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SINE TRIGONOMETRIC SPHERICAL FUZZY AGGREGATION OPERATORS AND THEIR APPLICATION IN DECISION SUPPORT SYSTEM, TOPSIS, VIKOR

Muhammad Qiyas and Saleem Abdullah

Abstract. Spherical fuzzy set (SFS) is also one of the fundamental concepts for address more uncertainties in decision problems than the existing structures of fuzzy sets, and thus its implementation was more substantial. The well-known sine trigonometric function maintains the periodicity and symmetry of the origin in nature and thus satisfies the expectations of the experts over the multi parameters. Taking this feature and the significance of the SFSs into the consideration, the main objective of the article is to describe some reliable sine trigonometric laws (STL) for SFSs. Associated with these laws, we develop new average and geometric aggregation operators to aggregate the Spherical fuzzy numbers (SFNs). Then, we presented a group decision- making (DM) strategy to address the multi-attribute group decision making (MAGDM) problem using the developed aggregation operators. In order to verify the value of the defined operators, a MAGDM strategy is provided along with an application for the selection of laptop. Moreover, a comparative study is also performed to present the effectiveness of the developed approach.

1. Introduction

Multiple attribute group decision making (MAGDM) method is one of the most relevant and evolving topics explaining how to choose the finest alternative with community of experts with some attributes. There are two relevant tasks in this system. The first is to define the context in which the values of the various parameters are effectively calculated, while the 2nd is to summarize the define information. Generally, the information describing the objects is taken mostly in the form of deterministic or crisp in nature. With the increasing complexity of a systems on a daily basis, however, it is difficult to aggregate the data, from the logbook, resources and experts, in the crisp form. Therefore, [58] developed the core concept of fuzzy set (FS), and also [1] work on it and further develop a new idea of intuitionistic fuzzy set (IFS), [57] developed the Pythagorean fuzzy sets (PyFSs), [45] was defined the idea of a hesitant fuzzy sets, which are used by scholars to communicate the information clearly. In IFS, it is observed that each object have two membership grades positive $\breve{E}$ and the negative $\breve{Z}$, which satisfying the condition $0 \leq \breve{E} + \breve{Z} \leq 1$, and for all

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$\breve{E}, \breve{Z}$ are lying in closed interval 0 and 1. However, in the Pythagorean fuzzy sets this constraint is relaxed from $\breve{E} + \breve{Z} \leq 1$, to $\breve{E}^2 + \breve{Z}^2 \leq 1$ for $\breve{E}, \breve{Z} \in [0, 1]$. Using this concept, many researcher have strongly addressed the define two critical tasks and discretion the techniques under the different aspects. The basic results of IFSs and Pythagorean fuzzy sets such as the operational laws [11] some exponential operational laws [17], some distance or similarity measures [18], [24], some information entropy [22]. Many researchers [54], [55], [16], [12], [32], [23], under IFS, defined some basic aggregation operators (AOs) , like as average and geometric, interactive AOs, Hamacher AOs. While for Pythagorean fuzzy sets, some basic operators are proposed by Peng & Yang [39]. To solve the MAGDM problems, Garg [13], [14] presented some basic concept of Einstein aggregation operators. Some extended aggregation operators dependent on intuitionistic and Pythagorean fuzzy information including the TOPSIS technique based on IF [20] and Pythagorean fuzzy [59], partitioned Bonferroni mean [37], Maclaurin symmetric mean [41], [19]. Apart form this, Yager et al. [56], intuitively developed the idea of q-rung orthopair fuzzy sets (q-ROFSs). Gao et al. [21],developed the basic idea of the continuities and differential of q-ROFSs. Peng et al. [40] presented exponential & logarithmic operation laws for q-ROFNs. Liu and Wang [33] developed weighted average and geometric aggregation operators for q-ROFNs.

