SOME REMARKS ON THE SUBORDINATION PRINCIPLE FOR ANALYTIC FUNCTIONS CONCERNED WITH ROGOSINSKI'S LEMMA

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Abstract. In this paper, we present a Schwarz lemma at the boundary for analytic functions at the unit disc, which generalizes classical Schwarz lemma for bounded analytic functions. For new inequalities, the results of Rogosinski's lemma, Subordination principle and Jack's lemma were used.

1. Introduction

Denote by $U = \{z : |z| < 1\}$ the unit disc in the complex plane $\mathbb{C}$ and let $w : U \to U$ be an analytic function with $w(0) = 0$. The Schwarz lemma tells us that $|w(z)| \leq |z|$ for all $z \in U$ and $|w'(0)| \leq 1$. In addition, if the equality $|w(z)| = |z|$ holds for any $z \neq 0$, or $|w'(0)| = 1$, then w is a rotation; that is $w(z) = ze^{i\theta}$, θ real ([5], p.329). A sharpened version of this is Rogosinski's Lemma [9], which say that for all $z \in U$

$$|w(z) - b_1| \leq r_1,$$

where

$$b_1 = \frac{zw'(0)\left(1 - |z|^2\right)}{1 - |z|^2 |w'(0)|^2} \quad and \quad r_1 = \frac{|z|^2 \left(1 - |w'(0)|^2\right)}{1 - |z|^2 |w'(0)|^2}.$$

Schwarz lemma has several applications in the field of electrical and electronics engineering. Use of positive real function and boundary analysis of these functions for circuit synthesis can be given as an exemplary application of the Schwarz lemma in electrical engineering. Furthermore, it is also used for analysis of transfer functions in control engineering and nulti-notch filter design in signal processing [12, 13].

We will use of the following definition and lemma to prove our result [5, 6].

Definition 1.1 (Subordination Principle). Let f and g be analytic functions in U. A function f is said to be *subordinate* to g, written as $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, analytic in U with $w(0) = 0$, $|w(z)| < 1$ such that $f(z) = g(w(z))$.

LEMMA 1.2 (Jack's Lemma). *Let $w(z)$ be a non-constant anaytic function in U with $w(0) = 0$. If*

$$|w(z_0)| = \max\left\{|w(z)| : |z| \le |z_0|\right\},$$

then there exists a real number $k \ge 1$ such that

$$\frac{z_0 w'(z_0)}{w(z_0)} = k.$$

Let $\mathcal{A}$ denote the class of functions $f(z) = 1 + a_1 z + a_2 z^2 + \dots$ that are analytic in U. Also, let $\mathcal{M}$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ satisfying

$$f(z) - \frac{1}{\alpha} z f'(z) \prec 1 + z, \ 0 < \alpha < 1, \ z \in U.$$

The certain anaytic functions which is in the class of $\mathcal{M}$ on the unit disc U are considered in this paper. The subject of the present paper is to discuss some properties of the function $f(z)$ which belongs to the class of $\mathcal{M}$ by applying Jack's Lemma and Rogosinski's Lemma.

Suppose that $f(z) \in \mathcal{M}$ and consider the following function

$$\phi(z) = \frac{1-\alpha}{\alpha} \left(f(z) - 1\right).$$

It is an analytic function in U and $\phi(0) = 0$. Now, let us show that $|\phi(z)| < 1$ in U. We suppose that there exists a $z_0 \in U$ such that

$$\max_{|z| \le |z_0|} |\phi(z)| = |\phi(z_0)| = 1.$$

From Jack's lemma, we obtain

$$\phi(z_0) = e^{i\theta} \text{ and } \frac{z_0 \phi'(z_0)}{\phi(z_0)} = k.$$

Therefore, we have that

$$
\begin{aligned}
\left|f(z_0) - \frac{1}{\alpha} z_0 f'(z_0) - 1\right| &= \left|1 + \frac{\alpha}{1-\alpha}\phi(z_0) - \frac{1}{\alpha} z_0 \frac{\alpha}{1-\alpha}\phi'(z_0) - 1\right| \\
&= \frac{\alpha}{1-\alpha}\left|e^{i\theta} - \frac{1}{\alpha}ke^{i\theta}\right| \\
&= \frac{\alpha}{1-\alpha}\left|1 - \frac{k}{\alpha}\right| \\
&\ge \frac{\alpha}{1-\alpha}\left(\frac{1}{\alpha} - 1\right) \\
&= 1.
\end{aligned}
$$

