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SHARPENING LOWER BOUND IN SOME INEQUALITIES FOR FRAMES IN HILBERT SPACES

Fahimeh Sultanzadeh, Mahmood Hassani, Mohsen Erfanian Omidvar, and Rajab Ali kamyabi Gol

Abstract. This paper aims to present a new lower bound for some inequalities related to Frames in Hilbert space. Some refinements of the inequalities for general frames and alternate dual frames under suitable conditions are given. These results refine the remarkable results obtained by Balan et al. and Gavruta.

1. Introduction and preliminary

Frame theory was introduced by Duffin and Schaeffer [6] as part of their research in the non-harmonic Fourier series. Frames are useful in some areas such as signal and image processing, data compression , and sampling theory. Their main advantage is that frames can be designed to be redundant while still providing reconstruction formulas. Due to their numerical stability, Parseval frames are of increasing interest in applications $[5, 7, 12]$. Let $(\mathbb{H}, \langle\cdot,\cdot\rangle)$ be a separable Hilbert space. We denote by $L(\mathbb{H})$ the algebra of all linear operators on $\mathbb{H}$. The space $l^2(I)$ is the set of $\{a_i\}_{i\in I}$ such that $a_i \in \mathbb{C}$ and $\sum_{i\in I} |a_i|^2 < \infty$ when I is a finite or countable set.

A frame for $\mathbb{H}$ is a family of vectors $F = \{f_i\}_{i\in I}$ in $\mathbb{H}$ which satisfies

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|^2, \quad \text{for every } f \in \mathbb{H}, \tag{1.1}$$

for positive constants $0 < A \leq B$. The optimal constants (maximal for A and minimal for B) are known as the upper and lower frame bounds, respectively. If $A = B$, then this frame is called an A-tight frame, and if $A = B = 1$, then it is called a Parseval frame.

If a family of vectors $F = \{f_i\}_{i\in I}$ satisfies the upper bound condition in (1.1), we call F a Bessel family. Associated with each frame $F = \{f_i\}_{i\in I}$, there are three linear and bounded operators:

$T : l^2(I) \to \mathbb{H}$, $\quad Tx = \sum_{i\in I}\langle x, e_i\rangle f_i,$ $\quad$ (synthesis operator)
$T^* : \mathbb{H} \to l^2(I),$ $\quad T^*(f) = \{\langle f, f_i\rangle\}_{i\in I},$ $\quad$ (analysis operator)
$S : \mathbb{H} \to \mathbb{H}$, $\quad Sf = TT^*f = \sum_{i\in I}\langle f, f_i\rangle f_i,$ $\quad$ (frame operator)
where $\{e_i\}_{i\in I}$ is the standard orthonormal basis of $l^2(I)$. The inequalities in (1.1)

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imply that S is a (positive) self-adjoint invertible operator, and it allows reconstruction of each vector $f \in \mathbb{H}$ in terms of the family F as follows:

$$f = \sum_{i\in I}\langle f, S^{-1}f_i\rangle f_i = \sum_{i\in I}\langle f, f_i\rangle S^{-1}f_i.$$

If F is a Parseval frame, that is, $S = id$, then the reconstruction formula resembles the Fourier series of f associated to an orthonormal basis $B = \{b_j\}_{j\in J}$ of $\mathbb{H}$:

$$f = \sum_{j\in J}\langle f, b_j\rangle b_j,$$

but the frame coefficients $\{\langle f, f_i\rangle\}_{i\in I}$ given by $F = \{f_i\}_{i\in I}$ allow us to reconstruct f even when some of these coefficients are corrupted; see [6].

The family $\{\tilde{f}_i\}_{i\in I}$, where $\tilde{f}_i = S^{-1}f_i, i \in I$, is also a frame for $\mathbb{H}$, called the canonical dual frame of the $F = \{f_i\}_{i\in I}$.

In general, the Bessel family $\{g_i\}_{i\in I}$ is called an alternative dual of the frame $F = \{f_i\}_{i\in I}$ if the following formula holds:

$$f = \sum_{i\in I}\langle f, g_i\rangle f_i, \quad \text{for all } f \in \mathbb{H}.$$

If $\{f_i\}_{i\in I}$ is a frame for $\mathbb{H}$, for every$J \subset I$, we define the operator

$$S_J f = \sum_{i\in J}\langle f, f_i\rangle f_i.$$

and denote $J^c = I \setminus J$. It follows that $S = S_J + S_{J^c}$. By this definition, it is clear that if $J_1 \subseteq J_2$, then $\|S_{J_1}f\| \leq \|S_{J_2}f\|$.

