}(f_1, \varphi)$ and $\rho_{g_1}^{(p,q)}(f_2, \varphi)$ exists and $g_1(z)$ satisfy the Property (G). Then $\rho_{g_1}^{(p,q)}(f_1 \cdot f_2, \varphi)$ $\leq \max\{\rho_{g_1}^{(p,q)}(f_1, \varphi), \rho_{g_1}^{(p,q)}(f_2, \varphi)\}$. The equality holds when f_1 and f_2 satisfy Property (X).*

Theorem 2.9. [3] *Let $f_1(z)$, $g_1(z)$ and $g_2(z)$ be any three entire functions of n-complex variables such that $\lambda_{g_1}^{(p,q)}(f_1, \varphi)$ and $\lambda_{g_2}^{(p,q)}(f_1, \varphi)$ exists and $g_1 \cdot g_2(z)$ satisfy the Property (G). Then $\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1, \varphi) \geq \min$ $\{\lambda_{g_1}^{(p,q)}(f_1, \varphi), \lambda_{g_2}^{(p,q)}(f_1, \varphi)\}$. The equality holds when $g_1(z)$ and $g_2(z)$ satisfy Property (X).*

Theorem 2.10. [3] *Let $f_1(z)$, $g_1(z)$ and $g_2(z)$ be any three entire functions of n-complex variables. Also let $f_1(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to at least any one of $g_1(z)$ or $g_2(z)$ and $g_1 \cdot g_2(z)$ satisfy the Property (G). Then $\rho_{g_1 \cdot g_2}^{(p,q)}(f_1, \varphi) \geq \min$*

$\{\rho_{g_1}^{(p,q)}(f_1,\varphi), \rho_{g_2}^{(p,q)}(f_1,\varphi)\}$. The equality holds when $g_1(z)$ and $g_2(z)$ satisfy Property (X).

Theorem 2.11. [3] *Let $f_1(z)$, $f_2(z)$, $g_1(z)$ and $g_2(z)$ be any four entire functions of n- complex variables. Also let $g_1 \cdot g_2(z)$ be satisfy the Property (G). Then*

$$\begin{aligned}\rho_{g_1 \cdot g_2}^{(p,q)}(f_1 \cdot f_2,\varphi) &= \max\left[\min\{\rho_{g_1}^{(p,q)}(f_1,\varphi), \rho_{g_1}^{(p,q)}(f_2,\varphi)\}\right., \\ &\quad \left.\min\{\rho_{g_2}^{(p,q)}(f_1,\varphi), \rho_{g2}^{(p,q)}(f_2,\varphi)\}\right]\end{aligned}$$

when the following two conditions hold:

(i) $f_1(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to at least any one of $g_1(z)$ or $g_2(z)$;

(ii) $f_2(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to at least any one of $g_1(z)$ or$g_2(z)$;

(iii) $f_1(z)$ and $f_2(z)$ satisfy Property (X); and

(iv) $g_1(z)$ and $g_2(z)$ satisfy Property (X).

Theorem 2.12. *[3]Let $f_1(z)$, $f_2(z)$, $g_1(z)$ and $g_2(z)$ be any four entire functions of n- complex variables. Also let $g_1 \cdot g_2(z), g_1(z)$ and $g_2(z)$ be satisfy the Property (G). Then*

$$\begin{aligned}\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1 \cdot f_2,\varphi) &\leq \min\left[\max\{\lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_1}^{(p,q)}(f_2,\varphi)\}\right., \\ &\quad \left.\max\{\lambda_{g_2}^{(p,q)}(f_1,\varphi), \lambda_{g2}^{(p,q)}(f_2,\varphi)\}\right]\end{aligned}$$

when the following two conditions hold:

(i) At least $f_1(z)$ or $f_2(z)$ is of regular(p,q)-φ relative Gol'dberg growth with respect to $g_1(z)$;

(ii) At least $f_1(z)$ or $f_2(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_2(z)$;

(iii) $f_1(z)$ and $f_2(z)$ satisfy Property (X); and

(iv) $g_1(z)$ and $g_2(z)$ satisfy Property (X)

3. Main Results

In this section we state the main results of the paper.

Theorem 3.1. *Let $f_1(z)$, $f_2(z)$, $g_1(z)$ and $g_2(z)$ be any four entire functions of n- complex variables and D be a bounded complete n-circular domain with center at origin in $\mathbb{C}^n$. Also let $\rho_{g_1}^{(p,q)}(f_1,\varphi)$,*

$\rho_{g_1}^{(p,q)}(f_2,\varphi)$, $\rho_{g_2}^{(p,q)}(f_1,\varphi)$ and $\rho_{g_2}^{(p,q)}(f_2,\varphi)$ are all non zero and finite where p and q are any two positive integers.
(A) If $\rho_{g_1}^{(p,q)}(f_i,\varphi) > \rho_{g_1}^{(p,q)}(f_j,\varphi)$ for $i = j = 1,2$ and $i \neq j$, then

$$\sigma_{g_1,D}^{(p,q)}(f_1 \pm f_2,\varphi) = \sigma_{g_1,D}^{(p,q)}(f_i,\varphi) \text{ and } \overline{\sigma}_{g_1,D}^{(p,q)}(f_1 \pm f_2,\varphi) = \overline{\sigma}_{g_1,D}^{(p,q)}(f_i,\varphi)\ .$$

(B) If $\rho_{g_i}^{(p,q)}(f_1,\varphi) < \rho_{g_j}^{(p,q)}(f_1,\varphi)$ with at least $f_1(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_j(z)$ for $i = j = 1,2$ and $i \neq j$, then

$$\sigma_{g_1 \pm g_2,D}^{(p,q)}(f_1,\varphi) = \sigma_{g_i,D}^{(p,q)}(f_1,\varphi) \text{ and } \overline{\sigma}_{g_1 \pm g_2,D}^{(p,q)}(f_1,\varphi) = \overline{\sigma}_{g_i,D}^{(p,q)}(f_1,\varphi)\ .$$

(C)Assume the functions $f_1(z)$, $f_2(z)$, $g_1(z)$ and $g_2(z)$ satisfy the following conditions:
(i) $\rho_{g_i}^{(p,q)}(f_1,\varphi) < \rho_{g_j}^{(p,q)}(f_1,\varphi)$ with at least $f_1(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_j(z)$ for $i = 1, 2$, $j = 1,2$ and $i \neq j$;
(ii) $\rho_{g_i}^{(p,q)}(f_2,\varphi) < \rho_{g_j}^{(p,q)}(f_2,\varphi)$ with at least $f_2(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_j(z)$ for $i = 1, 2$, $j = 1,2$ and $i \neq j$;
(iii) $\rho_{g_1}^{(p,q)}(f_i,\varphi) > \rho_{g_1}^{(p,q)}(f_j,\varphi)$ and $\rho_{g_2}^{(p,q)}(f_i,\varphi) > \rho_{g_2}^{(p,q)}(f_j,\varphi)$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$;
(iv)$\rho_{g_m}^{(p,q)}(f_l,\varphi) = \max[\min\{\rho_{g_1}^{(p,q)}(f_1,\varphi), \rho_{g_2}^{(p,q)}(f_1,\varphi)\}, \min\{\rho_{g_1}^{(p,q)}(f_2,\varphi), \rho_{g_2}^{(p,q)}(f_2,\varphi)\}] \mid l = m = 1,2$;then we have

$$\sigma_{g_1 \pm g_2,D}^{(p,q)}(f_1 \pm f_2,\varphi) = \sigma_{g_m,D}^{(p,q)}(f_l,\varphi) \text{ and } \sigma_{g_1 \pm g_2,D}^{(p,q)}(f_1 \pm f_2,\varphi) = \overline{\sigma}_{g_m,D}^{(p,q)}(f_l,\varphi)\ .$$

Proof. We obtain for all sufficiently large values of R that

$$\begin{aligned} (1) \qquad & M_{f_k,D}(R) \\ \leq\ & M_{g_l,D}(\exp^{[p-1]}\{(\sigma_{g_l,D}^{(p,q)}(f_k,\varphi) + \varepsilon)[\log^{[q-1]}\varphi(R)]^{\rho_{g_l}^{(p,q)}(f_k,\varphi)}\}), \end{aligned}$$

$$\begin{aligned} (2) \qquad & M_{f_k,D}(R) \\ \geq\ & M_{g_l,D}(\exp^{[p-1]}\{(\overline{\sigma}_{g_l,D}^{(p,q)}(f_k,\varphi) - \varepsilon)[\log^{[q-1]}\varphi(R)]^{\rho_{g_l}^{(p,q)}(f_k,\varphi)}\}) \end{aligned}$$

and for a sequence of values of R tending to infinity, we obtain that

$$\begin{aligned} (3) \qquad & M_{f_k,D}(R) \\ \geq\ & M_{g_l,D}(\exp^{[p-1]}\{(\sigma_{g_l,D}^{(p,q)}(f_k,\varphi) - \varepsilon)[\log^{[q-1]}\varphi(R)]^{\rho_{g_l}^{(p,q)}(f_k,\varphi)}\}) \end{aligned}$$

and

$$(4)\qquad M_{f_k,D}(R) \leq M_{g_l,D}(\exp^{[p-1]}\{(\overline{\sigma}^{(p,q)}_{g_l,D}(f_k,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R)]^{\rho^{(p,q)}_{g_l}(f_k,\varphi)}\}),$$

where $\varepsilon > 0$ is any arbitrary positive number $k = 1,\ 2$ and $l = 1, 2$.