While, the idea of IFSs and Pythagorean FSs are widely studied and implemented in various field. But their ability to express the information is still limited. Thus, it was still difficult for the experts and their corresponding information to convey the information in such sets. To overcome this information, the notion of the picture fuzzy sets (PFSs), which is defined by Cuong [8]. Thus, it was clearly noticed that PFS is the extented form of the IFSs, with accommodate some more ambiguities. In picture fuzzy sets, each object observed by defining three grades of the member named as membership $\breve{E}$, neutral $\breve{R}$ and non-membership $\breve{Z}$ with constraint that $\breve{E} + \breve{R} + \breve{Z} \leq 1$, for $\breve{E}, \breve{R}, \breve{Z} \in [0, 1]$. The definition of the PFS will convey opinions of experts like "yes" "abstain" "no" and "refusal" while avoiding missing evaluation details and encouraging the reliability of the acquired data with the actual environment for decision-making. Although the concept of PFSs are widely studied and applied in different fields. And their extension focus on the basic operational laws, which is the important aspect of the PFS as well as aggregation operators (AOs), which are an effective tools by the help of these AOs, we obtain raking of the alternatives by providing the comprehensive values to the alternatives. Wei [47], developed some operations of the PFS. Son [44], developed measuring analogousness in PFSs . Apart from these, several other kinds of the AOs of the PFSs have been developed such as logarithmic PF aggregation operators, which are presented by Khan [29] , Wang et.al, [49] presented PF normalized projection based VIKOR method, Wang et al. [50], develop PF Muirhead mean operators, Wei et al. [51], defined the idea of some q-ROF maclaurin symmetric mean operators. Wang [52], introduced similarity measure of q-ROFSs. Wei et al. [53], developed Bidirectional projection method for PFSs. Ashraf et al. [6], [2], [?] developed the idea of different approaches to MAGDM problems, picture fuzzy linguistic sets and exponential jensen PF divergence measure respectively. khan [30], presented PF aggregation based on Einstein operation. Qiyas et al. [42], presented linguistic PF Dombi aggregation operators. Cuong & Hai [9] defined some operations and dedined some picture fuzzy logic operators for fuzzy derivation forms. The properties of PF t-norm & conorm are examined by Cuong, Kreinovich & Ngan [10]. Phong et al. [38] analyze some design of PF relations. Akram et al. [7] proposed a decision making model under complex PF Hamacher AOs. Ahmad et al. [31] defined new operations on interval-valued picture fuzzy set, interval-valued picture fuzzy soft set and their applications. Garg [15] developed some picture fuzzy aggregation operators and an approach for multi-criteria decision-making. Lin et al. [35] proposed a novel picture fuzzy MCDM model based on extended MULTI-MOORA method to solve the site selection of car sharing station. Liu et al. [34] defined the similarity measures for interval-valued picture fuzzy sets and discussed their applications in decision making. Recently, Khan [29] defined the new concept about logarithmic operation laws for PFSs.

In order to address this limitation which PFN can not handle, Shahzaib et al. [3] defined the notion of Spherical fuzzy set (SFS) for the first time and identified some aggregation operators with the Spherical fuzzy information problem for MADM. In the SFS, all the membership degrees are gratifying the condition $0 \leq \left(\check{E}_{\check{I}}(r)\right)^2 + \left(\check{R}_{\check{I}}(r)\right)^2 + \left(\check{Z}_{\check{I}}(r)\right)^2 \leq 1$ rather than $0 \leq \check{E}_{\check{I}}(r) + \check{R}_{\check{I}}(r) + \check{Z}_{\check{I}}(r) \leq 1$ as in PFSs. Gundogdu et al. [28] specified the TOPSIS method for SFS and give example of multi-attribute decision problem. Huanhuan et al. [26] defined SLFS, which combines the concept of LFS with SFS. Ashraf et al. [4] using the Dombi method, described some SF aggregation operators and discussed their decision making application, also studied the presentation of SF t-norm and conorm in [5]. Jin Y et al. [25] developed some Spherical fuzzy logarithmic AOs based on entropy and their application in decision support systems. Rafiq et al. [43] introduced some cosine similarity measures of Spherical fuzzy sets and their applications in decision making. Zeng et al. [60] developed a Covering-based Spherical fuzzy rough set model hybrid with TOPSIS for MADM. Mahmood et al. [36] define a model for decision making and medical diagnosis problems using the concept of SFSs.

Among the above aspects, it is very clear that operational laws play main role model for any aggregation process. Besides these mathematical logarithmic functions another important feature is the sine trigonometry feature, which plays a main role during the fusion of the information. In this way, taking into consideration the advantages and usefulness of the sine trigonometric function, some new sine trigonometric operational laws need to be developed for SFSs and their behavior studied. Consequently, the paper's purpose is to develop some new operation laws for SFSs and also give the MAGDM algorithm for managing the information for SFSs evaluation. Describe several more sophisticated operational laws for SFSs as well as a novel entropy to remove the weight of the attributes to prevent subjective & objective aspects. Some more generalized functional aggregation operators are presented with help of the defined sine trigonometric operational laws ($STOLs$) for $SFNs$, many basic relations between the developed AOs are discussed and give a novel MAGDM technique depending on the developed operators to solve the group decision making problems. And finally, the proposed approach compared with the existing method.

So the goals and the motivations of this paper are as follows:

1. To present some more advanced operational laws for SFSs by combining the features of the ST and $SFNs$.
2. A novel entropy is presented to extract the attributes' weight for avoiding the influence of subjective and objective aspects.