This contradicts the $f(z) \in \mathcal{M}$. This means that there is no point $z_0 \in U$ such that $\max_{|z| \le |z_0|} |\phi(z)| = |\phi(z_0)| = 1$. Hence, we take $|\phi(z)| < 1$ in U. From the Schwarz lemma, we obtain

$$
\begin{aligned}
\phi(z) &= \frac{1-\alpha}{\alpha}\left(f(z) - 1\right) \\
&= \frac{1-\alpha}{\alpha}\left(1 + a_1 z + a_2 z^2 + \dots - 1\right) \\
&= \frac{1-\alpha}{\alpha}\left(a_1 z + a_2 z^2 + \dots\right),
\end{aligned}
$$

$$\frac{\phi(z)}{z} = \frac{1-\alpha}{\alpha}\left(a_1 + a_2 z + ...\right)$$

and

$$|a_1| \leq \frac{\alpha}{1-\alpha}.$$

We thus obtain the following lemma.

LEMMA 1.3. *If $f(z) \in \mathcal{M}$, then we have the inequality*

(1.1) $$|f'(0)| \leq \frac{\alpha}{1-\alpha}.$$

This result is sharp and the extremal function is

$$f(z) = \frac{1 - \alpha\,(1+z)}{1-\alpha}.$$

Since the area of applicability of Schwarz Lemma is quite wide, there exist many studies about it. Some of these studies, which is called the boundary version of Schwarz Lemma, are about being estimated from below the modulus of the derivative of the function at some boundary point of the unit disc. The boundary version of Schwarz Lemma is given as follows:

If w extends continuously to some boundary point c with $|c| = 1$, and if $|w(c)| = 1$ and $w'(c)$ exists, then $|w'(c)| \geq 1$, which is known as the Schwarz lemma on the boundary. In addition to conditions of the boundary Schwarz Lemma, if w fixes the point zero, that is $w(z) = c_1 z + c_2 z^2 + ...$, then the inequality

(1.2) $$|w'(c)| \geq \frac{2}{1 + |w'(0)|}.$$

is obtained [11]. Inequality (1.2) and its generalizations have important applications in geometric theory of functions and they are still hot topics in the mathematics literature [1–4, 7–14]. Mercer considers some Schwarz and Carathéodory inequalities at the boundary, as consequences of a lemma due to Rogosinski [9]. In addition, he obtain an new boundary Schwarz lemma , for analytic functions mapping the unit disk to itself [10].

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see, [15]).

LEMMA 1.4 (Julia-Wolff lemma). *Let w be an analytic function in U, $w(0) = 0$ and $w(U) \subset U$. If, in addition, the function w has an angular limit $w(c)$ at $c \in \partial U$, $|w(c)| = 1$, then the angular derivative $w'(c)$ exists and $1 \leq |w'(c)| \leq \infty$.*

2. Main Results

In this section, we discuss different versions of the boundary Schwarz lemma for $\mathcal{M}$ class. Assuming the existence of angular limit on a boundary point, we obtain some estimations from below for the moduli of derivatives of analytic functions from a certain class. We also show that these estimations are sharp.