Case I. Suppose that $\rho^{(p,q)}_{g_1}(f_1,\varphi) > \rho^{(p,q)}_{g_1}(f_2,\varphi)$ hold. Then for arbitrary $\varepsilon(> 0)$ and for all sufficiently large values of R,we get in view of (1) that

$M_{f_1\pm f_2,D}(R) \leq M_{f_1,D}(R) + M_{f_2,D}(R)$

$$(5)\quad \leq M_{g_1,D}(\exp^{[p-1]}\{(\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R)]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\})(1+A)$$

where $A = \frac{M_{g_1,D}(\exp^{[p-1]}\{(\sigma^{(p,q)}_{g_1,D}(f_2,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R)]^{\rho^{(p,q)}_{g_1}(f_2,\varphi)}\})}{M_{g_1,D}(\exp^{[p-1]}\{(\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R)]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\})}$ and in view of $\rho^{(p,q)}_{g_1}(f_1,\varphi) > \rho^{(p,q)}_{g_1}(f_2,\varphi)$, and for all sufficiently large values of R, we can make the term A sufficiently small. Hence for any $\alpha = 1+\varepsilon_1$, it follows from (5) for all sufficiently large values of R that

$$M_{f_1\pm f_2,D}(R) \leq M_{g_1,D}(\exp^{[p-1]}\{(\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R)]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\})(1+\varepsilon_1)$$

$$i.e.,\ M_{f_1\pm f_2,D}(R) \leq M_{g_1,D}(\exp^{[p-1]}\{(\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R)]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\})\cdot\alpha.$$

Since in view of Theorem 2.2, $\rho^{(p,q)}_{g_1}(f_1\pm f_2,\varphi) \leq \max\{\rho^{(p,q)}_{g_1}(f_1,\varphi), \rho^{(p,q)}_{g_1}(f_2,\varphi)\} = \rho^{(p,q)}_{g_1}(f_1,\varphi)$, letting $\alpha \to 1+$,and for all sufficiently large values of R, we get

$$\limsup_{R\to+\infty}\frac{\log^{[p-1]} M^{-1}_{g_1,D}(M_{f_1\pm f_2,D}(R))}{[\log^{[q-1]}\varphi(R)]^{\rho^{(p,q)}_{g_1}(f_1\pm f_2,\varphi)}} \leq \sigma^{(p,q)}_{g_1,D}(f_1,\varphi)$$

$$(6)\qquad i.e., \sigma^{(p,q)}_{g_1,D}(f_1\pm f_2,\varphi) \leq \sigma^{(p,q)}_{g_1,D}(f_1,\varphi)\ .$$

Next we take $f(z) = f_1(z)\pm f_2(z)$. Since $\rho^{(p,q)}_{g_1}(f_1,\varphi) > \rho^{(p,q)}_{g_1}(f_2,\varphi)$ hold. Then $\sigma^{(p,q)}_{g_1,D}(f,\varphi) = \sigma^{(p,q)}_{g_1,D}(f_1\pm f_2,\varphi) \leq \sigma^{(p,q)}_{g_1,D}(f_1,\varphi)$. Further, let $f_1(z) = (f(z)\pm f_2(z))$. Now, in view of Theorem 2.2 and $\rho^{(p,q)}_{g_1}(f_1,\varphi) > \rho^{(p,q)}_{g_1}(f_2,\varphi)$, we obtain that $\rho^{(p,q)}_{g_1}(f,\varphi) > \rho^{(p,q)}_{g_1}(f_2,\varphi)$ holds. Hence in view of (6) $\sigma_{g_1,D}^{(p,q)}(f_1,\varphi) \le \sigma_{g_1,D}^{(p,q)}(f,\varphi) = \sigma_{g_1,D}^{(p,q)}(f_1 \pm f_2,\varphi)$. Therefore $\sigma_{g_1,D}^{(p,q)}(f,\varphi) = \sigma_{g_1,D}^{(p,q)}(f_1,\varphi) \Rightarrow \sigma_{g_1,D}^{(p,q)}(f_1 \pm f_2,\varphi) = \sigma_{g_1,D}^{(p,q)}(f_1,\varphi)$.

Similarly, if we consider $\rho_{g_1}^{(p,q)}(f_1,\varphi) < \rho_{g_1}^{(p,q)}(f_2,\varphi)$, then one can easily verify that $\sigma_{g_1,D}^{(p,q)}(f_1 \pm f_2,\varphi) = \sigma_{g_1,D}^{(p,q)}(f_2,\varphi)$.

Case II. Let us consider that $\rho_{g_1}^{(p,q)}(f_1,\varphi) > \rho_{g_1}^{(p,q)}(f_2,\varphi)$ hold. Also let $\varepsilon(> 0)$ are arbitrary. Then we get from (1) and (4) for a sequence of values R_n tending to infinity that

$$M_{f_1 \pm f_2,D}(R_n) \le M_{f_1,D}(R_n) + M_{f_2,D}(R_n)$$

$$\le M_{g_1,D}(\exp^{[p-1]}\{(\overline{\sigma}_{g_1,D}^{(p,q)}(f_1,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R_n)]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\})(1+B) \tag{7}$$

where $B = \frac{M_{g_1,D}(\exp^{[p-1]}\{(\sigma_{g_1,D}^{(p,q)}(f_2,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R_n)]^{\rho_{g_1}^{(p,q)}(f_2,\varphi)}\})}{M_{g_1,D}(\exp^{[p-1]}\{(\overline{\sigma}_{g_1,D}^{(p,q)}(f_1,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R_n)]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\})}$ and in view of $\rho_{g_1}^{(p,q)}(f_1,\varphi) > \rho_{g_1}^{(p,q)}(f_2,\varphi)$, we can make the term B sufficiently small by taking n sufficiently large and therefore using the similar technique for as executed in the proof of Case I we get from (7) that $\overline{\sigma}_{g_1,D}^{(p,q)}(f_1 \pm f_2,\varphi) = \overline{\sigma}_{g_1,D}^{(p,q)}(f_1,\varphi)$ when $\rho_{g_1}^{(p,q)}(f_1,\varphi) > \rho_{g_1}^{(p,q)}(f_2,\varphi)$ hold. Likewise, if we consider $\rho_{g_1}^{(p,q)}(f_1,\varphi) < \rho_{g_1}^{(p,q)}(f_2,\varphi)$, then one can easily verify that $\overline{\sigma}_{g_1,D}^{(p,q)}(f_1 \pm f_2,\varphi) = \overline{\sigma}_{g_1,D}^{(p,q)}(f_2,\varphi)$.

Thus combining Case I and Case II, we obtain the first part of the theorem.

Case III. Let us consider that $\rho_{g_1}^{(p,q)}(f_1,\varphi) < \rho_{g_2}^{(p,q)}(f_1,\varphi)$ with at least $f_1(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_2(z)$. Therefore we obtain from (2) and (3) for a sequence of values R_n tending to infinity that

$$M_{g_1 \pm g_2,D}(R_n) \le M_{g_1,D}(R_n) + M_{g_2,D}(R_n),$$

$$\begin{aligned}
\Longrightarrow \quad & M_{g_1 \pm g_2,D}(\exp^{[p-1]}\{(\sigma_{g_1,D}^{(p,q)}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R_n)]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\}) \\
\le \quad & M_{g_1,D}(\exp^{[p-1]}\{(\sigma_{g_1,D}^{(p,q)}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R_n)]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\}) \\
& +M_{g_2,D}(\exp^{[p-1]}\{(\sigma_{g_1,D}^{(p,q)}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R_n)]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\})
\end{aligned}$$

$$(8)\qquad i.e., M_{g_1\pm g_2,D}(\exp^{[p-1]}\{(\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R_n)]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\})$$
$$\leq \ (1+C)M_{f_1,D}(R_n)$$

where $C=\frac{M_{g_2,D}(\exp^{[p-1]}\{(\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R_n)]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\})}{M_{g_2,D}(\exp^{[p-1]}\{(\overline{\sigma}^{(p,q)}_{g_2,D}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R_n)]^{\rho^{(p,q)}_{g_2}(f_1,\varphi)}\})}$.
Since $\rho^{(p,q)}_{g_1}(f_1,\varphi)<\rho^{(p,q)}_{g_2}(f_1,\varphi)$, we can make the term C sufficiently small ($<\varepsilon_1$ for $\varepsilon_1>0$) by taking n sufficiently large.Therefore for any $\alpha=1+\varepsilon_1$, we obtain in view of $C<\varepsilon_1$ from (8) and Theorem 2.4, for a sequence of values of R_n tending to infinity that

$$M_{g_1\pm g_2,D}(\exp^{[p-1]}\{(\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R)]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\})<\alpha M_{f_1,D}(R_n)$$

Therefore, making $\alpha\to 1+$, we obtain from above for a sequence of values R_n tending to infinity that

$$(\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R)]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}<\log^{[p-1]}M^{-1}_{g_1\pm g_2,D}M_{f_1,D}(R_n)\ .$$

Since $\varepsilon>0$ is arbitrary, we find that

$$(9)\qquad \sigma^{(p,q)}_{g_1\pm g_2,D}(f_1,\varphi)\geq\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)\ .$$

Now we may consider that $g(z)=g_1(z)\pm g_2(z)$. Also $\rho^{(p,q)}_{g_1}(f_1,\varphi)<\rho^{(p,q)}_{g_2}(f_1,\varphi)$ and at least $f_1(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_2(z)$. Then $\sigma^{(p,q)}_{g,D}(f_1,\varphi)=\sigma^{(p,q)}_{g_1\pm g_2,D}(f_1,\varphi)\geq\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)$. Further let $g_1(z)=(g(z)\pm g_2(z))$. Therefore in view of Theorem 2.4 and $\rho^{(p,q)}_{g_1}(f_1,\varphi)<\rho^{(p,q)}_{g_2}(f_1,\varphi)$, we obtain that $\rho^{(p.q)}_{g}(f_1,\varphi)<\rho^{(p,q)}_{g_2}(f_1,\varphi)$ as at least $f_1(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_2(z)$. Hence in view of (9), $\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)\geq\sigma^{(p,q)}_{g,D}(f_1,\varphi)=\sigma^{(p,q)}_{g_1\pm g_2,D}(f_1,\varphi)$. Therefore $\sigma^{(p,q)}_{g,D}(f_1,\varphi)=\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)\Rightarrow\sigma^{(p,q)}_{g_1\pm g_2,D}(f_1,\varphi)=\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)$.

Similarly if we consider $\rho^{(p,q)}_{g_1}(f_1,\varphi)>\rho^{(p,q)}_{g_2}(f_1,\varphi)$ with at least $f_1(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_1(z)$, then $\sigma^{(p,q)}_{g_1\pm g_2,D}(f_1,\varphi)=\sigma^{(p,q)}_{g_2,D}(f_1,\varphi)$.