3. To present some more generalized functional AOs with the help of the defined $STOLs$ for $SFNs$. Also, the several fundamental relations between the proposed AOs are derived to show its significance.
4. To present a novel MAGDM method based on the proposed operators to solve the group decisionmaking problems. The consistency of the proposed method is confirmed through these examples, and their evaluations are carried out in detail.

In second Section of the article, we can define some related to SFS. In Section 3, we define the new SFS operational laws based on sine trigonometric function and their properties. In Section 4, we presented a series of AOs along with their required properties, based on sine trigonometric operational laws. Section 5, provides the basic connection between the developed AOs. In Section 6, using the new aggregation operators, we introduce a new MAGDM approach and give detailed steps. An example in the field of medical line using the SFNs information are given in Section 7, and to validate the new method and comparative study is carried out with the current methods are also given. Finally the work is concluded in Section 8.

2. Preliminaries

Some fundamental ideas about Spherical fuzzy set (SFS) on the universal set $\breve{U}$ are discussed in this portion.

Definition 1. [3] Let $\breve{U}$ be the non-empty fixed sets. Then, the following set

$$\breve{I} = \left(\breve{u}, \breve{E}_{\breve{I}}(\breve{u}), \breve{R}_{\breve{I}}(\breve{u}), \breve{Z}_{\breve{I}}(\breve{u}) / \breve{u} \in \breve{U}\right). \tag{2.1}$$

Are said to be Spherical fuzzy set (SFS), where $\breve{E}_{\breve{I}}(\breve{u}), \breve{R}_{\breve{I}}(\breve{u}), \breve{I}_{\breve{I}}(\breve{u}) \in [0,1]$ are called as the grade of membership, positive, neutral and negative of the elements $\breve{u} \in \breve{U}$ to the set $\breve{I}$ respectively, where the following constraint has been fulfilled by $\breve{E}(\breve{u}), \breve{R}(\breve{u}), \breve{I}(\breve{u})$ for all $\breve{u} \in \breve{U}$.

$$0 \leq \breve{E}^2(\breve{u}) + \breve{R}^2(\breve{u}) + \breve{Z}^2(\breve{u}) \leq 1. \tag{2.2}$$

Furthermore, $\pi_{\breve{I}}(\breve{u}) = \sqrt{1 - \left(\breve{E}^2(\breve{u}) + \breve{R}^2(\breve{u}) + \breve{Z}^2(\breve{u})\right)}$ is referred as the refusal grade of $\breve{u} \in \breve{U}$ in $\breve{I}$. For convenience, $\left(\breve{E}_{\breve{I}}(\breve{u}), \breve{R}_{\breve{I}}(\breve{u}), \breve{Z}_{\breve{I}}(\breve{u})\right)$ is called as an Spherical fuzzy number (SFN).

Definition 2. [3] Let the three $SFNs$ are $\breve{I} = \left(\breve{E}_{\breve{I}}(\breve{u}), \breve{R}_{\breve{I}}(\breve{u}), \breve{Z}_{\breve{I}}(\breve{u})\right)$ $\breve{I}_1 = \left(\breve{E}_{\breve{I}_1}(\breve{u}), \breve{R}_{\breve{I}_1}(\breve{u}), \breve{Z}_{\breve{I}_1}(\breve{u})\right)$ and $\breve{I}_2 = \left(\breve{E}_{\breve{I}_2}(\breve{u}), \breve{R}_{\breve{I}_2}(\breve{u}), \breve{Z}_{\breve{I}_2}(\breve{u})\right)$. And also $\tilde{N} > 0$, is any scalar. Then,

1. $\breve{I}^c = \left\{\breve{Z}_{\breve{I}}(\breve{u}), \breve{R}_{\breve{I}}(\breve{u}), \breve{E}_{\breve{I}}(\breve{u})\right\}$;
2. $\breve{I}_1 \wedge \breve{I}_2 = \left\{\min\left(\breve{E}_{\breve{I}_1}(\breve{u}), \breve{E}_{\breve{I}_2}(\breve{u})\right), \min\left(\breve{R}_{\breve{I}_1}(\breve{u}), \breve{R}_{\breve{I}_2}(\breve{u})\right), \max\left(\breve{Z}_{\breve{I}_1}(\breve{u}), \breve{Z}_{\breve{I}_2}(\breve{u})\right)\right\}$;
3. $\breve{I}_1 \vee \breve{I}_2 = \left\{\max\left(\breve{E}_{\breve{I}_1}(\breve{u}), \breve{E}_{\breve{I}_2}(\breve{u})\right), \min\left(\breve{R}_{\breve{I}_1}(\breve{u}), \breve{R}_{\breve{I}_2}(\breve{u})\right), \min\left(\breve{Z}_{\breve{I}_1}(\breve{u}), \breve{Z}_{\breve{I}_2}(\breve{u})\right)\right\}$;
4. $\breve{I}_1 \oplus \breve{I}_2 = \left\{\sqrt{\breve{E}^2_{\breve{I}_1}(\breve{u}) + \breve{E}^2_{\breve{I}_2}(\breve{u}) - \breve{E}^2_{\breve{I}_1}(\breve{u}).\breve{E}^2_{\breve{I}_2}(\breve{u})}, \breve{R}_{\breve{I}_1}(\breve{u}).\breve{R}_{\breve{I}_2}(\breve{u}), \breve{Z}_{\breve{I}_1}(\breve{u}).\breve{Z}_{\breve{I}_2}(\breve{u})\right\}$;