THEOREM 2.1. *Let $f(z) \in \mathcal{M}$. Assume that, for some $c \in \partial U$, f has an angular limit $f(c)$ at c, $f(c) = \frac{1}{1-\alpha}$. Then we have the inequality*

(2.1) $$|f'(c)| \geq \frac{\alpha}{1-\alpha}.$$

Moreover, the equality in (2.1) occurs for the function

$$f(z) = \frac{1 - \alpha\,(1 + z)}{1 - \alpha}.$$

Proof. Let

$$\phi(z) = \frac{1 - \alpha}{\alpha}\,(f(z) - 1).$$

$\phi(z)$ is an analytic function in U, $\phi(0) = 0$ and $|\phi(z)| < 1$ for $z \in U$. Also, we take $|\phi(c)| = 1$ for $c \in \partial U$ and $f(c) = \frac{1}{1-\alpha}$. Therefore, from Schwarz lemma, we obtain $|\phi(z)| \leq |z|$ for $z \in U$ and

$$\left| \frac{\phi(z) - 1}{|z| - 1} \right| \geq \frac{1 - |\phi(z)|}{1 - |z|} \geq \frac{1 - |z|}{1 - |z|} = 1.$$

Without loss of generality, we will assume that $c = 1$. Passing to angular limit in the last equality yields

$$|\phi'(1)| \geq 1$$

and

$$|f'(1)| \geq \frac{\alpha}{1 - \alpha}.$$

Now, we shall show that the inequality (2.1) is sharp. Let

$$f(z) = \frac{1 - \alpha\,(1 + z)}{1 - \alpha}.$$

Then

$$f'(z) = \frac{-\alpha}{1 - \alpha}$$

and

$$|f'(1)| = \frac{\alpha}{1 - \alpha}.$$

$\square$

The inequality (2.1) can be strengthened as below by taking into account a_1 which is second coefficient in the expansion of the function $f(z) = 1 + a_1 z + a_2 z^2 + \dots$.

THEOREM 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$(2.2) \qquad |f'(c)| \geq \frac{1}{1 - \alpha}\left(\frac{2\alpha^2}{\alpha + (1 - \alpha)\,|f'(0)|} \right).$$

The inequality (2.2) is sharp with equality for the function

$$f(z) = \frac{1 - \alpha\,(1 + z)}{1 - \alpha}.$$

Proof. Let $\phi(z)$ be the same as in the proof of Theorem 2.1. So, from Rogosinski's lemma, we obtain

$$|\phi(z) - b_1| \leq r_1,$$

where

$$b_1 = \frac{z\phi'(0)\,\left(1 - |z|^2\right)}{1 - |z|^2\,|\phi'(0)|^2}, \quad r_1 = \frac{|z|^2\,\left(1 - |\phi'(0)|^2\right)}{1 - |z|^2\,|\phi'(0)|^2}.$$

Without loss of generality, we will assume that $c = 1$. Thus, we obtain

$$\left| \frac{\phi(z) - 1}{z - 1} \right| \geq \frac{1 - |b_1| - r_1}{1 - |z|} = \frac{1 - \frac{|z||\phi'(0)|\left(1 - |z|^2\right)}{1 - |z|^2|\phi'(0)|^2} - \frac{|z|^2\left(1 - |\phi'(0)|^2\right)}{1 - |z|^2|\phi'(0)|^2}}{1 - |z|}$$

$$= \frac{1 - |z|^2 |\phi'(0)|^2 - |z| |\phi'(0)| \left(1 - |z|^2\right) - |z|^2 \left(1 - |\phi'(0)|^2\right)}{(1 - |z|) \left(1 - |z|^2 |\phi'(0)|^2\right)}$$

$$= \frac{\left(1 - |z|^2\right) \left(^2 \left(1 - |z| |\phi'(0)|\right)\right)}{(1 - |z|) \left(1 - |z|^2 |\phi'(0)|^2\right)}$$

$$= \frac{1 + |z|}{1 + |z| |\phi'(0)|}.$$

Passing to the angular limit in the last inequality yields

$$|\phi'(1)| \geq \frac{2}{1 + |\phi'(0)|}.$$

Since

$$|\phi'(1)| = \frac{1 - \alpha}{\alpha} |f'(1)|$$

and

$$|\phi'(0)| = \frac{1 - \alpha}{\alpha} |f'(0)|$$

we get

$$|f'(1)| \geq \frac{1}{1 - \alpha} \left(\frac{2\alpha^2}{\alpha + (1 - \alpha) |f'(0)|} \right).$$