Case IV. In this case suppose that $\rho^{(p,q)}_{g_1}(f_1,\varphi)<\rho^{(p,q)}_{g_2}(f_1,\varphi)$ with at least $f_1(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_2(z)$. Therefore from (2), we get for all sufficiently large values of R that

$$M_{g_1 \pm g_2, D}(R) \leq M_{g_1, D}(R) + M_{g_2, D}(R),$$

$$\begin{aligned}
\Longrightarrow \quad & M_{g_1 \pm g_2, D}(\exp^{[p-1]}\{(\overline{\sigma}^{(p,q)}_{g_1,D}(f_1,\varphi) - \varepsilon)[\log^{[q-1]}\varphi(R)]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\}) \\
\leq \quad & M_{g_1, D}(\exp^{[p-1]}\{(\overline{\sigma}^{(p,q)}_{g_1,D}(f_1,\varphi) - \varepsilon)[\log^{[q-1]}\varphi(R)]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\}) \\
& + M_{g_2, D}(\exp^{[p-1]}\{(\overline{\sigma}^{(p,q)}_{g_1,D}(f_1,\varphi) - \varepsilon)[\log^{[q-1]}\varphi(R)]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\})
\end{aligned}$$

$$\begin{aligned}
(10) \quad & i.e., M_{g_1 \pm g_2, D}(\exp^{[p-1]}\{(\overline{\sigma}^{(p,q)}_{g_1,D}(f_1,\varphi) - \varepsilon)[\log^{[q-1]}\varphi(R)]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\}) \\
\leq \quad & (1 + D) M_{f_1, D}(R)
\end{aligned}$$

where $D = \frac{M_{g_2,D}(\exp^{[p-1]}\{(\overline{\sigma}^{(p,q)}_{g_1,D}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R)]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\})}{M_{g_2,D}(\exp^{[p-1]}\{(\overline{\sigma}^{(p,q)}_{g_2,D}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R)]^{\rho^{(p,q)}_{g_2}(f_1,\varphi)}\})}$ and now in view of $\rho^{(p,q)}_{g_1}(f_1,\varphi) < \rho^{(p,q)}_{g_2}(f_1,\varphi)$, we can make the term D sufficiently small by taking R sufficiently large and therefore using the similar technique for as executed in the proof of Case III we get from (10) that $\overline{\sigma}^{(p,q)}_{g_1 \pm g_2, D}(f_1,\varphi) = \overline{\sigma}^{(p,q)}_{g_1,D}(f_1,\varphi)$ where $\rho^{(p,q)}_{g_1}(f_1,\varphi) < \rho^{(p,q)}_{g_2}(f_1,\varphi)$ and at least $f_1(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_2(z)$.

Similarly, if we consider $\rho^{(p,q)}_{g_1}(f_1,\varphi) > \rho^{(p,q)}_{g_2}(f_1,\varphi)$ with at least $f_1(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_1(z)$, then $\overline{\sigma}^{(p,q)}_{g_1 \pm g_2, D}(f_1,\varphi) = \overline{\sigma}^{(p,q)}_{g_2,D}(f_1,\varphi)$.

Thus combining Case III and Case IV, we obtain the second part of the theorem.

The third part of the theorem is a natural consequence of Theorem 2.5 and the first part and second part of the theorem. Hence its proof is omitted □

Theorem 3.2. *Let $f_1(z)$, $f_2(z)$, $g_1(z)$ and $g_2(z)$ be any four entire functions of n- complex variables and D be a bounded complete n-circular domain with center at origin in $\mathbb{C}^n$. Also let $\lambda^{(p,q)}_{g_1}(f_1,\varphi)$, $\lambda^{(p,q)}_{g_1}(f_2,\varphi)$, $\lambda^{(p,q)}_{g_2}(f_1,\varphi)$ and $\lambda^{(p,q)}_{g_2}(f_2,\varphi)$ are all non zero and finite where p and q are any two positive integers.*
(A) If $\lambda^{(p,q)}_{g_1}(f_i,\varphi) > \lambda^{(p,q)}_{g_1}(f_j,\varphi)$ with at least $f_j(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_1(z)$ for $i = j = 1, 2$ and $i \neq j$, then

$$\tau^{(p,q)}_{g_1,D}(f_1 \pm f_2,\varphi) = \tau^{(p,q)}_{g_1,D}(f_i,\varphi) \text{ and } \overline{\tau}^{(p,q)}_{g_1,D}(f_1 \pm f_2,\varphi) = \overline{\tau}^{(p,q)}_{g_1,D}(f_i,\varphi)\ .$$

(B) If $\lambda_{g_i}^{(p,q)}(f_1,\varphi) < \lambda_{g_j}^{(p,q)}(f_1,\varphi)$ *for* $i = j = 1,2$ *and* $i \neq j$*, then*

$$\tau_{g_1 \pm g_2,D}^{(p,q)}(f_1,\varphi) = \tau_{g_i,D}^{(p,q)}(f_1,\varphi) \text{ and } \overline{\tau}_{g_1 \pm g_2,D}^{(p,q)}(f_1,\varphi) = \overline{\tau}_{g_i,D}^{(p,q)}(f_1,\varphi)\ .$$

(C) Assume the functions $f_1(z), f_2(z), g_1(z)$ *and* $g_2(z)$ *satisfy the following conditions:*
(i) $\rho_{g_1}^{(p,q)}(f_i,\varphi) > \rho_{g_1}^{(p,q)}(f_j,\varphi)$ *with at least* $f_j(z)$ *is of regular* (p,q)-φ *relative Gol'dberg growth with respect to* $g_1(z)$ *for* $i = j = 1,2$ *and* $i \neq j$*;*
(ii) $\rho_{g_2}^{(p,q)}(f_i,\varphi) > \rho_{g_2}^{(p,q)}(f_j,\varphi)$ *with at least* $f_j(z)$ *is of regular* (p,q)-φ *relative Gol'dberg growth with respect to* $g_2(z)$ *for* $i = j = 1,2$ *and* $i \neq j$*;*
(iii) $\rho_{g_i}^{(p,q)}(f_1,\varphi) < \rho_{g_j}^{(p,q)}(f_1,\varphi)$ *and* $\rho_{g_i}^{(p,q)}(f_2,\varphi) < \rho_{g_j}^{(p,q)}(f_2,\varphi)$ *holds simultaneously for* $i = j = 1,2$ *and* $i \neq j$*;*
(iv) $\lambda_{g_m}^{(p,q)}(f_l,\varphi) = \min[\max\{\lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_1}^{(p,q)}(f_2,\varphi)\},$
$\max\{\lambda_{g_2}^{(p,q)}(f_1,\varphi), \lambda_{g_2}^{(p,q)}(f_2,\varphi)\}] \mid l = m = 1,2$*; then we have*

$$\tau_{g_1 \pm g_2,D}^{(p,q)}(f_1 \pm f_2,\varphi) = \tau_{g_m,D}^{(p,q)}(f_l,\varphi) \text{ and } \overline{\tau}_{g_1 \pm g_2,D}^{(p,q)}(f_1 \pm f_2,\varphi) = \overline{\tau}_{g_m,D}^{(p,q)}(f_l,\varphi)\ .$$

Proof. For any arbitrary positive number $\varepsilon(>0)$, we have for all sufficiently large values of R that

$$\begin{aligned} (11) \qquad & M_{f_k,D}(R) \\ & \leq M_{g_l,D}(\exp^{[p-1]}\{(\overline{\tau}_{g_l,D}^{(p,q)}(f_k,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R)]^{\lambda_{g_l}^{(p,q)}(f_k,\varphi)}\}), \end{aligned}$$

$$\begin{aligned} (12) \qquad & M_{f_k,D}(R) \\ & \geq M_{g_l,D}(\exp^{[p-1]}\{(\tau_{g_l,D}^{(p,q)}(f_k,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R)]^{\lambda_{g_l}^{(p,q)}(f_k,\varphi)}\}) \end{aligned}$$

and for a sequence of values of R tending to infinity we obtain that

$$\begin{aligned} (13) \qquad & M_{f_k,D}(R) \\ & \geq M_{g_l,D}(\exp^{[p-1]}\{(\overline{\tau}_{g_l,D}^{(p,q)}(f_k,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R)]^{\lambda_{g_l}^{(p,q)}(f_k,\varphi)}\}) \end{aligned}$$

and

$$\begin{aligned} (14) \qquad & M_{f_k,D}(R) \\ & \leq M_{g_l,D}(\exp^{[p-1]}\{(\tau_{g_l,D}^{(p,q)}(f_k,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R)]^{\lambda_{g_l}^{(p,q)}(f_k,\varphi)}\}), \end{aligned}$$

where $k = 1,2$ and $l = 1,2$.

Case I. Let $\lambda_{g_1}^{(p,q)}(f_1,\varphi) > \lambda_{g_1}^{(p,q)}(f_2,\varphi)$ with at least $f_2(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_1(z)$. Also let $\varepsilon(>0)$

be arbitrary. Hence we get in view of (11) and (14) for a sequence of values R_n tending to infinity that

$$M_{f_1 \pm f_2, D}(R_n) \leq M_{f_1,D}(R_n) + M_{f_2,D}(R_n)$$

$$\begin{aligned} \leq \; & M_{g_1,D}(\exp^{[p-1]}\{(\tau^{(p,q)}_{g_1,D}(f_1,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R_n)]^{\lambda^{(p,q)}_{g_1}(f_1,\varphi)}\}) \\ & +M_{g_1,D}(\exp^{[p-1]}\{(\overline{\tau}^{(p,q)}_{g_1,D}(f_2,{}^{(p,q)}_{g_1}(f_2,\varphi)\}) \end{aligned}$$

(15) $i.e., M_{f_1 \pm f_2, D}(R_n)$

$$\leq \; M_{g_1,D}(\exp^{[p-1]}\{(\tau^{(p,q)}_{g_1,D}(f_1,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R_n)]^{\lambda^{(p,q)}_{g_1}(f_1,\varphi)}\})(1+E),$$

where $E = \frac{M_{g_1,D}(\exp^{[p-1]}\{(\overline{\tau}^{(p,q)}_{g_1,D}(f_2,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R_n)]^{\lambda^{(p,q)}_{g_1}(f_2,\varphi)}\})}{M_{g_1,D}(\exp^{[p-1]}\{(\tau^{(p,q)}_{g_1,D}(f_1,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R_n)]^{\lambda^{(p,q)}_{g_1}(f_1,\varphi)}\})}$ and in view of $\lambda^{(p,q)}_{g_1}(f_1,\varphi) > \lambda^{(p,q)}_{g_1}(f_2,\varphi)$, we can make the term E sufficiently small by taking n sufficiently large. Therefore with the help of Theorem 2.1 and using the similar technique of Case I of Theorem 3.1, we get from (15) that

$$\tau^{(p,q)}_{g_1,D}(f_1 \pm f_2,\varphi) \leq \tau^{(p,q)}_{g_1,D}(f_1,\varphi). \tag{16}$$

Further, we may consider that $f(z) = f_1(z) \pm f_2(z)$. Also suppose that $\lambda^{(p,q)}_{g_1}(f_1,\varphi) > \lambda^{(p,q)}_{g_1}(f_2,\varphi)$ and at least $f_2(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_1(z)$. Then $\tau^{(p,q)}_{g_1,D}(f,\varphi) = \tau^{(p,q)}_{g_1,D}(f_1 \pm f_2,\varphi) \leq \tau^{(p,q)}_{g_1,D}(f_1,\varphi)$. Now let $f_1(z) = (f(z) \pm f_2(z))$. Therefore in view of Theorem 2.1, $\lambda^{(p,q)}_{g_1}(f_1,\varphi) > \lambda^{(p,q)}_{g_1}(f_2,\varphi)$ and at least $f_2(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_1(z)$, we obtain that $\lambda^{(p,q)}_{g_1}(f,\varphi) > \lambda^{(p,q)}_{g_1}(f_2,\varphi)$ holds. Hence in view of (16), $\tau^{(p,q)}_{g_1,D}(f_1,\varphi) \leq \tau^{(p,q)}_{g_1,D}(f,\varphi) = \tau^{(p,q)}_{g_1,D}(f_1 \pm f_2,\varphi)$. Therefore $\tau^{(p,q)}_{g_1,D}(f,\varphi) = \tau^{(p,q)}_{g_1,D}(f_1,\varphi) \Rightarrow \tau^{(p,q)}_{g_1,D}(f_1 \pm f_2,\varphi) = \tau^{(p,q)}_{g_1,D}(f_1,\varphi)$.