5. $\breve{I}_1 \otimes \breve{I}_2 = \left\{ \begin{array}{c} \breve{E}_{\breve{I}_1}(\breve{u}) . \breve{E}_{\breve{I}_2}(\breve{u}), \sqrt{\breve{R}^2_{\breve{I}_1}(\breve{u}) + \breve{R}^2_{\breve{I}_2}(\breve{u}) - \breve{R}^2_{\breve{I}_1}(\breve{u}) . \breve{R}^2_{\breve{I}_2}(\breve{u})}, \\ \sqrt{\breve{Z}^2_{\breve{I}_1}(\breve{u}) + \breve{Z}^2_{\breve{I}_2}(\breve{u}) - \breve{Z}^2_{\breve{I}_1}(\breve{u}) . \breve{Z}^2_{\breve{I}_2}(\breve{u})} \end{array} \right\}$;

6. $\tilde{N}\breve{I} = \left\{ \sqrt{1 - \left(1 - \breve{E}^2_{\breve{I}}(\breve{u})\right)^{\tilde{N}}}, \left(\breve{R}_{\breve{I}}(\breve{u})\right)^{\tilde{N}}, \left(\breve{Z}_{\breve{I}}(\breve{u})\right)^{\tilde{N}} \right\}$;

7. $\left(\breve{I}\right)^{\tilde{N}} = \left\{ \left(\breve{E}_{\breve{I}}(\breve{u})\right)^{\tilde{N}}, \sqrt{1 - \left(1 - \breve{R}^2_{\breve{I}}(\breve{u})\right)^{\tilde{N}}}, \sqrt{1 - \left(1 - \breve{Z}^2_{\breve{I}}(\breve{u})\right)^{\tilde{N}}} \right\}$.

Definition 3. [15] Let $\breve{I} = \left(\breve{E}_{\breve{I}}(\breve{u}), \breve{R}_{\breve{I}}(\breve{u}), \breve{Z}_{\breve{I}}(\breve{u})\right)$ be the SFN. The score and accuracy function are then described as, follows;

$$\bar{S}c\left(\breve{I}\right) = \breve{E}_{\breve{I}}(\breve{u}) - \breve{R}_{\breve{I}}(\breve{u}) - \breve{Z}_{\breve{I}}(\breve{u}), \text{ where } \bar{S}c\left(\breve{I}\right) \in [-1, 1], \tag{2.3}$$

$$\bar{H}c\left(\breve{I}\right) = \breve{E}_{\breve{I}}(\breve{u}) + \breve{R}_{\breve{I}}(\breve{u}) + \breve{Z}_{\breve{I}}(\breve{u}), \text{ where } \bar{H}c\left(\breve{I}\right) \in [0, 1]. \tag{2.4}$$

Definition 4. [15] Let the two SFNs are $\breve{I}_1 = \left(\breve{E}_{\breve{I}_1}(\breve{u}), \breve{R}_{\breve{I}_1}(\breve{u}), \breve{Z}_{\breve{I}_1}(\breve{u})\right)$ and $\breve{I}_2 = \left(\breve{E}_{\breve{I}_2}(\breve{u}), \breve{R}_{\breve{I}_2}(\breve{u}), \breve{Z}_{\breve{I}_2}(\breve{u})\right)$. Then, the rules for comparison can be defined as if the score function i.e.,

· $\bar{S}c\left(\breve{I}_1\right) > \bar{S}c\left(\breve{I}_2\right)$, then $\breve{I}_1 > \breve{I}_2$, and if the score function i.e.,
· $\bar{S}c\left(\breve{I}_1\right) = \bar{S}c\left(\breve{I}_2\right)$, and $\bar{H}c\left(\breve{I}_1\right) > \bar{H}c\left(\breve{I}_2\right)$, then $\breve{I}_1 > \breve{I}_2$,
· if $\bar{H}c\left(\breve{I}_1\right) = \bar{H}c\left(\breve{I}_2\right)$, then $\breve{I}_1 = \breve{I}_2$.

3. New Sine Trigonometric Operational Laws ($STOLs$) for SFSs

We will define some operational laws for SFNs in this portion.