For more details we refer to [2–4, 8, 11]. In [1], Balan et al. proved the following identity for Parseval frames:

$$\sum_{i\in J}|\langle f, f_i\rangle|^2 - \|\sum_{i\in J}\langle f, f_i\rangle f_i\|^2 = \sum_{i\in J^c}|\langle f, f_i\rangle|^2 - \|\sum_{i\in J^c}\langle f, f_i\rangle f_i\|^2. \tag{1.2}$$

Moreover, in [1], the following inequality was obtained:

$$\frac{3}{4}\|f\|^2 \leq \sum_{i\in J}|\langle f, f_i\rangle|^2 + \|\sum_{i\in J^c}\langle f, f_i\rangle f_i\|^2. \tag{1.3}$$

See [9, 10] for further details. In fact, the identity (1.2) was obtained as a particular case from the following result for general frames:

$$\sum_{i\in J}|\langle f, f_i\rangle|^2 + \sum_{i\in I}|\langle S_{J^c}f, \tilde{f}_i\rangle|^2 = \sum_{i\in J^c}|\langle f, f_i\rangle|^2 + \sum_{i\in I}|\langle S_J f, \tilde{f}_i\rangle|^2. \tag{1.4}$$

Inequality (1.3) leads us to introduce, for a Parseval frame, the numbers

$$v_+(F;J) = \sup_{f\neq 0}\frac{\|\sum_{i\in J}\langle f, f_i\rangle f_i\|^2 + \sum_{i\in J^c}|\langle f, f_i\rangle|^2}{\|f\|^2},$$

$$v_-(F;J) = \inf_{f\neq 0}\frac{\|\sum_{i\in J}\langle f, f_i\rangle f_i\|^2 + \sum_{i\in J^c}|\langle f, f_i\rangle|^2}{\|f\|^2}.$$

Recall that $v_+(F;J)$ is called the upper index of F relative to J, and $v_-(F;J)$ the lower index of F relative to J.

Gavruta [9] presented the basic properties of these indexes.

Balan et al. [1] and Gavruta [9] established several identities and inequalities for Hilbert spaces frames. Zou and Jiang [13] presented a refinement of the well-known arithmetic-geometric mean inequality. In the present paper, we use this improved inequality in some inequalities for Parseval frames and get new inequalities. Thereafter, we show improvements of the inequalities for general frames. However, our main focus will be on Parseval frames because of their importance in applications, particularly signal processing. Finally, we obtain improvements of the inequalities for alternative dual frames too.

2. Main results

The results of this paper are organized as follows: first, we improve the left-hand-side of inequality (1.3). Thereafter in Lemma 2.5 improvements for self-adjoint operators are given, which we apply for general frames. Finally, in Theorems 2.9 and 2.10, we obtain the results for alternate dual frames.

The well-known arithmetic-geometric mean inequality says that if $a, b \geq 0$, then $\sqrt{ab} \leq \dfrac{a+b}{2}$. A refinement of this inequality is given in the following lemma:

Lemma 2.1. [13] *If $a, b \geq 0$, then*

$$(1+\frac{(\ln a-\ln b)^2}{8})\sqrt{ab} \leq \frac{a+b}{2}.$$

We will need the following important result from operator theory [1].

Lemma 2.2. *If $S, T \in L(\mathbb{H})$ satisfying $S+T=Id$, then $S-T=S^2-T^2$.*

Proof. The proof follows from

$$S-T=S-(id-S)=2S-id=S^2-(id-2S+S^2)=S^2-(id-S)^2=S^2-T^2.$$

□

Theorem 2.1. *If $\{f_i\}_{i\in I}$ is a Parseval frame for Hilbert space $\mathbb{H}$ with frame operator S, then, for every $\emptyset \neq J \subset I$, it follows that*

$$(\frac{3+\lambda}{4+\lambda})\|f\|^2 \leq \sum_{i\in J}|\langle f, f_i\rangle|^2+\|\sum_{i\in J^c}\langle f, f_i\rangle f_i\|^2, \quad \text{for } f\in\mathbb{H}, \tag{2.1}$$

where $\lambda=\inf_{f\in\mathbb{H}}(\ln\|S_Jf\|-\ln\|S_{J^c}f\|)^2$.