Similarly, if we consider $\lambda^{(p,q)}_{g_1}(f_1,\varphi) < \lambda^{(p,q)}_{g_1}(f_2,\varphi)$ with at least $f_1(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_1(z)$ then one can easily verify that $\tau^{(p,q)}_{g_1,D}(f_1 \pm f_2,\varphi) = \tau^{(p,q)}_{g_1,D}(f_2,\varphi)$.

Case II. Let us consider that $\lambda^{(p,q)}_{g_1}(f_1,\varphi) > \lambda^{(p,q)}_{g_1}(f_2,\varphi)$ with at least $f_2(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_1(z)$. Also let $\varepsilon(>0)$ be arbitrary. Therefore we get in view of (11) for all sufficiently large values of R that

$$M_{f_1\pm f_2,D}(R) \le M_{f_1,D}(R) + M_{f_2,D}(R)$$

$$\begin{aligned} \le\ & M_{g_1,D}(\exp^{[p-1]}\{(\overline{\tau}^{(p,q)}_{g_1,D}(f_1,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R)]^{\lambda^{(p,q)}_{g_1}(f_1,\varphi)}\}) \\ & + M_{g_1,D}(\exp^{[p-1]}\{(\overline{\tau}^{(p,q)}_{g_1,D}(f_2,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R)]^{\lambda^{(p,q)}_{g_1}(f_2,\varphi)}\}) \end{aligned}$$

$$\begin{aligned} (17)\quad & i.e., M_{f_1\pm f_2,D}(R) \\ \le\ & M_{g_1,D}(\exp^{[p-1]}\{(\overline{\tau}^{(p,q)}_{g_1,D}(f_1,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R)]^{\lambda^{(p,q)}_{g_1}(f_1,\varphi)}\})(1+F), \end{aligned}$$

where $F = \frac{M_{g_1,D}(\exp^{[p-1]}\{(\overline{\tau}^{(p,q)}_{g_1,D}(f_2,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R)]^{\lambda^{(p,q)}_{g_1}(f_2,\varphi)}\})}{M_{g_1,D}(\exp^{[p-1]}\{(\overline{\tau}^{(p,q)}_{g_1,D}(f_1,\varphi)+\varepsilon)[\log^{[q-1]}\varphi(R)]^{\lambda^{(p,q)}_{g_1}(f_1,\varphi)}\})}$ and in view of $\lambda^{(p,q)}_{g_1}(f_1,\varphi) > \lambda^{(p,q)}_{g_1}(f_2,\varphi)$, one can make the term F sufficiently small by taking R sufficiently large and therefore for similar reasoning of Case I we get from (17) that $\overline{\tau}^{(p_i,q)}_{g_1,D}(f_1\pm f_2,\varphi) = \overline{\tau}^{(p,q)}_{g_1,D}(f_1,\varphi)$ when $\lambda^{(p,q)}_{g_1}(f_1,\varphi) > \lambda^{(p,q)}_{g_1}(f_2,\varphi)$ and at least $f_2(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_1(z)$.

Likewise, if we consider $\lambda^{(p,q)}_{g_1}(f_1,\varphi) < \lambda^{(p,q)}_{g_1}(f_2,\varphi)$ with at least $f_1(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_1(z)$ then one can easily verify that $\overline{\tau}^{(p,q)}_{g_1,D}(f_1\pm f_2,\varphi) = \overline{\tau}^{(p,q)}_{g_1,D}(f_2,\varphi)$

Thus combining Case I and Case II, we obtain the first part of the theorem.

Case III. Let us consider that $\lambda^{(p,q)}_{g_1}(f_1,\varphi) < \lambda^{(p,q)}_{g_2}(f_1,\varphi)$. Now we get from (12) for all sufficiently large values of R that

$$M_{g_1\pm g_2,D}(R) \le M_{g_1,D}(R) + M_{g_2,D}(R)$$

$$\begin{aligned} \Rightarrow\ & M_{g_1\pm g_2,D}(\exp^{[p-1]}\{(\tau^{(p,q)}_{g_1,D}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R)]^{\lambda^{(p,q)}_{g_1}(f_1,\varphi)}\}) \\ \le\ & M_{g_1,D}(\exp^{[p-1]}\{(\tau^{(p,q)}_{g_1,D}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R)]^{\lambda^{(p,q)}_{g_1}(f_1,\varphi)}\}) \\ & + M_{g_2,D}(\exp^{[p-1]}\{(\tau^{(p,q)}_{g_1,D}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R)]^{\lambda^{(p,q)}_{g_1}(f_1,\varphi)}\}) \end{aligned}$$

$$\begin{aligned} (18)\quad & M_{g_1\pm g_2,D}(\exp^{[p-1]}\{(\tau^{(p,q)}_{g_1,D}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R)]^{\lambda^{(p,q)}_{g_1}(f_1,\varphi)}\}) \\ \le\ & (1+G)M_{f_1,D}(R) \end{aligned}$$

where $G = \frac{M_{g_2,D}(\exp^{[p-1]}\{(\tau^{(p,q)}_{g_1,D}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R)]^{\lambda^{(p,q)}_{g_1}(f_1,\varphi)}\})}{M_{g_2,D}(\exp^{[p-1]}\{(\tau^{(p,q)}_{g_2,D}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R)]^{\lambda^{(p,q)}_{g_2}(f_1,\varphi)}\})}$,

and as $\lambda^{(p,q)}_{g_1}(f_1,\varphi) < \lambda^{(p,q)}_{g_2}(f_1,\varphi)$, we can make the term G sufficiently small by taking R sufficiently large. Now with the help of Theorem 2.3 and using the similar technique of Case III of Theorem 3.1, we get from (18) that

$$\tau^{(p,q)}_{g_1\pm g_2,D}(f_1,\varphi) \geq \tau^{(p,q)}_{g_1,D}(f_1,\varphi)\ . \tag{19}$$

Again, we may consider that $g(z) = g_1(z)\pm g_2(z)$. As $\lambda^{(p,q)}_{g_1}(f_1,\varphi) < \lambda^{(p,q)}_{g_2}(f_1,\varphi)$, so $\tau^{(p,q)}_{g,D}(f_1,\varphi) = \tau^{(p,q)}_{g_1\pm g_2,D}(f_1,\varphi) \geq \tau^{(p,q)}_{g_1,D}(f_1,\varphi)$. Also let $g_1(z) = (g(z)\pm g_2(z))$. Therefore in view of Theorem 2.3 and $\lambda^{(p,q)}_{g_1}(f_1,\varphi) < \lambda^{(p,q)}_{g_2}(f_1,\varphi)$ we obtain that $\lambda^{(p.q)}_{g}(f_1,\varphi) < \lambda^{(p,q)}_{g_2}(f_1,\varphi)$ holds. Hence in view of (19) $\tau^{(p,q)}_{g_1,D}(f_1,\varphi) \geq \tau^{(p,q)}_{g,D}(f_1,\varphi) = \tau^{(p,q)}_{g_1\pm g_2,D}(f_1,\varphi)$. Therefore $\tau^{(p,q)}_{g,D}(f_1,\varphi) = \tau^{(p,q)}_{g_1,D}(f_1,\varphi) \Rightarrow \tau^{(p,q)}_{g_1\pm g_2,D}(f_1,\varphi) = \tau^{(p,q)}_{g_1,D}(f_1,\varphi)$.

Likewise, if we consider that $\lambda^{(p,q)}_{g_1}(f_1,\varphi) > \lambda^{(p,q)}_{g_2}(f_1,\varphi)$, then one can easily verify that $\tau^{(p,q)}_{g_1\pm g_2,D}(f_1,\varphi) = \tau^{(p,q)}_{g_2,D}(f_1,\varphi)$.