Definition 5. Let the SFN is $\breve{I} = \left(\breve{E}_{\breve{I}}(\breve{u}), \breve{R}_{\breve{I}}(\breve{u}), \breve{Z}_{\breve{I}}(\breve{u})\right)$. Then, we define a $STOLs$ of a Spherical fuzzy set as;

$$\sin \breve{I} = \left\{ \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}}(\breve{u})\right)\right), 1 - \sin\left(\frac{\pi}{2}\left(1 - \breve{R}_{\breve{I}}(\breve{u})\right)\right), 1 - \sin\left(\frac{\pi}{2}\left(1 - \breve{Z}_{\breve{I}}(\breve{u})\right)\right) \right\}. \tag{3.1}$$

From the above definition it is clear that the $\sin \breve{I}$ is also SFS, and also satisfied the following condition of the SFS as, the membership, neutral and nonmembership degrees of SFS are define respectively

$$\sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}}(\breve{u})\right)\right) : \breve{U} \dashrightarrow [0, 1], \text{ such that } 0 \le \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}}(\breve{u})\right)\right) \le 1,$$

$$1 - \sin\left(\frac{\pi}{2}\left(1 - \breve{R}_{\breve{I}}(\breve{u})\right)\right) : \breve{U} \dashrightarrow [0, 1], \text{ such that } 0 \le 1 - \sin\left(\frac{\pi}{2}\left(1 - \breve{R}_{\breve{I}}(\breve{u})\right)\right) \le 1,$$

$$1 - \sin\left(\frac{\pi}{2}\left(1 - \breve{Z}_{\breve{I}}(\breve{u})\right)\right) : \breve{U} \dashrightarrow [0, 1], \text{ such that } 0 \le 1 - \sin\left(\frac{\pi}{2}\left(1 - \breve{Z}_{\breve{I}}(\breve{u})\right)\right) \le 1,$$

Therefore,

$$\sin \breve{I} = \left\{ \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}}(\breve{u})\right)\right), 1 - \sin\left(\frac{\pi}{2}\left(1 - \breve{R}_{\breve{I}}(\breve{u})\right)\right), 1 - \sin\left(\frac{\pi}{2}\left(1 - \breve{Z}_{\breve{I}}(\breve{u})\right)\right) \right\}$$

is SFS.

Definition 6. Let $\breve{I}=\left(\breve{E}_{\breve{I}}(\breve{u}),\breve{R}_{\breve{I}}(\breve{u}),\breve{Z}_{\breve{I}}(\breve{u})\right)$ be a SFN. Then,

$$\sin\breve{I}=\left\{\sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}}(\breve{u})\right)\right),1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}}(\breve{u})\right)\right),1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}}(\breve{u})\right)\right)\right\}, \tag{3.2}$$

is known as sine trigonometric (ST) operator and their value is known as sine trigonometric SFN.

Definition 7. Let the collection of $SFNs$ are $\breve{I}=\left(\breve{E}_{\breve{I}}(\breve{u}),\breve{R}_{\breve{I}}(\breve{u}),\breve{Z}_{\breve{I}}(\breve{u})\right)$, $\breve{I}_1=\left(\breve{E}_{\breve{I}_1}(\breve{u}),\breve{R}_{\breve{I}_1}(\breve{u}),\breve{Z}_{\breve{I}_1}(\breve{u})\right)$ and $\breve{I}_2=\left(\breve{E}_{\breve{I}_2}(\breve{u}),\breve{R}_{\breve{I}_2}(\breve{u}),\breve{Z}_{\breve{I}_2}(\breve{u})\right)$. Then, we define the following operational laws where $\Theta>0$ is any scalar.

