Korean J. Math. **29** (2021), No. 2, pp. 267–269
http://dx.doi.org/10.11568/kjm.2021.29.2.267

ON THE REDUCTION OF AN IWASAWA MODULE

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Abstract. A finitely generated torsion module M for $\mathbb{Z}_p[[T, T_2, \cdots, T_d]]$ is pseudo-null if M/TM is pseudo-null over $\mathbb{Z}_p[[T_2, \cdots, T_d]]$. This result is used as a tool to prove the generalized Greenberg's conjecture in certain cases. The converse may not be true. In this paper, we give examples of pseudo-null Iwasawa modules whose reduction are not pseudo-null.

1. Introduction

Fix a prime number p and let k be a number field. Suppose that K_d is a $\mathbb{Z}_p^d$-extension of k, so $K_d = \cup_{n\geq 0} k_n$ with $k_n \subset k_{n+1}$ and $Gal(k_n/k) \simeq (\mathbb{Z}/p^n\mathbb{Z})^d$. Denote by L_n the p-Hilbert class field of k_n and write $L_{K_d} = \cup_{n\geq 0} L_n$. Then let

$$Y_{K_d} = Gal(L_{K_d}/K_d).$$

It is well-known that Y_{K_d} is a finitely generated torsion module for $\Lambda_d = \mathbb{Z}_p[[Gal(K_d/k)]]$. A finitely generated torsion Λ_d-module M is called pseudo-null(written by $M \sim 0$) if M has two relatively prime annihilators in Λ_d. Denote by $\tilde{k}$ the composite of all $\mathbb{Z}_p$-extensions of k. Generalized Greenberg's conjecture claims that $Y_{\tilde{k}} \sim 0$. In certain cases, generalized Greenberg's conjecture is proved by some authors [1, 3]. In those cases, the following theorem is a basic tool to attack the conjecture:

$$\text{If } Y_{K_d}/TY_{K_d} \sim 0 \text{ then } Y_{K_d} \sim 0$$

Here Y_{K_d}/TY_{K_d} is viewed as a $\mathbb{Z}_p[[Gal(K_{d-1}/k)]]$-module where $k \subset K_{d-1} \subset K_d$, γ is a topological generator of $Gal(K_d/K_{d-1})$, and $T = \gamma - 1$. In this paper, we give explicit number fields k such that the converse of the above theorem does not hold. In other words, we give examples of k such that

$$Y_{K_d} \sim 0, \quad \text{but} \quad Y_{K_d}/TY_{K_d} \not\sim 0.$$

2. Proof of Theorems

Denote by k_c the cyclotomic $\mathbb{Z}_p$-extension of a number field k. When k is an imaginary quadratic field, a theorem of Minardi assures us to find easily k which we are looking for.

Received May 15, 2020. Revised January 13, 2021. Accepted April 1, 2021.
2010 Mathematics Subject Classification: 11R23.
Key words and phrases: Iwasawa invariants, generalized Greenberg conjecture, bi-quadratic fields.

Theorem 2.1. *Let k be an imaginary quadratic field with the class number h_k not divisible by p. Moreover assume that $\lambda_p(k) \geq 1$ when only one prime of k exists above p or $\lambda_p(k) \geq 2$ when p splits in k. Then the cyclotomic $\mathbb{Z}_p$-extension k_c satisfies the followings:*

$$(1) Y_{\tilde{k}} \sim 0$$

$$(2) Y_{\tilde{k}}/TY_{\tilde{k}} \not\sim 0$$

where γ is a topological generator of $Gal(\tilde{k}/k_c)$.

Proof. By Minardi [3, Proposition 3.A], since $p \nmid h_k$, we see that

$$Y_{\tilde{k}} \sim 0.$$

Note that the fixed field of $TY_{\tilde{k}}$ is the maximal subfield L_0 of $L_{\tilde{k}}$ which is abelian over k_c and $Y_{\tilde{k}}/TY_{\tilde{k}} \simeq Gal(L_0/\tilde{k})$. By the assumption on λ-invariant, $Y_{\tilde{k}}/TY_{\tilde{k}}$ is not finite, i.e., not pseudo-null. □

Example 1. When $k = \mathbb{Q}(\sqrt{-3})$ and $p = 13$, $p \nmid h_k$, p splits in k and $\lambda_p = 2$.