Proof. Since

$$\|f\|^2=\|S_Jf+S_{J^c}f\|^2 \leq \|S_Jf\|^2+\|S_{J^c}f\|^2+2\|S_Jf\|\|S_{J^c}f\|,$$

by applying Lemma 2.1, we have

$$\|f\|^2 \leq \|S_Jf\|^2+\|S_{J^c}f\|^2+\frac{\|S_Jf\|^2+\|S_{J^c}f\|^2}{1+\dfrac{(\ln\|S_Jf\|-\ln\|S_{J^c}f\|)^2}{2}}.$$

Put $\lambda=\inf_{f\in\mathbb{H}}(\ln\|S_Jf\|-\ln\|S_{J^c}f\|)^2$, then

$$\|f\|^2 \leq (\|S_Jf\|^2+\|S_{J^cf}\|^2)(1+\frac{2}{2+\lambda}),$$

and we have

$$\langle(\frac{2+\lambda}{4+\lambda})idf, f\rangle \leq \langle(S_J^2+S_{J^c}^2)f, f\rangle.$$

This implies that

$$(\frac{2+\lambda}{4+\lambda})id \leq S_J^2 + S_{J^c}^2.$$

So,

$$(\frac{2+\lambda}{4+\lambda}+1)id \leq S_J + S_{J^c}^2 + S_{J^c} + S_J^2.$$

Now by applying Lemma 2.2, it follows that

$$(\frac{3+\lambda}{4+\lambda})id \leq S_J + S_{J^c}^2.$$

Hence

$$(\frac{3+\lambda}{4+\lambda})\|f\|^2 \leq \langle S_J f, f\rangle + \langle S_{J^c} f, S_{J^c} f\rangle = \sum_{i\in J} |\langle f, f_i\rangle|^2 + \|\sum_{i\in J^c}\langle f, f_i\rangle f_i\|^2.$$

□

Note that for $\lambda = 0$, inequality (2.1) is the same as inequality (1.3) and that for $\lambda > 0$, (2.1) is an improvement of (1.3).

Corollary 2.1. *Let $F = \{f_i\}_{i\in I}$ be a Parseval frame and let $J \subset I$. Then*

$$\frac{3+\lambda}{4+\lambda} \leq v_-(F; J) \leq v_+(F; J) \leq 1;$$

where $\lambda = \inf_{f\in\mathbb{H}}(\ln\|S_J f\| - \ln\|S_{J^c} f\|)^2$.

Proof. By using Theorem 2.3 and that F is a Parseval frame, we have

$$(\frac{3+\lambda}{4+\lambda})\|f\|^2 \leq \sum_{i\in J} |\langle f, f_i\rangle|^2 + \|\sum_{i\in J^c}\langle f, f_i\rangle f_i\|^2 \leq \|f\|^2.$$

So

$$\frac{3+\lambda}{4+\lambda} \leq \frac{\sum_{i\in J} |\langle f, f_i\rangle|^2 + \|\sum_{i\in J^c}\langle f, f_i\rangle f_i\|^2}{\|f\|^2} \leq 1,$$

hence

$$\frac{3+\lambda}{4+\lambda} \leq v_-(F; J) \leq v_+(F; J) \leq 1.$$

□

In the following lemma, we give an improvement of the inequality proved in [9, Theorem 2.1], under some conditions.

Lemma 2.3. *Let $T_1, T_2 \in L(\mathbb{H})$ be self-adjoint operators satisfying $T_1 + T_2 = id$, such that $T_1 \geq \frac{k}{k+1}id$, or $T_1 \leq \frac{1}{k+1}id$, where $k \in \mathbb{N}$. Then*

$$\frac{k^2+k+1}{(k+1)^2}\|f\|^2 \leq \langle T_1 f, f\rangle + \|T_2 f\|^2 = \langle T_2 f, f\rangle + \|T_1 f\|^2, \quad \textit{for } f \in \mathbb{H}. \tag{2.2}$$