1. $$\sin\breve{I}_1\oplus\sin\breve{I}_2=\left\{\begin{array}{c}\sqrt{1-\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_1}(\breve{u})\right)\right)\right).\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_2}(\breve{u})\right)\right)\right)},\\ \left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_1}(\breve{u})\right)\right)\right).\left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_2}(\breve{u})\right)\right)\right),\\ \left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_1}(\breve{u})\right)\right)\right).\left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_2}(\breve{u})\right)\right)\right)\end{array}\right\};$$
2. $$\sin\breve{I}_1\otimes\sin\breve{I}_2=\left\{\begin{array}{c}\left(\sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_1}(\breve{u})\right)\right)\right).\left(\sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_2}(\breve{u})\right)\right)\right),\\ \sqrt{1-\left(\sin\left(\frac{\pi}{2}\left(1-\breve{R}^2_{\breve{I}_1}(\breve{u})\right)\right)\right).\left(\sin\left(\frac{\pi}{2}\left(1-\breve{R}^2_{\breve{I}_2}(\breve{u})\right)\right)\right)},\\ \sqrt{1-\left(\sin\left(\frac{\pi}{2}\left(1-\breve{Z}^2_{\breve{I}_1}(\breve{u})\right)\right)\right).\left(\sin\left(\frac{\pi}{2}\left(1-\breve{Z}^2_{\breve{I}_2}(\breve{u})\right)\right)\right)}\end{array}\right\};$$
3. $$\tilde{N}\sin\breve{I}=\left\{\begin{array}{c}\sqrt{1-\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}}(\breve{u})\right)\right)\right)^{\tilde{N}}},\left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}}(\breve{u})\right)\right)\right)^{\tilde{N}}\\ \left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}}(\breve{u})\right)\right)\right)^{\tilde{N}}\end{array}\right\};$$
4. $$\left(\sin\breve{I}\right)^{\tilde{N}}=\left\{\begin{array}{c}\left(\sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}}(\breve{u})\right)\right)\right)^{\tilde{N}},\sqrt{1-\left(\sin\left(\frac{\pi}{2}\left(1-\breve{R}^2_{\breve{I}}(\breve{u})\right)\right)\right)^{\tilde{N}}}\\ \sqrt{1-\left(\sin\left(\frac{\pi}{2}\left(1-\breve{Z}^2_{\breve{I}}(\breve{u})\right)\right)\right)^{\tilde{N}}}\end{array}\right\};$$

3.1. Some basic properties of $STOLs$ of $SFNs$. Some fundamental properties of sine trigonometric SFN are discussed in this portion, using the sine trigonometric operational laws $(STOLs)$.

Theorem 1. *Let a collection of $SFNs$ are $\breve{I}_{\hat{J}}=\left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}),\breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}),\breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)$, where $\hat{J}=1,...,3$. Then,*

1. $\sin\breve{I}_1\oplus\sin\breve{I}_2=\sin\breve{I}_2\oplus\sin\breve{I}_1$
2. $\sin\breve{I}_1\otimes\sin\breve{I}_2=\sin\breve{I}_2\otimes\sin\breve{I}_1$
3. $\left(\sin\breve{I}_1\oplus\sin\breve{I}_2\right)\oplus\sin\breve{I}_3=\sin\breve{I}_1\oplus\left(\sin\breve{I}_2\oplus\sin\breve{I}_3\right)$
4. $\left(\sin\breve{I}_1\otimes\sin\breve{I}_2\right)\otimes\sin\breve{I}_3=\sin\breve{I}_1\otimes\left(\sin\breve{I}_2\otimes\sin\breve{I}_3\right)$

Proof. Here, we solve the first two parts using the $STOLs$ (sine trigonometric operation laws) define in Definition (7), and the proof of the other two part are similar to the first parts, so we omit here, we get

1.

$$\sin \breve{I}_1 \oplus \sin \breve{I}_2 = \begin{pmatrix} \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_1}(\breve{u})\right)\right), \\ 1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_1}(\breve{u})\right)\right), \\ 1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_1}(\breve{u})\right)\right) \end{pmatrix} \oplus \begin{pmatrix} \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_2}(\breve{u})\right)\right), \\ 1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_2}(\breve{u})\right)\right), \\ 1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_2}(\breve{u})\right)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{1-\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_1}(\breve{u})\right)\right)\right).\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_2}(\breve{u})\right)\right)\right)}, \\ \left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_1}(\breve{u})\right)\right)\right).\left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_2}(\breve{u})\right)\right)\right), \\ \left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_1}(\breve{u})\right)\right)\right).\left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_2}(\breve{u})\right)\right)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{1-\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_2}(\breve{u})\right)\right)\right).\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_1}(\breve{u})\right)\right)\right)}, \\ \left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_2}(\breve{u})\right)\right)\right).\left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_1}(\breve{u})\right)\right)\right), \\ \left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_2}(\breve{u})\right)\right)\right).\left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_1}(\breve{u})\right)\right)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_2}(\breve{u})\right)\right), \\ 1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_2}(\breve{u})\right)\right), \\ 1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_2}(\breve{u})\right)\right) \end{pmatrix} \oplus \begin{pmatrix} \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_1}(\breve{u})\right)\right), \\ 1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_1}(\breve{u})\right)\right), \\ 1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_1}(\breve{u})\right)\right) \end{pmatrix}$$

$$= \sin \breve{I}_2 \oplus \sin \breve{I}_1$$

therefore, from the above

$$\sin \breve{I}_1 \oplus \sin \breve{I}_2 = \sin \breve{I}_2 \oplus \sin \breve{I}_1$$

2.