Now, we shall show that the inequality (2.2) is sharp. Let

$$f(z) = \frac{1 - \alpha (1 + z)}{1 - \alpha}.$$

Then, we take

$$|f'(1)| = \frac{-\alpha}{1 - \alpha}.$$

On the other hand, we obtain

$$1 + a_1 z + a_2 z^2 + \ldots = \frac{1 - \alpha (1 + z)}{1 - \alpha},$$

$$a_1 z + a_2 z^2 + \ldots = \frac{1 - \alpha (1 + z)}{1 - \alpha} - 1,$$

$$a_1 z + a_2 z^2 + \ldots = \frac{-\alpha z}{1 - \alpha}$$

and

$$a_1 + a_2 z + \ldots = \frac{-\alpha}{1 - \alpha}.$$

Passing to limit in the last equality yields $|a_1| = \frac{\alpha}{1 - \alpha}$. Thus, we obtain

$$\frac{1}{1 - \alpha} \left(\frac{2\alpha^2}{\alpha + (1 - \alpha) |f'(0)|} \right) = \frac{1}{1 - \alpha} \left(\frac{2\alpha^2}{\alpha + (1 - \alpha) \frac{\alpha}{1 - \alpha}} \right)$$

$$= \frac{\alpha}{1 - \alpha}.$$

$\square$

In the following theorem, inequality (2.2) has been strenghened by adding the consecutive terms $a_1 = f'(0)$ and $a_2 = \frac{f''(0)}{2!}$ of $f(z)$ function.

THEOREM 2.3. *Let $f(z) \in \mathcal{M}$. Assume that, for some $c \in \partial U$, f has an angular limit $f(c)$ at c, $f(c) = \frac{1}{1-\alpha}$. Then we have the inequality*

$$(2.3) \qquad |f'(c)| \geq \frac{\alpha}{1-\alpha}\left(1 + \frac{4\left(\alpha - (1-\alpha)\,|f''(0)|\right)^2}{2\left(\alpha^2 - (1-\alpha)^2\,|f'(0)|^2\right) + \alpha\,(1-\alpha)\,|f''(0)|}\right).$$

Proof. Let $\phi(z)$ be the same as in the proof of Theorem 2.1. Let us consider the function

$$l(z) = \frac{\phi(z)}{z}$$

and

$$u(z) = \frac{l(z) - l(0)}{1 - \overline{l(0)}l(z)}$$

The function $u(z)$ is analytic in U, $u(0) = 0$, $|u(z)| < 1$ for $|z| < 1$ and

$$u'(0) = \frac{l'(0)}{\left(1 - |l(0)|^2\right)} = \frac{\phi''(0)}{2\left(1 - |\phi'(0)|^2\right)}.$$

From Rogosinski's Lemma and [8,9], we have

$$(2.4) \qquad\qquad\qquad |\phi(z) - b_2| \leq r_2,$$

where

$$b_2 = \frac{z\,|\phi'(0)|\,(1-\beta^2)}{1 - \beta^2\,|\phi'(0)|^2}, \quad r_2 = \frac{\beta\,|z|\left(1 - |\phi'(0)|^2\right)}{1 - \phi^2\,|\phi'(0)|^2}, \quad \beta = |z|\,\frac{|z| + |u'(0)|}{1 + |z|\,|u'(0)|}.$$