Case IV. In this case further we consider $\lambda^{(p,q)}_{g_1}(f_1) < \lambda^{(p,q)}_{g_2}(f_1)$. Now we obtain from (12) and (13) for a sequence of values of R tending to infinity that

$$M_{g_1\pm g_2,D}(R_n) \leq M_{g_1,D}(R_n) + M_{g_2,D}(R_n)$$

$$\begin{aligned} & M_{g_1\pm g_2,D}(\exp^{[p-1]}\{(\overline{\tau}^{(p,q)}_{g_1,D}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R_n)]^{\lambda^{(p,q)}_{g_1}(f_1,\varphi)}\}) \\ \leq\ & M_{g_1,D}(\exp^{[p-1]}\{(\overline{\tau}^{(p,q)}_{g_1,D}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R_n)]^{\lambda^{(p,q)}_{g_1}(f_1,\varphi)}\}) \\ & +M_{g_2,D}(\exp^{[p-1]}\{(\overline{\tau}^{(p,q)}_{g_1,D}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R_n)]^{\lambda^{(p,q)}_{g_1}(f_1,\varphi)}\}) \end{aligned}$$

$$\begin{aligned} & \textit{i.e.,}\ M_{g_1\pm g_2,D}(\exp^{[p-1]}\{(\overline{\tau}^{(p,q)}_{g_1,D}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R_n)]^{\lambda^{(p,q)}_{g_1}(f_1,\varphi)}\}) \\ \leq\ & (1+H)M_{f_1,D}(R_n) \end{aligned} \tag{20}$$

where $H = \frac{M_{g_2,D}(\exp^{[p-1]}\{(\overline{\tau}^{(p,q)}_{g_1,D}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R_n)]^{\lambda^{(p,q)}_{g_1}(f_1,\varphi)}\})}{M_{g_2,D}(\exp^{[p-1]}\{(\overline{\tau}^{(p,q)}_{g_2,D}(f_1,\varphi)-\varepsilon)[\log^{[q-1]}\varphi(R_n)]^{\lambda^{(p,q)}_{g_2}(f_1,\varphi)}\})}$, and in view of $\lambda^{(p,q)}_{g_1}(f_1,\varphi) < \lambda^{(p,q)}_{g_2}(f_1,\varphi)$, we can make the term H sufficiently small by taking n sufficiently large and therefore using the similar technique as executed in the proof of Case IV of Theorem 3.1, we get from (20) that

$\overline{\tau}^{(p,q)}_{g_1\pm g_2,D}(f_1,\varphi)=\overline{\tau}^{(p,q)}_{g_1,D}(f_1,\varphi)$ when $\lambda^{(p,q)}_{g_1}(f_1,\varphi)<\lambda^{(p,q)}_{g_2}(f_1,\varphi)$. Similarly, if we consider that $\lambda^{(p,q)}_{g_1}(f_1,\varphi)>\lambda^{(p,q)}_{g_2}(f_1,\varphi)$, then one can easily verify that $\overline{\tau}^{(p,q)}_{g_1\pm g_2,D}(f_1,\varphi)=\overline{\tau}^{(p,q)}_{g_2,D}(f_1,\varphi)$.

Thus combining Case III and Case IV, we obtain the second part of the theorem.

The proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 2.6 and the above cases. □

THEOREM 3.3. *Let $f_1(z)$, $f_2(z)$, $g_1(z)$ and $g_2(z)$ be any four entire functions of n- complex variables and D be a bounded complete n-circular domain with center at origin in $\mathbb{C}^n$and p, q are any two positive integers.*
(A) The following condition is assumed to be satisfied:
(i) Either $\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)\neq\sigma^{(p,q)}_{g_1,D}(f_2,\varphi)$ or $\overline{\sigma}^{(p,q)}_{g_1,D}(f_1,\varphi)\neq\overline{\sigma}^{(p,q)}_{g_1,D}(f_2,\varphi)$ holds, then

$$\rho^{(p,q)}_{g_1}(f_1\pm f_2,\varphi)=\rho^{(p,q)}_{g_1}(f_1,\varphi)=\rho^{(p,q)}_{g_1}(f_2,\varphi)\ .$$

(B) The following conditions are assumed to be satisfied:
(i) Either $\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)\neq\sigma^{(p,q)}_{g_2,D}(f_1,\varphi)$ or $\overline{\sigma}^{(p,q)}_{g_1,D}(f_1,\varphi)\neq\overline{\sigma}^{(p,q)}_{g_2,D}(f_1,\varphi)$ holds;
(ii) $f_1(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to at least any one of $g_1(z)$ or $g_2(z)$, then

$$\rho^{(p,q)}_{g_1\pm g_2}(f_1,\varphi)=\rho^{(p,q)}_{g_1}(f_1,\varphi)=\rho^{(p,q)}_{g_2}(f_1,\varphi)\ .$$

Proof. Case I. Suppose that $\rho^{(p,q)}_{g_1}(f_1,\varphi)=\rho^{(p,q)}_{g_1}(f_2,\varphi)$ $(0<\rho^{(p,q)}_{g_1}(f_1,\varphi),\rho^{(p,q)}_{g_1}(f_2,\varphi)<\infty)$. Now in view of Theorem 2.2 it is easy to see that $\rho^{(p,q)}_{g_1}(f_1\pm f_2,\varphi)\le\rho^{(p,q)}_{g_1}(f_1,\varphi)=\rho^{(p,q)}_{g_1}(f_2,\varphi)$. If possible let

$$\rho^{(p,q)}_{g_1}(f_1\pm f_2,\varphi)<\rho^{(p,q)}_{g_1}(f_1,\varphi)=\rho^{(p,q)}_{g_1}(f_2,\varphi)\ . \tag{21}$$

Let $\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)\neq\sigma^{(p,q)}_{g_1,D}(f_2,\varphi)$. Then in view of the first part of Theorem 3.1 and (21) we obtain that $\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)=\sigma^{(p,q)}_{g_1,D}(f_1\pm f_2\mp f_2,\varphi)=\sigma^{(p,q)}_{g_1,D}(f_2,\varphi)$ which is a contradiction. Hence $\rho^{(p,q)}_{g_1}(f_1\pm f_2,\varphi)=\rho^{(p,q)}_{g_1}(f_1,\varphi)=\rho^{(p,q)}_{g_1}(f_2,\varphi)$. Similarly with the help of the first part of Theorem 3.1, one can obtain the same conclusion under the hypothesis $\overline{\sigma}^{(p,q)}_{g_1,D}(f_1)\neq\overline{\sigma}^{(p,q)}_{g_1,D}(f_2)$. This proves the first part of the theorem.

Case II. Let us consider that $\rho^{(p,q)}_{g_1}(f_1,\varphi)=\rho^{(p,q)}_{g_2}(f_1,\varphi)$ $(0<\rho^{(p,q)}_{g_1}(f_1,\varphi),\rho^{(p,q)}_{g_2}(f_1,\varphi)<\infty)$ and $f_1(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to at least any one of $g_1(z)$ or $g_2(z)$ and $(g_1(z)\pm$

$g_2(z))$. Therefore in view of Theorem 2.4, it follows that $\rho^{(p,q)}_{g_1 \pm g_2}(f_1, \varphi) \geq \rho^{(p,q)}_{g_1}(f_1, \varphi) = \rho^{(p,q)}_{g_2}(f_1, \varphi)$ and if possible let

$$\rho^{(p,q)}_{g_1 \pm g_2}(f_1, \varphi) > \rho^{(p,q)}_{g_1}(f_1, \varphi) = \rho^{(p,q)}_{g_2}(f_1, \varphi) \ . \tag{22}$$

Let us consider that $\sigma^{(p,q)}_{g_1,D}(f_1, \varphi) \neq \sigma^{(p,q)}_{g_2,D}(f_1, \varphi)$. Then. in view of the proof of the second part of Theorem 3.1 and (22) we obtain that $\sigma^{(p,q)}_{g_1,D}(f_1, \varphi) = \sigma^{(p,q)}_{g_1 \pm g_2 \mp g_2,D}(f_1, \varphi) = \sigma^{(p,q)}_{g_2,D}(f_1, \varphi)$ which is a contradiction. Hence $\rho^{(p,q)}_{g_1 \pm g_2}(f_1, \varphi) = \rho^{(p,q)}_{g_1}(f_1, \varphi) = \rho^{(p,q)}_{g_2}(f_1, \varphi)$. Again, in view of the proof of second part of Theorem 3.1 one can derive the same conclusion for the condition $\overline{\sigma}^{(p,q)}_{g_1,D}(f_1, \varphi) \neq \overline{\sigma}^{(p,q)}_{g_2,D}(f_1, \varphi)$ and therefore the second part of the theorem is established. □

Theorem 3.4. *Let $f_1(z)$, $f_2(z)$, $g_1(z)$ and $g_2(z)$ be any four entire functions of n- complex variables and D be a bounded complete n-circular domain with center at origin in $\mathbb{C}^n$ and p, q are any two positive integers.*

(A) The following conditions are assumed to be satisfied:

(i) $f_1(z) \pm f_2(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to at least any one of $g_1(z)$ or $g_2(z)$;

(ii) Either $\sigma^{(p,q)}_{g_1,D}(f_1 \pm f_2, \varphi) \neq \sigma^{(p,q)}_{g_2,D}(f_1 \pm f_2, \varphi)$ or $\overline{\sigma}^{(p,q)}_{g_1,D}(f_1 \pm f_2, \varphi) \neq \overline{\sigma}^{(p,q)}_{g_2,D}(f_1 \pm f_2, \varphi)$;

(iii) Either $\sigma^{(p,q)}_{g_1,D}(f_1, \varphi) \neq \sigma^{(p,q)}_{g_1,D}(f_2, \varphi)$ or $\overline{\sigma}^{(p,q)}_{g_1,D}(f_1, \varphi) \neq \overline{\sigma}^{(p,q)}_{g_1,D}(f_2, \varphi)$;

(iv) Either $\sigma^{(p,q)}_{g_2,D}(f_1, \varphi) \neq \sigma^{(p,q)}_{g_2,D}(f_2, \varphi)$ or $\overline{\sigma}^{(p,q)}_{g_2,D}(f_1, \varphi) \neq \overline{\sigma}^{(p,q)}_{g_2,D}(f_2, \varphi)$; then

$$\begin{aligned} \rho^{(p,q)}_{g_1 \pm g_2}(f_1 \pm f_2, \varphi) &= \rho^{(p,q)}_{g_1}(f_1, \varphi) \\ &= \rho^{(p,q)}_{g_1}(f_2, \varphi) = \rho^{(p,q)}_{g_2}(f_1, \varphi) = \rho^{(p,q)}_{g_2}(f_2, \varphi) \ . \end{aligned}$$

(B) The following conditions are assumed to be satisfied:

(i) $f_1(z)$ and $f_2(z)$ are of regular (p,q)-φ relative Gol'dberg growth with respect to at least any one of $g(z)$ or $g_2(z)$;

(ii) Either $\sigma^{(p,q)}_{g_1 \pm g_2,D}(f_1, \varphi) \neq \sigma^{(p,q)}_{g_1 \pm g_2,D}(f_2, \varphi)$ or $\overline{\sigma}^{(p,q)}_{g_1 \pm g_2,D}(f_1, \varphi) \neq \overline{\sigma}^{(p,q)}_{g_1 \pm g_2,D}(f_2, \varphi)$;

(iii) Either $\sigma^{(p,q)}_{g_1,D}(f_1, \varphi) \neq \sigma^{(p,q)}_{g_2,D}(f_1, \varphi)$ or $\sigma^{(p,q)}_{g_1,D}(f_1, \varphi) \neq \overline{\sigma}^{(p,q)}_{g_2,D}(f_1, \varphi)$;

(iv) Either $\sigma^{(p,q)}_{g_1,D}(f_2, \varphi) \neq \sigma^{(p,q)}_{g_2,D}(f_2, \varphi)$ or $\overline{\sigma}^{(p,q)}_{g_1,D}(f_2, \varphi) \neq \overline{\sigma}^{(p,q)}_{g_2,D}(f_2, \varphi)$;

then

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2, \varphi) = \rho_{g_1}^{(p,q)}(f_1, \varphi) = \rho_{g_1}^{(p,q)}(f_2, \varphi) = \rho_{g_2}^{(p,q)}(f_1, \varphi) = \rho_{g_2}^{(p,q)}(f_2, \varphi) .$$

We omit the proof of Theorem 3.4 as it is a natural consequence of Theorem 3.3.