Proof. From our assumptions, we have

$$\begin{aligned}\langle T_2 f, f\rangle + \|T_1 f\|^2 &= \langle (id - T_1)f, f\rangle + \langle T_1^2 f, f\rangle \\ &= \langle (T_1^2 - T_1 + id)f, f\rangle \\ &= \langle T_1 f, f\rangle + \langle (id - T_1)^2 f, f\rangle \\ &= \langle T_1 f, f\rangle + \|T_2 f\|^2.\end{aligned}$$

For every $k \in \mathbb{N}$, we can write

$$\langle (T_1^2 - T_1 + id)f, f\rangle = \langle (T_1^2 - T_1 + \frac{k}{(k+1)^2} id)f, f\rangle + \langle (\frac{k^2+k+1}{(k+1)^2} id)f, f\rangle.$$

If

$$\langle T_1 f, f\rangle \leq \langle (\frac{1}{2} - \frac{1}{2}\sqrt{1 - \frac{4k}{(k+1)^2}})f, f\rangle = \langle \frac{1}{k+1} f, f\rangle,$$

or

$$\langle T_1 f, f\rangle \geq \langle (\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4k}{(k+1)^2}})f, f\rangle = \langle (\frac{k}{k+1} f, f\rangle,$$

we have

$$\langle (T_1^2 - T_1 + \frac{k}{(k+1)^2} id)f, f\rangle \geq 0.$$

So

$$\langle (T_1^2 - T_1 + id)f, f\rangle \geq \frac{k^2+k+1}{(k+1)^2}\|f\|^2.$$

Therefore,

$$\langle T_1 f, f\rangle + \|T_2 f\|^2 = \langle T_2 f, f\rangle + \|T_1 f\|^2 \geq \frac{k^2+k+1}{(k+1)^2}\|f\|^2.$$

$\square$

REMARK 2.1. Notice that for $k = 1$, inequality (2.2) is the same as the inequality proved in [9, Theorem 2.1] and for $k > 1$, from the inequality $\frac{3}{4} < \frac{k^2+k+1}{(k+1)^2}$, it follows that inequality (2.2) is an improvement for it.
For $k = 1, 2, 3, 4, 5, \ldots$, the correspondence values of $1 - \frac{k}{(k+1)^2}$ or $\frac{k^2+k+1}{(k+1)^2}$ are $0.75 < 0.78 < 0.81 < 0.84 < 0.86 < \ldots.$, respectively.
Hence by increasing k, we see that $1 - \frac{k}{(k+1)^2}$ is rapidly approaching to 1. Therefore inequality (2.2) is better in the application and we use it for frames.

THEOREM 2.2. *Let $\{f_i\}_{i\in I}$ be a frame for Hilbert space $\mathbb{H}$ with frame operator S and canonical dual frame $\{\tilde{f}_i\}_{i\in I}$. For every $\emptyset \neq J \subset I$, if $0 < S^{-\frac{1}{2}} S_J S^{-\frac{1}{2}} \leq \frac{1}{k+1} id$, or $S^{-\frac{1}{2}} S_J S^{-\frac{1}{2}} \geq \frac{k}{k+1} id$, where $k \in \mathbb{N}$, then*

$$\begin{aligned} \frac{k^2+k+1}{(k+1)^2} \sum_{i\in I} |\langle f, f_i\rangle|^2 &\leq \sum_{i\in J} |\langle f, f_i\rangle|^2 + \sum_{i\in I} |\langle S_{J^c} f, \tilde{f}_i\rangle|^2 \\ &= \sum_{i\in J^c} |\langle f, f_i\rangle|^2 + \sum_{i\in I} |\langle S_J f, \tilde{f}_i\rangle|^2, \quad \text{for } f \in \mathbb{H}. \end{aligned} \tag{2.3}$$

Proof. For every $J \subset I$, we have $S_J + S_{J^c} = S$, and hence $S^{-\frac{1}{2}} S_J S^{-\frac{1}{2}} + S^{-\frac{1}{2}} S_{J^c} S^{-\frac{1}{2}} = id$. By our assumptions and taking $T_1 = S^{-\frac{1}{2}} S_J S^{-\frac{1}{2}}$, $T_2 = S^{-\frac{1}{2}} S_{J^c} S^{-\frac{1}{2}}$, and $S^{\frac{1}{2}} f$ instead of f in Lemma 2.5, we get