Without loss of generality, we will assume that $c = 1$. So, from (2.4), we obtain

$$\left|\frac{\phi(z) - 1}{z - 1}\right| \geq \frac{1 - |b_2| - r_2}{1 - |z|} = \frac{1 - \frac{|z||\phi'(0)|\left(1-\beta^2\right)}{1-\beta^2|\phi'(0)|^2} - \frac{\beta|z|\left(1-|\phi'(0)|^2\right)}{1-\beta^2|\phi'(0)|^2}}{1 - |z|}$$

$$= \frac{1 - \beta^2\,|\phi'(0)|^2 - |z|\,|\phi'(0)|\,(1-\beta^2) - \beta\,|z|\left(1 - |\phi'(0)|^2\right)}{\left(1 - |z|\right)\left(1 - \beta^2\,|\phi'(0)|^2\right)}$$

$$= \frac{\left(1 - \beta\,|\phi'(0)|\right)\left(1 + \beta\,|\phi'(0)| - |z|\,|\phi'(0)| - \beta\,|z|\right)}{\left(1 - |z|\right)\left(1 - \beta^2\,|\phi'(0)|^2\right)}$$

$$= \frac{1 + \beta\,|\phi'(0)| - |z|\,|\phi'(0)| - \beta\,|z|}{\left(1 - |z|\right)\left(1 + \beta\,|\phi'(0)|\right)}.$$

Since $\beta = |z|\frac{|z|+|u'(0)|}{1+|z||u'(0)|}$, we take

$$\left|\frac{\phi(z)-1}{z-1}\right| \geq \frac{1 + |z|\frac{|z|+|u'(0)|}{1+|z||u'(0)|}|\phi'(0)| - |z||\phi'(0)| - |z|\frac{|z|+|u'(0)|}{1+|z||u'(0)|}|z|}{(1-|z|)\left(1 + |z|\frac{|z|+|u'(0)|}{1+|z||u'(0)|}|\phi'(0)|\right)}$$

$$= \frac{1 - |z|^3 + |z||u'(0)|(1-|z|) - |z||\phi'(0)|(1-|z|) + |z||\phi'(0)||u'(0)|(1-|z|)}{(1-|z|)\left(1 + |z||u'(0)| + |z|^2|\phi'(0)| + |z||\phi'(0)||u'(0)|\right)}$$

$$= \frac{\left(1 + |z| + |z|^2\right) + |z||u'(0)| - |z||\phi'(0)| + |z||u'(0)||\phi'(0)|}{1 + |z||u'(0)| + |z|^2|\phi'(0)| + |z||\phi'(0)||u'(0)|}.$$

Passing to the angular limit in the last inequality yields

$$|\phi'(1)| \geq \frac{3 + |u'(0)| - |\phi'(0)| + |u'(0)|\,|\phi'(0)|}{1 + |u'(0)| + |\phi'(0)| + |u'(0)|\,|\phi'(0)|}$$
$$= \frac{3 + |u'(0)| - |\phi'(0)| + |u'(0)|\,|\phi'(0)|}{(1 + |u'(0)|)\,(1 + |\phi'(0)|)}.$$

A little manipulation gives

$$|\varphi'(1)| \geq 1 + \frac{2\,(1 - |\phi'(0)|)^2}{(1 + |u'(0)|)\,(1 - |\phi'(0)|^2)}$$
$$= 1 + \frac{4\,(1 - |\phi'(0)|)^2}{2\,(1 - |\phi'(0)|^2) + |\phi''(0)|}.$$

Since

$$|\phi'(1)| = \frac{1 - \alpha}{\alpha}\,|f'(1)|,$$

$$|\phi'(0)| = \frac{1 - \alpha}{\alpha}\,|f'(0)|$$

and

$$|\phi''(0)| = \frac{1 - \alpha}{\alpha}\,|f'(0)|,$$

we obtain

$$|f'(1)| \geq \frac{\alpha}{1 - \alpha}\left(1 + \frac{4\,(\alpha - (1 - \alpha)\,|f''(0)|)^2}{2\,(\alpha^2 - (1 - \alpha)^2\,|f'(0)|^2) + \alpha\,(1 - \alpha)\,|f''(0)|}\right).$$

$$\square$$

If $f(z) - z$ a have zeros different from $z = 0$, taking into account these zeros, the inequality (2.3) can be strengthened in another way. This is given by the following Theorem.