THEOREM 3.5. *Let $f_1(z)$, $f_2(z)$, $g_1(z)$ and $g_2(z)$ be any four entire functions of n- complex variables and D be a bounded complete n-circular domain with center at origin in $\mathbb{C}^n$.*
(A) The following conditions are assumed to be satisfied:
(i) At least any one of $f_1(z)$ or $f_2(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_1(z)$ where p and q are any two positive integers;
(ii) Either $\tau_{g_1,D}^{(p,q)}(f_1, \varphi) \neq \tau_{g_1,D}^{(p,q)}(f_2, \varphi)$ or $\overline{\tau}_{g_1,D}^{(p,q)}(f_1, \varphi) \neq \overline{\tau}_{g_1,D}^{(p,q)}(f_2, \varphi)$ holds, then

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2, \varphi) = \lambda_{g_1}^{(p,q)}(f_1, \varphi) = \lambda_{g_1}^{(p,q)}(f_2, \varphi) .$$

(B) The following conditions are assumed to be satisfied:
(i) $f_1(z)$, $g_1(z)$ and $g_2(z)$ be any three entire functions such that $\lambda_{g_1}^{(p,q)}(f_1, \varphi)$ and $\lambda_{g_2}^{(p,q)}(f_1, \varphi)$ exists where p and q are any two positive integers;
(ii) Either $\tau_{g_1,D}^{(p,q)}(f_1, \varphi) \neq \tau_{g_2,D}^{(p,q)}(f_1, \varphi)$ or $\overline{\tau}_{g_1,D}^{(p,q)}(f_1, \varphi) \neq \overline{\tau}_{g_2,D}^{(p,q)}(f_1, \varphi)$ holds, then

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1, \varphi) = \lambda_{g_1}^{(p,q)}(f_1, \varphi) = \lambda_{g_2}^{(p,q)}(f_1, \varphi) .$$

Proof. Case I. Let $\lambda_{g_1}^{(p,q)}(f_1, \varphi) = \lambda_{g_1}^{(p,q)}(f_2, \varphi)$ $(0 < \lambda_{g_1}^{(p,q)}(f_1, \varphi), \lambda_{g_1}^{(p,q)}(f_2, \varphi) < \infty)$ and at least $f_1(z)$ or $f_2(z)$ and $(f_1(z) \pm f_2(z))$ are of regular (p,q)-φ relative Gol'dberg growth with respect to $g_1(z)$. Now, in view of Theorem 2.1, it is easy to see that $\lambda_{g_1}^{(p,q)}(f_1 \pm f_2, \varphi) \leq \lambda_{g_1}^{(p,q)}(f_1, \varphi) = \lambda_{g_1}^{(p,q)}(f_2, \varphi)$. If possible let

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2, \varphi) < \lambda_{g_1}^{(p,q)}(f_1, \varphi) = \lambda_{g_1}^{(p,q)}(f_2, \varphi) . \tag{23}$$

Let $\tau_{g_1,D}^{(p,q)}(f_1, \varphi) \neq \tau_{g_1,D}^{(p,q)}(f_2, \varphi)$. Then in view of the proof of the first part of Theorem 3.2 and (23) we obtain that $\tau_{g_1,D}^{(p,q)}(f_1, \varphi) = \tau_{g_1,D}^{(p,q)}(f_1 \pm f_2 \mp f_2, \varphi) = \tau_{g_1,D}^{(p,q)}(f_2, \varphi)$ which is a contradiction. Hence $\lambda_{g_1}^{(p,q)}(f_1 \pm f_2, \varphi) = \lambda_{g_1}^{(p,q)}(f_1, \varphi) = \lambda_{g_1}^{(p,q)}(f_2, \varphi)$. Similarly in view of the proof of the first part of Theorem 3.2 , one can establish the same conclusion under the hypothesis $\overline{\tau}_{g_1,D}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_1,D}^{(p,q)}(f_2,\varphi)$. This proves the first part of the theorem.

Case II. Let us consider that $\lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi)$ $(0 < \lambda_{g_1}^{(p,q)}(f_1,\varphi), \lambda_{g_2}^{(p,q)}(f_1,\varphi) < \infty$. Therefore in view of Theorem 2.3, it follows that $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) \geq \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi)$ and if possible let

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) > \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi) \ . \tag{24}$$

Suppose $\tau_{g_1,D}^{(p,q)}(f_1,\varphi) \neq \tau_{g_2,D}^{(p,q)}(f_1,\varphi)$. Then in view of the second part of Theorem 3.2 and (24), we obtain that

$$\tau_{g_1,D}^{(p,q)}(f_1,\varphi) = \tau_{g_1 \pm g_2 \mp g_2,D}^{(p,q)}(f_1,\varphi) = \tau_{g_2,D}^{(p,q)}(f_1,\varphi)$$

which is a contradiction. Hence $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1,\varphi) = \lambda_{g_1}^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi)$. Analogously with the help of the second part of Theorem 3.2, the same conclusion can also be derived under the condition

$$\overline{\tau}_{g_1,D}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_2,D}^{(p,q)}(f_1,\varphi)$$

and therefore the second part of the theorem is established. □

Theorem 3.6. *Let $f_1(z)$, $f_2(z)$, $g_1(z)$ and $g_2(z)$ be any four entire functions of n- complex variables and D be a bounded complete n-circular domain with center at origin in $\mathbb{C}^n$.*
(A) The following conditions are assumed to be satisfied:
(i) At least any one of $f_1(z)$ or $f_2(z)$ is of regular (p,q)-φ relative Gol'dberg growth with respect to $g_1(z)$ and $g_2(z)$ where p, q are any two positive integers
(iii) Either $\tau_{g_1,D}^{(p,q)}(f_1,\varphi) \neq \tau_{g_1,D}^{(p,q)}(f_2,\varphi)$ or $\overline{\tau}_{g_1,D}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_1,D}^{(p,q)}(f_2,\varphi)$;
(iv) Either $\tau_{g_2,D}^{(p,q)}(f_1,\varphi) \neq \tau_{g_2,D}^{(p,q)}(f_2,\varphi)$ or $\overline{\tau}_{g_2,D}^{(p,q)}(f_1,\varphi) \neq \overline{\tau}_{g_2,D}^{(p,q)}(f_2,\varphi)$; then

$$\begin{aligned}\lambda_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2,\varphi) &= \lambda_{g_1}^{(p,q)}(f_1,\varphi) \\ &= \lambda_{g_1}^{(p,q)}(f_2,\varphi) = \lambda_{g_2}^{(p,q)}(f_1,\varphi) = \lambda_{g_2}^{(p,q)}(f_2,\varphi) \ .\end{aligned}$$

(B) The following conditions are assumed to be satisfied:
(i) At least any one of $f_1(z)$ or $f_2(z)$ are of regular (p,q)-φ relative Gol'dberg growth with respect to $g_1(z) \pm g_2(z)$ where p and q are any two positive integers;
(ii) Either $\tau_{g_1 \pm g_2,D}^{(p,q)}(f_1,\varphi) \neq \tau_{g_1 \pm g_2,D}^{(p,q)}(f_2,\varphi)$

or $\overline{\tau}^{(p,q)}_{g_1 \pm g_2, D}(f_1,\varphi) \neq \overline{\tau}^{(p,q)}_{g_1 \pm g_2, D}(f_2,\varphi)$ *holds;*
(iii) *Either* $\tau^{(p,q)}_{g_1,D}(f_1,\varphi) \neq \tau^{(p,q)}_{g_2,D}(f_1,\varphi)$ *or* $\overline{\tau}^{(p,q)}_{g_1,D}(f_1,\varphi) \neq \overline{\tau}^{(p,q)}_{g_2,D}(f_1,\varphi)$ *holds;*
(iv) *Either* $\tau^{(p,q)}_{g_1,D}(f_2,\varphi) \neq \tau^{(p,q)}_{g_2,D}(f_2,\varphi)$ *or* $\overline{\tau}^{(p,q)}_{g_1,D}(f_2,\varphi) \neq \overline{\tau}^{(p,q)}_{g_2,D}(f_2,\varphi)$ *holds, then*

$$\begin{aligned}\lambda^{(p,q)}_{g_1 \pm g_2}(f_1 \pm f_2,\varphi) &= \lambda^{(p,q)}_{g_1}(f_1,\varphi)\\ &= \lambda^{(p,q)}_{g_1}(f_2,\varphi) = \lambda^{(p,q)}_{g_2}(f_1,\varphi) = \lambda^{(p,q)}_{g_2}(f_2,\varphi)\ .\end{aligned}$$

We omit the proof of Theorem 3.6 as it is a natural consequence of Theorem 3.5.