$$\frac{k^2+k+1}{(k+1)^2}\|S^{\frac{1}{2}} f\|^2 \leq \langle S^{-\frac{1}{2}} S_J f, S^{\frac{1}{2}} f\rangle + \|S^{-\frac{1}{2}} S_{J^c} f\|^2 = \langle S^{-\frac{1}{2}} S_{J^c} f, S^{\frac{1}{2}} f\rangle + \|S^{-\frac{1}{2}} S_J f\|^2.$$

or equivalently,

$$\frac{k^2+k+1}{(k+1)^2}\langle Sf,f\rangle \leq \langle S_J f,f\rangle + \langle S^{-1}S_{J^c}f, S_{J^c}f\rangle = \langle S_{J^c}f,f\rangle + \langle S^{-1}S_J f, S_J f\rangle.$$

Therefore

$$\begin{aligned}\frac{k^2+k+1}{(k+1)^2}\sum_{i\in I}|\langle f,f_i\rangle|^2 &\leq \sum_{i\in J}|\langle f,f_i\rangle|^2 + \sum_{i\in I}|\langle S_{J^c}f,\tilde{f}_i\rangle|^2\\ &= \sum_{i\in J^c}|\langle f,f_i\rangle|^2 + \sum_{i\in I}|\langle S_J f,\tilde{f}_i\rangle|^2.\end{aligned}$$

□

For $k=1$, inequality (2.3) is the same as the inequality proved in [9, Theorem 2.2] and for $k>1$ it is an improvement for it.

In the following, we give an improvement for alternate dual frames. We first improve an inequality given in [9] for operators under conditions.

Lemma 2.4. *If $T_1, T_2 \in L(\mathbb{H})$ satisfy $T_1+T_2 = id$ and $\operatorname{Re} T_1 \geq \frac{k^2+k+1}{(k+1)^2} id$ where $k\in\mathbb{N}$, then*

$$\frac{k^2+k+1}{(k+1)^2} id \leq T_1^*T_1 + \frac{1}{2}(T_2^*+T_2) = T_2^*T_2 + \frac{1}{2}(T_1^*+T_1). \tag{2.4}$$

Proof. From our assumptions, we have

$$\begin{aligned}T_1^*T_1 + \frac{1}{2}(T_2^*+T_2) &= T_1^*T_1 + \frac{1}{2}(id - T_1^* + id - T_1)\\ &= T_1^*T_1 - \frac{1}{2}(T_1^*+T_1) + id\\ &= (id-T_1^*)(id-T_1) + \frac{1}{2}(T_1^*+T_1)\\ &= T_2^*T_2 + \frac{1}{2}(T_1^*+T_1).\end{aligned}$$

And also, $T_2^*T_2 + \frac{1}{2}(T_1^*+T_1) = T_2^*T_2 + \operatorname{Re} T_1 \geq \frac{k^2+k+1}{(k+1)^2} id.$

□

Note that, for $k=1$, inequality (2.4) is the same as the inequality in [9, Theorem 3.1] and for every $k>1$, inequality (2.4) is its improvement.

Theorem 2.3. *Let $\{f_i\}_{i\in I}$ be a frame for Hilbert space $\mathbb{H}$ and let $\{g_i\}_{i\in I}$ be an alternate dual frame of $\{f_i\}_{i\in I}$.*
For every $J\subset I$ and $f\in\mathbb{H}$, if $\operatorname{Re}\langle(\sum_{i\in J}\langle f,g_i\rangle f_i), f\rangle \geq \frac{k^2+k+1}{(k+1)^2}\langle f,f\rangle$, where $k\in\mathbb{N}$, then

$$\begin{aligned}\frac{k^2+k+1}{(k+1)^2}\|f\|^2 &\le \operatorname{Re}\sum_{i\in J}\langle f,g_i\rangle\overline{\langle f,f_i\rangle}+\|\sum_{i\in J^c}\langle f,g_i\rangle f_i\|^2\\ &= \operatorname{Re}\sum_{i\in J^c}\langle f,g_i\rangle\overline{\langle f,f_i\rangle}+\|\sum_{i\in J}\langle f,g_i\rangle f_i\|^2.\end{aligned}\tag{2.5}$$