THEOREM 2.4. *Let $f(z) \in \mathcal{M}$. Assume that, for some $c \in \partial U$, f has an angular limit $f(c)$ at c, $f(c) = \frac{1}{1-\alpha}$. Let $z_1, z_2, ..., z_n$ be zeros points of the function $f(z) - z$ in U that are different from zero. Then we have the inequality*

$$|f'(c)| \geq \frac{\alpha}{1-\alpha}\left(1 + \sum_{i=1}^{n} \frac{1 - |z_i|^2}{|c - z_i|^2}\right. \tag{2.5}$$

$$\left. + \frac{2\left(\alpha \prod_{i=1}^{n} |z_i| - ((1-\alpha)|a_2|)\right)^2}{\left(\alpha \prod_{i=1}^{n} |z_i|\right)^2 - ((1-\alpha)|a_2|)^2 + \alpha(1-\alpha) \prod_{i=1}^{n} |z_i|\left|a_2 + a_1 \sum_{i=1}^{n} \frac{1 - |z_i|^2}{z_i}\right|}\right).$$

Proof. Let $\phi(z)$ be as in the proof of Theorem 2.1 and $z_1, z_2, ..., z_n$ be zeros points of the function $f(z) - z$ in U that are different from zero. Let

$$B(z) = z \prod_{i=1}^{n} \frac{z - z_i}{1 - \overline{z_i} z}.$$

$B(z)$ is an analytic function in U and $|B(z)| < 1$ for $|z| < 1$. By the maximum principle for each $z \in U$, we have $|\phi(z)| \le |B(z)|$. Consider the function

$$\varphi(z) = \frac{\phi(z)}{B(z)} = \frac{\frac{1-\alpha}{\alpha}(f(z)-1)}{z\prod_{i=1}^{n}\frac{z-a_i}{1-\overline{a_i}z}}$$

$$= \frac{1-\alpha}{\alpha}\frac{a_1 z + a_2 z^2 + \ldots}{z\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}},$$

$$= \frac{1-\alpha}{\alpha}\frac{a_1 + a_2 z + \ldots}{\prod_{i=1}^{n}\frac{z-z_i}{1-\overline{z_i}z}}.$$

$u(z)$ is analytic in U and $|u(z)| < 1$ for $z \in U$. In particular, we have

$$|\varphi(0)| = \frac{1-\alpha}{\alpha}\frac{|a_1|}{\prod_{i=1}^{n}|z_i|}$$

and

$$|\varphi'(0)| = \frac{1-\alpha}{\alpha}\frac{\left|a_2 + a_1\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{\prod_{i=1}^{n}|z_i|}.$$

Moreover, with the simple calculations, we get

$$\frac{c\phi'(c)}{\phi(c)} = |\phi'(c)| \ge |B'(c)| = \frac{cB'(c)}{B(c)}$$

and

$$|B'(c)| = 1 + \sum_{i=1}^{n}\frac{1-|z_i|^2}{|c-z_i|^2}.$$

The auxiliary function

$$t(z) = \frac{\varphi(z)-\varphi(0)}{1-\overline{\varphi(0)}\varphi(z)}$$

is analytic in the unit disc U, $t(0) = 0$, $|t(z)| < 1$ for $z \in U$ and $|t(c)| = 1$ for $c \in \partial U$. From (1.2), we obtain

$$\frac{2}{1+|t'(0)|} \le |t'(c)| = \frac{1+|\varphi(0)|^2}{\left|1-\overline{\varphi(0)}\varphi(c)\right|^2}|\varphi'(c)|$$

$$\le \frac{1+|\varphi(0)|}{1-|\varphi(0)|}\{|\phi'(c)| - |B'(c)|\}.$$

Since

$$|t'(0)| = \frac{|\varphi'(0)|}{1-|\varphi(0)|^2} = \frac{\frac{1-\alpha}{\alpha}\frac{\left|a_2+a_1\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{\prod_{i=1}^{n}|z_i|}}{1-\left(\frac{1-\alpha}{\alpha}\frac{|a_1|}{\prod_{i=1}^{n}|z_i|}\right)^2}$$