$$\sin \breve{I}_1 \otimes \sin \breve{I}_2 = \begin{pmatrix} \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_1}(\breve{u})\right)\right), \\ 1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_1}(\breve{u})\right)\right), \\ 1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_1}(\breve{u})\right)\right) \end{pmatrix} \otimes \begin{pmatrix} \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_2}(\breve{u})\right)\right), \\ 1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_2}(\breve{u})\right)\right), \\ 1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_2}(\breve{u})\right)\right) \end{pmatrix}$$

$$= \begin{pmatrix} \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_1}(\breve{u})\right)\right).\sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_2}(\breve{u})\right)\right), \sqrt{1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}^2_{\breve{I}_1}(\breve{u})\right)\right).\sin\left(\frac{\pi}{2}\left(1-\breve{R}^2_{\breve{I}_2}(\breve{u})\right)\right)}, \\ \sqrt{1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}^2_{\breve{I}_1}(\breve{u})\right)\right).\sin\left(\frac{\pi}{2}\left(1-\breve{Z}^2_{\breve{I}_2}(\breve{u})\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_2}(\breve{u})\right)\right).\sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_1}(\breve{u})\right)\right), \sqrt{1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}^2_{\breve{I}_2}(\breve{u})\right)\right).\sin\left(\frac{\pi}{2}\left(1-\breve{R}^2_{\breve{I}_1}(\breve{u})\right)\right)}, \\ \sqrt{1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}^2_{\breve{I}_2}(\breve{u})\right)\right).\sin\left(\frac{\pi}{2}\left(1-\breve{Z}^2_{\breve{I}_1}(\breve{u})\right)\right)} \end{pmatrix}$$

$$= \begin{pmatrix} \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_2}(\breve{u})\right)\right), \\ 1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_2}(\breve{u})\right)\right), \\ 1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_2}(\breve{u})\right)\right) \end{pmatrix} \otimes \begin{pmatrix} \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_1}(\breve{u})\right)\right), \\ 1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_1}(\breve{u})\right)\right), \\ 1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_1}(\breve{u})\right)\right) \end{pmatrix}$$

$$= \sin \breve{I}_1 \otimes \sin \breve{I}_2$$

therefore, from the above solution

$$\sin \breve{I}_1 \otimes \sin \breve{I}_2 = \sin \breve{I}_2 \otimes \sin \breve{I}_1$$

□

THEOREM 2. *Let a collection of SFNs are* $\breve{I}_{\hat{J}} = \left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)$ *where* $\hat{J} = 1, 2$. *And also* $\tilde{N}, \tilde{N}_1, \tilde{N}_2 > 0$ *be the real number. Then,*

1. $\tilde{N}\left(\sin \breve{I}_1 \oplus \sin \breve{I}_2\right) = \tilde{N}\sin \breve{I}_1 \oplus \tilde{N}\sin \breve{I}_2$
2. $\left(\sin \breve{I}_1 \otimes \sin \breve{I}_2\right)^{\tilde{N}} = \left(\sin \breve{I}_1\right)^{\tilde{N}} \otimes \left(\sin \breve{I}_2\right)^{\tilde{N}}$
3. $\tilde{N}_1 \sin \breve{I} \oplus \tilde{N}_2 . \sin \breve{I} = \left(\tilde{N}_1 \oplus \tilde{N}_2\right) \sin \breve{I}$
4. $\left(\sin \breve{I}\right)^{\tilde{N}_1} \otimes \left(\sin \breve{I}\right)^{\tilde{N}_2} = \left(\sin \breve{I}\right)^{\tilde{N}_1 \oplus \tilde{N}_2}$
5. $\left(\left(\sin \breve{I}\right)^{\tilde{N}_1}\right)^{\tilde{N}_2} = \left(\sin \breve{I}\right)^{\tilde{N}_1 . \tilde{N}_2}$

Proof. Here, we will prove the first part of the above theorem only by using the *STOLs* define in Definition (7), while rest can be proven similarly. As we know,

1.

$$\sin \breve{I}_1 = \left\{\sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_1}(\breve{u})\right)\right), 1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_1}(\breve{u})\right)\right), 1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_1}(\breve{u})\right)\right)\right\}$$

and

$$\sin \breve{I}_2 = \left\{\sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_2}(\breve{u})\right)\right), 1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_2}(\breve{u})\right)\right), 1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_2}(\breve{u})\right)\right)\right\}$$

by using the *STOLs*, we have

$$\sin \breve{I}_1 \oplus \sin \breve{I}_2 = \begin{pmatrix} \sqrt{1-\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_1}(\breve{u})\right)\right)\right).\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_2}(\breve{u})\right)\right)\right)}, \\ \left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_1}(\breve{u})\right)\right)\right).\left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_2}(\breve{u})\right)\right)\right), \\ \left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_1}(\breve{u})\right)\right)\right).\left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_2}(\breve{u})\right)\right)\right) \end{pmatrix}$$