Proof. For every $J\subset I$ define the bounded linear operator Z_J on $\mathbb{H}$ by

$$Z_Jf:=\sum_{i\in J}\langle f,g_i\rangle f_i.$$

By the Cauchy-Schwarz inequality, it follows that this series converges unconditionally. Since $Z_J+Z_{J^c}=id$, by Lemma 2.8, for every $f\in\mathbb{H}$, we have

$$\begin{aligned}\frac{k^2+k+1}{(k+1)^2}\langle f,f\rangle &\le \langle Z_J^*Z_Jf,f\rangle+\frac{1}{2}\langle (Z_{J^c}^*+Z_{J^c})f,f\rangle\\ &= \langle Z_{J^c}^*Z_{J^c}f,f\rangle+\frac{1}{2}\langle (Z_J^*+Z_J)f,f\rangle,\end{aligned}$$

or

$$\begin{aligned}\frac{k^2+k+1}{(k+1)^2}\|f\|^2 &\le \|K_Jf\|^2+\frac{1}{2}(\overline{\langle Z_{J^c}f,f\rangle}+\langle Z_{J^c}f,f\rangle)\\ &= \|Z_{J^c}f\|^2+\frac{1}{2}(\overline{\langle Z_Jf,f\rangle}+\langle Z_Jf,f\rangle).\end{aligned}$$

Hence

$$\begin{aligned}\frac{k^2+k+1}{(k+1)^2}\|f\|^2 &\le \|\sum_{i\in J}\langle f,g_i\rangle f_i\|^2+\operatorname{Re}\langle\sum_{i\in J^c}\langle f,g_i\rangle f_i,f\rangle\\ &= \|\sum_{i\in J^c}\langle f,g_i\rangle f_i\|^2+\operatorname{Re}\langle\sum_{i\in J}\langle f,g_i\rangle f_i,f\rangle,\end{aligned}$$

and the proof is completed. □

Note that, for $k=1$, inequality (2.5) is the same as the inequality proved in [9, Theorem 3.2] and for every $k>1$, inequality (2.5) is its improvement.
Finally we show a more general result.

THEOREM 2.4. *Let $\{f_i\}_{i\in I}$ be a frame for Hilbert space $\mathbb{H}$ and let $\{g_i\}_{i\in I}$ be an alternate dual frame of $\{f_i\}_{i\in I}$.*
For every $f\in\mathbb{H}$, if $\operatorname{Re}\langle(\sum_{i\in J}\langle f,g_i\rangle f_i),f\rangle\ge\frac{k^2+k+1}{(k+1)^2}\langle f,f\rangle$, where $k\in\mathbb{N}$, then for any bounded sequence $\{w_i\}_{i\in I}$,

$$\begin{aligned}\frac{k^2+k+1}{(k+1)^2}\|f\|^2 &\le \operatorname{Re}\sum_{i\in I}w_i\langle f,g_i\rangle\overline{\langle f,f_i\rangle}+\|\sum_{i\in I}(1-w_i)\langle f,g_i\rangle f_i\|^2\\ &= \operatorname{Re}\sum_{i\in I}(1-w_i)\langle f,g_i\rangle\overline{\langle f,f_i\rangle}+\|\sum_{i\in I}w_i\langle f,g_i\rangle f_i\|^2.\end{aligned}$$

Proof. In Lemma 2.8, we put

$$T_1f=\sum_{i\in I}w_i\langle f,g_i\rangle f_i,\quad T_2f=\sum_{i\in I}(1-w_i)\langle f,g_i\rangle f_i.$$

Now, the result follows from Theorem 2.9 if we take $J \subset I$ and

$$w_i = \begin{cases} 1, & \text{for } i \in J, \\ 0, & \text{for } i \in J^c. \end{cases}$$

□

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Fahimeh Sultanzadeh
Department of Mathematics, Mashhad Branch, Islamic Azad University,
Mashhad, Iran.
E-mail: `fsultanzadeh@gmail.com`

Mahmood Hassani
Department of Mathematics, Mashhad Branch, Islamic Azad University,
Mashhad, Iran.
E-mail: `mhassanimath@gmail.com`

Mohsen Erfanian Omidvar
Department of Mathematics, Mashhad Branch, Islamic Azad University,
Mashhad, Iran.
E-mail: `math.erfanian@gmail.com`

Rajab Ali kamyabi Gol
Department of Pure Mathematics, Ferdowsi University of Mashhad,
Mashhad, Iran.
E-mail: `kamyabi@um.ac.ir`