$$= \alpha(1-\alpha)\prod_{i=1}^{n}|z_i|\frac{\left|a_2+a_1\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{\left(\alpha\prod_{i=1}^{n}|z_i|\right)^2-((1-\alpha)|a_2|)^2},$$

we get

$$\frac{2}{1+\alpha(1-\alpha)\prod_{i=1}^{n}|z_i|\frac{\left|a_2+a_1\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}{\left(\alpha\prod_{i=1}^{n}|z_i|\right)^2-((1-\alpha)|a_2|)^2}}$$

$$\leq \frac{1+\frac{1-\alpha}{\alpha}\frac{|a_1|}{\prod_{i=1}^{n}|z_i|}}{1-\frac{1-\alpha}{\alpha}\frac{|a_1|}{\prod_{i=1}^{n}|z_i|}}\left\{\frac{1-\alpha}{\alpha}|f'(c)|-1-\sum_{i=1}^{n}\frac{1-|z_i|^2}{|c-z_i|^2}\right\},$$

$$\frac{2\left[\left(\alpha\prod_{i=1}^{n}|z_i|\right)^2-((1-\alpha)|a_2|)^2\right]}{\left(\alpha\prod_{i=1}^{n}|z_i|\right)^2-((1-\alpha)|a_2|)^2+\alpha(1-\alpha)\prod_{i=1}^{n}|z_i|\left|a_2+a_1\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}$$

$$\leq \frac{\alpha\prod_{i=1}^{n}|z_i|+(1-\alpha)|a_1|}{\alpha\prod_{i=1}^{n}|z_i|-(1-\alpha)|a_1|}\left\{\frac{1-\alpha}{\alpha}|f'(c)|-1-\sum_{i=1}^{n}\frac{1-|z_i|^2}{|c-z_i|^2}\right\},$$

$$\frac{2\left[\left(\alpha\prod_{i=1}^{n}|z_i|\right)-((1-\alpha)|a_2|)\right]^2}{\left(\alpha\prod_{i=1}^{n}|z_i|\right)^2-((1-\alpha)|a_2|)^2+\alpha(1-\alpha)\prod_{i=1}^{n}|z_i|\left|a_2+a_1\sum_{i=1}^{n}\frac{1-|z_i|^2}{z_i}\right|}$$

$$\leq \frac{1-\alpha}{\alpha}|f'(c)|-1-\sum_{i=1}^{n}\frac{1-|z_i|^2}{|c-z_i|^2}$$

and

$$|f''(c)| \geq \frac{\alpha}{1-\alpha} \left(1 + \sum_{i=1}^{n} \frac{1 - |z_i|^2}{|c - z_i|^2} \right.$$

$$\left. + \frac{2 \left(\alpha \prod_{i=1}^{n} |z_i| - ((1-\alpha)|a_2|) \right)^2}{\left(\alpha \prod_{i=1}^{n} |z_i| \right)^2 - ((1-\alpha)|a_2|)^2 + \alpha (1-\alpha) \prod_{i=1}^{n} |z_i| \left| a_2 + a_1 \sum_{i=1}^{n} \frac{1-|z_i|^2}{z_i} \right|} \right).$$

$$\square$$

THEOREM 2.5. *Let $f(z) \in \mathcal{M}$. Assume that, for $1 \in \partial U$, f has an angular limit $f(1)$ at 1, $f(1) = \frac{1}{1-\alpha}$. Then we have the inequality*

$$(2.6) \quad f'(1) \geq \frac{\alpha}{1-\alpha} \left(1 + \frac{|\alpha - (1-\alpha)f'(0)|^2}{\alpha^2 - |(1-\alpha)f'(0)|^2} \frac{2}{1 + \Re \left(\frac{\alpha - (1-\alpha)\overline{f'(0)}}{\alpha - (1-\alpha)f'(0)} \frac{\frac{1-\alpha}{2\alpha} f''(0)}{1 - \left| \frac{1-\alpha}{\alpha} f'(0) \right|^2} \right)} \right).$$