Theorem 3.7. *Let $f_1(z)$, $f_2(z)$, $g_1(z)$ and $g_2(z)$ be any four entire functions of n- complex variables and D be a bounded complete n-circular domain with center at origin in $\mathbb{C}^n$. Also let $\rho^{(p,q)}_{g_1}(f_1,\varphi)$, $\rho^{(p,q)}_{g_1}(f_2,\varphi)$, $\rho^{(p,q)}_{g_2}(f_1,\varphi)$ and $\rho^{(p,q)}_{g_2}(f_2,\varphi)$ are all non zero.*
(A) Assume the functions f_1, f_2 and g_1 satisfy the following conditions:
(i) g_1 satisfies the Property (G) and
(ii) f_1 and f_2 satisfy Property (X); then

$$\sigma^{(p,q)}_{g_1,D}(f_1 \cdot f_2,\varphi) = \sigma^{(p,q)}_{g_1,D}(f_i,\varphi) \text{ and } \overline{\sigma}^{(p,q)}_{g_1,D}(f_1 \cdot f_2,\varphi) = \overline{\sigma}^{(p,q)}_{g_1,D}(f_i,\varphi)\,.$$

(B) (i) If ${\rho_{g_i}}^{(p,q)}(f_1,\varphi) < {\rho_{g_j}}^{(p,q)}(f_1,\varphi)$ with at least f_1 is of regular (p,q)-φ relative Gol'dberg growth with respect to g_j for i, $j = 1,2$ and $i \neq j$,
(ii) f_1 satisfies the Property (G),
(iii) g_1 and g_2 are satisfy Property (X), then

$$\sigma^{(p,q)}_{g_1 \cdot g_2,D}(f_1,\varphi) = {\sigma_{g_{i},D}}^{(p,q)}(f_1,\varphi) \text{ and } \overline{\sigma}^{(p,q)}_{g_1 \cdot g_2,D}(f_1,\varphi) = {\overline{\sigma}_{g_{i},D}}^{(p,q)}(f_1,\varphi).$$

(C) Assume the functions f_1, f_2, g_1 and g_2 satisfy the following conditions:
$(i) g_1 \cdot g_2$, f_1 and f_2 are satisfy the Property (G);
(ii) f_1 and f_2 satisfy Property (X);
(iii) g_1 and g_2 satisfy Property (X);
(iv) $\rho^{(p,q)}_{g_i}(f_1,\varphi) < \rho^{(p,q)}_{g_j}(f_1,\varphi)$ with at least f_1 is of regular (p,q)-φ relative Gol'dberg growth with respect to g_j for $i = 1$, 2, $j = 1,2$ and $i \neq j$;
(v) $\rho^{(p,q)}_{g_i}(f_2,\varphi) < \rho^{(p,q)}_{g_j}(f_2,\varphi)$ with at least f_2 is of regular (p,q)-φ relative Gol'dberg growth with respect to g_j for $i = 1$, 2, $j = 1,2$ and $i \neq j$;
(vi) $\rho^{(p,q)}_{g_1}(f_i,\varphi) < \rho^{(p,q)}_{g_1}(f_j,\varphi)$ and $\rho^{(p,q)}_{g_2}(f_i,\varphi) < \rho^{(p,q)}_{g_2}(f_j,\varphi)$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$;

$(vii)\rho^{(p,q)}_{g_m}(f_l,\varphi)=\max[\min\{\rho^{(p,q)}_{g_1}(f_1,\varphi),\rho^{(p,q)}_{g_2}(f_1,\varphi)\}$, $\min\{\rho^{(p,q)}_{g_1}(f_2,\varphi),\rho^{(p,q)}_{g_2}(f_2,\varphi)\}]\mid l,m=1,2$; *then we have*

$$\sigma^{(p,q)}_{g_1\cdot g_2,D}(f_1\cdot f_2,\varphi)=\sigma^{(p,q)}_{g_m,D}(f_l,\varphi) \text{ and } \overline{\sigma}^{(p,q)}_{g_1\cdot g_2,D}(f_1\cdot f_2,\varphi)=\overline{\sigma}^{(p,q)}_{g_m,D}(f_l,\varphi).$$

Proof. Case I. Let g_1 satisfies the Property (G) and f_1 and f_2 satisfy Property (X). We have using (1) for all sufficiently large values of R and for any arbitrary $\varepsilon>0$ that

$$M_{f_1\cdot f_2,D}(R)\le M_{f_1,D}(R)\cdot M_{f_2,D}(R)$$

$$\begin{aligned}\le\;& M_{g_1,D}(\exp^{[p-1]}\{(\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)+\varepsilon)\left[\log^{[q-1]}\varphi(R)\right]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\})\\ &\cdot M_{g_1,D}(\exp^{[p-1]}\{(\sigma^{(p,q)}_{g_1,D}(f_2,\varphi)+\varepsilon)\left[\log^{[q-1]}\varphi(R)\right]^{\rho^{(p,q)}_{g_1}(f_2,\varphi)}\})\end{aligned}$$

Suppose that $max\left\{\rho^{(p,q)}_{g_1}(f_1,\varphi),\rho^{(p,q)}_{g_1}(f_2,\varphi)\right\}=\rho^{(p,q)}_{g_1}(f_1,\varphi)$, then

$$\begin{aligned}& M_{f_1\cdot f_2,D}(R)\\ <\;& (M_{g_1,D}(\exp^{[p-1]}\{(\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)+\varepsilon)\left[\log^{[q-1]}\varphi(R)\right]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\}))^2\end{aligned}$$

Now in view of Theorem 2.8 and for $\varepsilon_1>0$,

$$\begin{aligned}& i.e. M_{f_1\cdot f_2,D}(R)\\ <\;& (M_{g_1,D}(\exp^{[p-1]}\{(\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)+\varepsilon)\left[\log^{[q-1]}\varphi(R)\right]^{\rho^{(p,q)}_{g_1}(f_1\cdot f_2,\varphi)+\varepsilon_1}\}))^2\end{aligned}$$

Since g_1 satisfies the Property (G), we have

$$\begin{aligned}(25)\quad & M_{f_1\cdot f_2,D}(R)\\ <\;& M_{g_1,D}(\exp^{[p-1]}\{(\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)+\varepsilon)\left[\log^{[q-1]}\varphi(R)\right]^{\rho^{(p,q)}_{g_1}(f_1\cdot f_2,\varphi)+\varepsilon_1}\})^{\delta},\end{aligned}$$

where $\delta>1$.

Since $\varepsilon,\varepsilon_1>0$ is arbitrary, we obtain from (25) by letting $\delta\longrightarrow 1+$ that

$$\sigma^{(p,q)}_{g_1,D}(f_1\cdot f_2,\varphi)\le\sigma^{(p,q)}_{g_1,D}(f_1,\varphi)\ . \tag{26}$$

Again since f_1 and f_2 are satisfy Property (X), then using (1) for all sufficiently large values of R and for any arbitrary $\varepsilon > 0$,we have

$$\begin{aligned} & M_{f_1,D}(R) \\ < \;& M_{f_1 \cdot f_2,D}(R) \\ \leq \;& M_{g_1,D}(\exp^{[p-1]}\{(\sigma^{(p,q)}_{g_1,D}(f_1 \cdot f_2,\varphi) + \varepsilon)\left[\log^{[q-1]}\varphi(R)\right]^{\rho^{(p,q)}_{g_1}(f_1 \cdot f_2,\varphi)}\}) \end{aligned}$$

In view of Theorem 2.8,

$$\begin{aligned} & M_{f_1,D}(R) \\ \leq \;& M_{g_1,D}(\exp^{[p-1]}\{(\sigma^{(p,q)}_{g_1,D}(f_1 \cdot f_2,\varphi) + \varepsilon)\left[\log^{[q-1]}\varphi(R)\right]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\}) \end{aligned}$$

which implies

$$\sigma^{(p,q)}_{g_1,D}(f_1 \cdot f_2,\varphi) \geq \sigma^{(p,q)}_{g_1,D}(f_1,\varphi) \tag{27}$$

Hence from from (26) and (27) , we conclude that

$$\sigma^{(p,q)}_{g_1,D}(f_1 \cdot f_2,\varphi) = \sigma^{(p,q)}_{g_1,D}(f_1,\varphi).$$

Similarly, if we consider $\rho^{(p,q)}_{g_1}(f_1,\varphi) < \rho^{(p,q)}_{g_1}(f_2,\varphi)$, then one can verify that

$$\sigma^{(p,q)}_{g_1,D}(f_1 \cdot f_2,\varphi) = \sigma^{(p,q)}_{g_1,D}(f_2,\varphi).$$

Case II. Let $\rho^{(p,q)}_{g_1}(f_1,\varphi) > \rho^{(p,q)}_{g_1}(f_2,\varphi)$ and g_1 satisfy the Property (G). Now for any arbitrary $\varepsilon > 0$, like Case I, we have from (1) and (4) for a sequence of values of R_n tending to infinity that

$$M_{f_1 \cdot f_2,D}(R_n) \leq M_{f_1,D}(R_n) \cdot M_{f_2,D}(R_n)$$

$$\begin{aligned} \leq \;& M_{g_1,D}(\exp^{[p-1]}\{(\overline{\sigma}^{(p,q)}_{g_1,D}(f_1,\varphi) + \varepsilon)\left[\log^{[q-1]}\varphi(R_n)\right]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\}) \\ & \cdot M_{g_1,D}(\exp^{[p-1]}\{(\overline{\sigma}^{(p,q)}_{g_1,D}(f_2,\varphi) + \varepsilon)\left[\log^{[q-1]}\varphi(R_n)\right]^{\rho^{(p,q)}_{g_1}(f_2,\varphi)}\}) \end{aligned}$$

Now using the similar technique for a sequence of values of R_n tending to infinity as explored in the proof of Case I, one can easily verify from above that $\overline{\sigma}^{(p,q)}_{g_1,D}(f_1 \cdot f_2,\varphi) = \overline{\sigma}^{(p,q)}_{g_1,D}(f_1,\varphi)$ under the conditions specified in the theorem. Similarly, if we consider $\rho^{(p,q)}_{g_1}(f_1,\varphi) < \rho^{(p,q)}_{g_1}(f_2,\varphi)$, then one can also verify that $\overline{\sigma}^{(p,q)}_{g_1,D}(f_1 \cdot f_2,\varphi) = \overline{\sigma}^{(p,q)}_{g_1,D}(f_2,\varphi)$. Therefore the first part of theorem follows from Case I and Case II.