Next, we give an example of k with $[k : \mathbb{Q}] > 2$. We state theorems needed for our construction. When k is a real quadratic field, Taya gives a necessary and sufficient condition for triviality of Y_{k_c}.

Theorem 2.2. *(= [4, Theorem1]) Let d be a square-free integer with $d \equiv 1$ (mod 3) and $d > 0$. Put $k+ = \mathbb{Q}(\sqrt{d})$ and $k- = \mathbb{Q}(\sqrt{-3d})$. For the cyclotomic $\mathbb{Z}_3$-extension $k+_c$ of $k+$, denoted by $k+_n$ the n-th layer in $k+_c/k+$ and by $A+_n$ the 3-Sylow subgroup of the ideal class group of $k+_n$. Then $A+_n$ is trivial for all integers $n \geq 0$ if and only if the class number h_{k-} of $k-$ is not divisible by 3.*

Fujii proves that generalized Greenberg conjecture holds for certain CM fields.

Theorem 2.3. *(= [1, Theorem1]) Let k be a CM-field of degree greater than or equal to 4. Let p be an odd prime which splits completely in $k/\mathbb{Q}$. Suppose that Leopoldt's conjecture holds for p and k^+, $p \nmid h_k$ and that all of Iwasawa invariants of the cyclotomic $\mathbb{Z}_p$-extension of k^+ are trivial. Then $Y_{\tilde{k}}$ is pseudo-null.*

For $i \geq 2$, if $Y_{K_{i+1}} \sim 0$, then $Y_{K_i} \sim 0$ for infinitely many subextensions $K_i \subset K_{i+1}$(See [2]). Here we need more subtle theorem for our purpose.

Theorem 2.4. *(= [3, Corollary 2 in chapter 4]) Suppose that k is a complex abelian extension of $\mathbb{Q}$ with $[k : \mathbb{Q}] > 2$. If $Y_{\tilde{k}} \sim 0$, then there is an infinite number of $\mathbb{Z}_p^2$-extensions K/k with $k_c \subset K$ and $Y_K \sim 0$.*

Now, by following idea of Minardi [3], we prove that $k = \mathbb{Q}(\sqrt{7}, \sqrt{-2})$ is the desired number field. From now on $p = 3$.

Theorem 2.5. *Let $k = \mathbb{Q}(\sqrt{7}, \sqrt{-2})$. Then there exists a $\mathbb{Z}_p^2$-extension K_2 of k satisfying the followings:*

$$(1) k \subset K_1 \subset K_2$$

$$(2) Y_{K_2} \sim 0$$

$$(3) Y_{K_2}/TY_{K_2} \not\sim 0$$

where γ is a topological generator of $Gal(K_2/K_1)$.

Proof. Note that $\tilde{k}$ is a $\mathbb{Z}_p^3$-extension of k. Denote by $k+ = \mathbb{Q}(\sqrt{7})$ the maximal real subfield of k. By Theorem 2.2, $A+_n$ is trivial for all integers $n \geq 0$ since the class number $h_{k-} = h_{\mathbb{Q}}(\sqrt{-21})$ is 4. The quadratic subfields of k are $\mathbb{Q}(\sqrt{7}), \mathbb{Q}(\sqrt{-2}), \mathbb{Q}(\sqrt{-14})$. The prime p splits completely in each quadratic subfields of k, hence p splits completely in k The product of class numbers of quadratic subfields is 4, so h_k is not divisible by p. Therefore, by Theorem 2.2 and Theorem 2.3, we see that

$$Y_{\tilde{k}} \sim 0.$$

By Theorem 2.4, we can choose a $\mathbb{Z}_p^2$-extension K_2/k with $K_1(= k_c) \subset K_2$ and $Y_{K_2} \sim 0$. Since p splits completely in k and primes above p are totally ramified in K_1/k, the extension $\tilde{k}/K_1$ is unramified everywhere. Therefore the fixed field of TY_{K_2} contains $\tilde{k}$. So Y_{K_2}/TY_{K_2} is not finite, i.e., not pseudo-null. This completes the proof. □

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