Proof. Let $\phi(z)$ be as in the proof of Theorem 2.1. So, from the hypothesis, we have

$$\phi(1) = \frac{1-\alpha}{\alpha} (f(1) - 1) = \frac{1-\alpha}{\alpha} \left(\frac{1}{1-\alpha} - 1 \right) = 1$$

and

$$\phi(1) = 1,$$

where 1 is a boundary fixed point of $\phi(z)$. Also, we have

$$\phi(z) = \frac{1-\alpha}{\alpha} (f(z) - 1) = \frac{1-\alpha}{\alpha} \left(a_1 z + a_2 z^2 + ... \right)$$
$$= d_1 z + d_2 z^2 + d_3 z^3 + ...$$

Consider the function

$$\Gamma(z) = \frac{1 - \overline{d_1}}{d_1 - 1} \frac{d_1 z - \phi(z)}{z - \overline{d_1} \phi(z)}.$$

$\Gamma(z)$ is analytic in U and $|\Gamma(z)| < 1$ for $z \in U$ and 1 is a boundary fixed point of $\Gamma(z)$. That is, $\Gamma(1) = 1$. Also, with the simple calculations, we obtain

$$\Gamma'(1) = \frac{1 - |d_1|^2}{|1 - d_1|^2} (\phi'(1) - 1).$$

On the other hand, we take

$$\Gamma'(0) = \frac{1 - \overline{d_1}}{1 - d_1} \frac{d_2}{1 - |d_1|^2}.$$

In particular, from (1.2), we have

$$(2.7) \quad \Gamma'(1) \geq \frac{2}{1 + \Re \Gamma'(0)}.$$

Let us substitute the values of $\Gamma'(1)$ and $\Gamma'(0)$ into (2.7). Therefore, we take

$$\frac{1-|d_1|^2}{|1-d_1|^2}\left(\phi'(1)-1\right) \geq \frac{2}{1+\Re\left(\frac{1-\overline{d_1}}{1-d_1}\frac{d_2}{1-|d_1|^2}\right)}$$

and

$$\phi'(1) \geq 1+\frac{|1-d_1|^2}{1-|d_1|^2}\frac{2}{1+\Re\left(\frac{1-\overline{d_1}}{1-d_1}\frac{d_2}{1-|d_1|^2}\right)}.$$

Since

$$\phi'(1)=\frac{1-\alpha}{\alpha}\left|f'(1)\right|,\ d_1=\frac{1-\alpha}{\alpha}f'(0),\ d_2=\frac{1-\alpha}{2\alpha}f''(0),$$

we obtain

$$f'(1) \geq \frac{\alpha}{1-\alpha}\left(1+\frac{\left|1-\frac{1-\alpha}{\alpha}f'(0)\right|^2}{1-\left|\frac{1-\alpha}{\alpha}f'(0)\right|^2}\frac{2}{1+\Re\left(\frac{1-\overline{\frac{1-\alpha}{\alpha}f'(0)}}{1-\frac{1-\alpha}{\alpha}f'(0)}\frac{\frac{1-\alpha}{2\alpha}f''(0)}{1-\left|\frac{1-\alpha}{\alpha}f'(0)\right|^2}\right)}\right)$$

and

$$f'(1) \geq \frac{\alpha}{1-\alpha}\left(1+\frac{|\alpha-(1-\alpha)f'(0)|^2}{\alpha^2-|(1-\alpha)f'(0)|^2}\frac{2}{1+\Re\left(\frac{\alpha-(1-\alpha)\overline{f'(0)}}{\alpha-(1-\alpha)f'(0)}\frac{\frac{1-\alpha}{2\alpha}f''(0)}{1-\left|\frac{1-\alpha}{\alpha}f'(0)\right|^2}\right)}\right).$$

$\square$

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