Case III. Let f_1 satisfy the Property (G) and $\rho_{g_1}^{(p,q)}(f_1,\varphi) < \rho_{g_2}^{(p,q)}(f_1,\varphi)$ with f_1 is of regular (p,q)-φ relative Gol'dberg growth with respect to at least any one of g_1 or g_2.For a sequence of values of R_n tending to infinity that

$$M_{g_1\cdot g_2,D}(R_n) \le M_{g_1,D}(R_n)\cdot M_{g_2,D}(R_n),$$

$$\begin{aligned} & i.e., M_{g_1\cdot g_2,D}(\exp^{[p-1]}\{(\sigma_{g_1,D}^{(p,q)}(f_1,\varphi)-\varepsilon)\left[\log^{[q-1]}\varphi(R_n)\right]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\}) \\ \le\ & M_{g_1,D}(\exp^{[p-1]}\{(\sigma_{g_1,D}^{(p,q)}(f_1,\varphi)-\varepsilon)\left[\log^{[q-1]}\varphi(R_n)\right]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\}) \\ & \cdot M_{g_2,D}(\exp^{[p-1]}\{(\sigma_{g_1,D}^{(p,q)}(f_1,\varphi)-\varepsilon)\left[\log^{[q-1]}\varphi(R_n)\right]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\}), \end{aligned}$$

Now $\rho_{g_1}^{(p,q)}(f_1,\varphi) < \rho_{g_2}^{(p,q)}(f_1,\varphi)$ implies

$$\frac{M_{g_2,D}(\exp^{[p-1]}\{(\sigma_{g_1,D}^{(p,q)}(f_1,\varphi)-\varepsilon)\left[\log^{[q-1]}\varphi(R_n)\right]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\})}{M_{g_2,D}(\exp^{[p-1]}\{(\overline{\sigma}_{g_2,D}^{(p,q)}(f_1,\varphi)-\varepsilon)\left[\log^{[q-1]}\varphi(R_n)\right]^{\rho_{g_2}^{(p,q)}(f_1,\varphi)}\})} < 1,$$

so

$$\begin{aligned} & i.e., M_{g_1\cdot g_2,D}(\exp^{[p-1]}\{(\sigma_{g_1,D}^{(p,q)}(f_1,\varphi)-\varepsilon)\left[\log^{[q-1]}\varphi(R_n)\right]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\}) \\ <\ & M_{g_1,D}(\exp^{[p-1]}\{(\sigma_{g_1,D}^{(p,q)}(f_1,\varphi)-\varepsilon)\left[\log^{[q-1]}\varphi(R_n)\right]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\}) \\ & \cdot M_{g_2,D}(\exp^{[p-1]}\{(\overline{\sigma}_{g_2,D}^{(p,q)}(f_1,\varphi)-\varepsilon)\left[\log^{[q-1]}\varphi(R_n)\right]^{\rho_{g_2}^{(p,q)}(f_1,\varphi)}\}) \end{aligned}$$

Now we have in view of (2) and (3) for a sequence of values of R_n tending to infinity that

$$\begin{aligned} (28)\quad & i.e., M_{g_1\cdot g_2,D}(\exp^{[p-1]}\{(\sigma_{g_1,D}^{(p,q)}(f_1,\varphi)-\varepsilon)\left[\log^{[q-1]}\varphi(R_n)\right]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\}) \\ <\ & [M_{f_1,D}(R_n)]^2 \end{aligned}$$

Since f_1 satisfy the Property (G), we have from (28) for any $\delta > 1$,

$$\begin{aligned} (29)\quad & M_{g_1\cdot g_2,D}(\exp^{[p-1]}\{(\sigma_{g_1,D}^{(p,q)}(f_1,\varphi)-\varepsilon)\left[\log^{[q-1]}\varphi(R_n)\right]^{\rho_{g_1}^{(p,q)}(f_1,\varphi)}\}) \\ <\ & \left[M_{f_1,D}\left(R_n^{\delta}\right)\right] \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows from (29) by taking $\delta \to 1+$ and for a sequence of values R_n tending to infinity that

$$\sigma^{(p,q)}_{g_1 \cdot g_2, D}(f_1, \varphi) \geq \sigma^{(p,q)}_{g_1, D}(f_1, \varphi). \tag{30}$$

Next since g_1 and g_2 are satisfy Property (X), then for all sufficiently large values of R, we have from(1) and Theorem2.10

$$M_{g_1, D}(R) < M_{g_1 \cdot g_2, D}(R)$$

$$\begin{aligned}
& i.e.,\ M_{g_1 \cdot g_2, D}(\exp^{[p-1]}\{(\sigma^{(p,q)}_{g_1, D}(f_1, \varphi) + \varepsilon)\left[\log^{[q-1]} \varphi(R)\right]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\}) \\
> \ & M_{g_1, D}(\exp^{[p-1]}\{(\sigma^{(p,q)}_{g_1, D}(f_1, \varphi) + \varepsilon)\left[\log^{[q-1]} \varphi(R)\right]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\}) \\
\geq \ & M_{f_1, D}(R)
\end{aligned}$$

So

$$\sigma^{(p,q)}_{g_1, D}(f_1, \varphi) \geq \sigma^{(p,q)}_{g_1 \cdot g_2, D}(f_1, \varphi) \tag{31}$$

Hence from (30) and (31), we conclude that

$$\sigma^{(p,q)}_{g_1 \cdot g_2, D}(f_1, \varphi) = \sigma^{(p,q)}_{g_1, D}(f_1, \varphi).$$

Similarly, if we consider $\rho^{(p,q)}_{g_1}(f_1, \varphi) > \rho^{(p,q)}_{g_2}(f_1, \varphi)$, then one can verify that $\sigma^{(p,q)}_{g_1 \cdot g_2, D}(f_1, \varphi) = \sigma^{(p,q)}_{g_2, D}(f_1, \varphi)$.

Case IV. Suppose f_1 satisfy the Property (G). Also let $\rho^{(p,q)}_{g_1}(f_1, \varphi) < \rho^{(p,q)}_{g_2}(f_1, \varphi)$ with f_1 is of regular (p, q)-φ relative Gol'dberg growth with respect to at least any one of g_1 or g_2. Therefore like Case I and in view of (2), we obtain for all sufficiently large values of R that

$$M_{g_1 \cdot g_2, D}(R) \leq M_{g_1, D}(R) \cdot M_{g_2, D}(R)$$

$$\begin{aligned}
\Rightarrow \quad & M_{g_1 \cdot g_2, D}(\exp^{[p-1]}\{(\overline{\sigma}^{(p,q)}_{g_1, D}(f_1, \varphi) - \varepsilon)\left[\log^{[q-1]} \varphi(R)\right]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\}) \\
\leq \quad & M_{g_1, D}(\exp^{[p-1]}\{(\overline{\sigma}^{(p,q)}_{g_1, D}(f_1, \varphi) - \varepsilon)\left[\log^{[q-1]} \varphi(R)\right]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\}) \\
& \cdot M_{g_2, D}(\exp^{[p-1]}\{(\overline{\sigma}^{(p,q)}_{g_1, D}(f_1, \varphi) - \varepsilon)\left[\log^{[q-1]} \varphi(R)\right]^{\rho^{(p,q)}_{g_1}(f_1,\varphi)}\})
\end{aligned}$$

Now using the similar technique for all sufficiently large values of R as explored in the proof of Case III, one can easily verify that

$\overline{\sigma}^{(p,q)}_{g_1\cdot g_2,D}(f_1,,\varphi)=\overline{\sigma}^{(p,q)}_{g_1,D}(f_1,\varphi)$ under the conditions specified in the theorem. Likewise, if we consider $\rho^{(p,q)}_{g_1}(f_1,\varphi)>\rho^{(p,q)}_{g_2}(f_1,\varphi)$ with at least f_1 is of regular (p,q)-φ relative Gol'dberg growth with respect to g_1, then one can verify that $\overline{\sigma}^{(p,q)}_{g_1\cdot g_2,D}(f_1,\varphi)=\overline{\sigma}^{(p,q)}_{g_2,D}(f_1,\varphi)$. Therefore the second part of theorem follows from Case III and Case IV.

Proof of the third part of the Theorem is omitted as it can be carried out in view of Theorem 2.11 and the above cases. □

THEOREM 3.8. *Let $f_1(z)$, $f_2(z)$, $g_1(z)$ and $g_2(z)$ be any four entire functions of n- complex variables and D be a bounded complete n-circular domain with center at origin in $\mathbb{C}^n$. Also let $\lambda^{(p,q)}_{g_1}(f_1,\varphi)$, $\lambda^{(p,q)}_{g_1}(f_2,\varphi)$, $\lambda^{(p,q)}_{g_2}(f_1,\varphi)$ and $\lambda^{(p,q)}_{g_2}(f_2,\varphi)$ are all non zero and finite.*
(A) Assume the functions f_1,f_2 and g_1 satisfy the following conditions:
(i) At least f_1 or f_2 is of regular (p,q)-φ relative Gol'dberg growth with respect to g_1 and g_1 satisfy the Property (G) and
(ii) f_1 and f_2 satisfy Property (X); then

$$\tau^{(p,q)}_{g_1,D}(f_1\cdot f_2,\varphi)=\tau^{(p,q)}_{g_1,D}(f_i,\varphi)\ \text{ and }\ \overline{\tau}^{(p,q)}_{g_1,D}(f_1\cdot f_2,\varphi)=\overline{\tau}^{(p,q)}_{g_1,D}(f_i,\varphi).$$

(B) Assume the functions g_1,g_2 and f_1 satisfy the following conditions:
(i) f_1 satisfy the Property (G) and
(ii) g_1 and g_2 satisfy Property (X); then

$$\tau^{(p,q)}_{g_1\cdot g_2,D}(f_1,\varphi)=\tau^{(p,q)}_{g_i,D}(f_1,\varphi)\ \text{ and }\ \overline{\tau}^{(p,q)}_{g_1\cdot g_2,D}(f_1,\varphi)=\overline{\tau}^{(p,q)}_{g_i,D}(f_1,\varphi).$$

(C) Assume the functions f_1,f_2, g_1 and g_2 satisfy the following conditions:
(i) $g_1\cdot g_2$, f_1 and f_2 are satisfy the Property (G);
(ii) f_1 and f_2 satisfy Property (X);
(iii) g_1 and g_2 satisfy Property (X);
(iv) At least f_1 or f_2 is of regular (p,q)-φ relative Gol'dberg growth with respect to g_1 for $i=1,2$, $j=1,2$ and $i\neq j$;
(v) At least f_1 or f_2 is of regular (p,q)-φ relative Gol'dberg growth with respect to g_2 for $i=1,2$, $j=1,2$ and $i\neq j$;
(vi) $\lambda^{(p,q)}_{g_m}(f_l,\varphi)=\min[\max\{\lambda^{(p,q)}_{g_1}(f_1,\varphi),\lambda^{(p,q)}_{g_1}(f_2,\varphi)\}$, $\max\{\lambda^{(p,q)}_{g_2}(f_1,\varphi),\lambda^{(p,q)}_{g_2}(f_2,\varphi)\}]\mid l,m=1,2$; then

$$\tau^{(p,q)}_{g_1\cdot g_2,D}(f_1\cdot f_2,\varphi)=\tau^{(p,q)}_{g_m,D}(f_l,\varphi)\ \text{ and }\overline{\tau}^{(p,q)}_{g_1\cdot g_2,D}(f_1\cdot f_2,\varphi)=\overline{\tau}^{(p,q)}_{g_m,D}(f_l,\varphi).$$

We omit the proof of Theorem 3.8 as it is a natural consequence of Theorem 3.7 and Theorem.

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