but it is given in statement of the Theorem that $\breve{R} > 0$, again we use the Definition (7), we have

$$\begin{aligned} & \tilde{N}\left(\sin \breve{I}_1 \oplus \sin \breve{I}_2\right) \\ = & \begin{pmatrix} \sqrt{1-\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_1}(\breve{u})\right)\right)\right)^{\tilde{N}}.\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_2}(\breve{u})\right)\right)\right)^{\tilde{N}}}, \\ \left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_1}(\breve{u})\right)\right)\right)^{\tilde{N}}.\left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_2}(\breve{u})\right)\right)\right)^{\tilde{N}}, \\ \left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_1}(\breve{u})\right)\right)\right)^{\tilde{N}}.\left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_2}(\breve{u})\right)\right)\right)^{\tilde{N}} \end{pmatrix} \\ = & \begin{pmatrix} \sqrt{1-\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_1}(\breve{u})\right)\right)\right)^{\tilde{N}}}, \\ \left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_1}(\breve{u})\right)\right)\right)^{\tilde{N}}, \\ \left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_1}(\breve{u})\right)\right)\right)^{\tilde{N}} \end{pmatrix} \oplus \begin{pmatrix} \sqrt{1-\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_2}(\breve{u})\right)\right)\right)^{\tilde{N}}}, \\ \left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_2}(\breve{u})\right)\right)\right)^{\tilde{N}}, \\ \left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_2}(\breve{u})\right)\right)\right)^{\tilde{N}} \end{pmatrix} \\ = & \tilde{N}\sin \breve{I}_1 \oplus \tilde{N}\sin \breve{I}_2 \end{aligned}$$

□

Corollary 1. *Let a collection of two SFNs are* $\breve{I}_{\hat{\jmath}} = \left(\breve{E}_{\breve{I}_{\hat{\jmath}}}(\breve{u}), \breve{R}_{\breve{I}_{\hat{\jmath}}}(\breve{u}), \breve{Z}_{\breve{I}_{\hat{\jmath}}}(\breve{u})\right)$ *where* $\hat{J} = 1, 2$, *such that* $\breve{E}_{\breve{I}_1}(\breve{u}) \geq \breve{E}_{\breve{I}_2}(\breve{u}), \breve{R}_{\breve{I}_1}(\breve{u}) \leq \breve{R}_{\breve{I}_2}(\breve{u})$ *and* $\breve{Z}_{\breve{I}_1}(\breve{u}) \leq \breve{Z}_{\breve{I}_2}(\breve{u})$. *Then show that* $\sin \breve{I}_1 \geq \sin \breve{I}_2$.

Proof. Let $\breve{I}_1 = \left(\breve{E}_{\breve{I}_1}(\breve{u}), \breve{R}_{\breve{I}_1}(\breve{u}), \breve{Z}_{\breve{I}_1}(\breve{u})\right)$ and $\breve{I}_2 = \left(\breve{E}_{\breve{I}_2}(\breve{u}), \breve{R}_{\breve{I}_2}(\breve{u}), \breve{Z}_{\breve{I}_2}(\breve{u})\right)$ are the SFN with condition $\breve{E}_1(\breve{u}) \geq \breve{E}_{\breve{I}_2}(\breve{u})$, since in the closed interval $\left[0, \frac{\pi}{2}\right]$, sin is an increasing function, thus we have $\sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_1}(\breve{u})\right)\right) \geq \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_2}(\breve{u})\right)\right)$. But also given that $\breve{R}_1(\breve{u}) \leq \breve{R}_{\breve{I}_2}(\breve{u})$ which implies that $\left(1 - \breve{R}_{\breve{I}_1}(\breve{u})\right) \geq \left(1 - \breve{R}_{\breve{I}_2}(\breve{u})\right)$, since in closed interval $\left[0, \frac{\pi}{2}\right]$, sine is an increasing function, thus we have $\sin\left(\frac{\pi}{2}\left(1 - \breve{R}_{\breve{I}_1}(\breve{u})\right)\right) \geq \sin\left(\frac{\pi}{2}\left(1 - \breve{R}_{\breve{I}_2}(\breve{u})\right)\right)$, which implies that $1-\sin\left(\frac{\pi}{2}\left(1 - \breve{R}_{\breve{I}_1}(\breve{u})\right)\right) \leq 1-\sin\left(\frac{\pi}{2}\left(1 - \breve{R}_{\breve{I}_2}(\breve{u})\right)\right)$, similarly $\breve{Z}_{\breve{I}_1}(\breve{u}) \leq \breve{Z}_{\breve{I}_2}(\breve{u})$ which implies that $\left(1 - \breve{Z}_{\breve{I}_1}(\breve{u})\right) \geq \left(1 - \breve{Z}_{\breve{I}_2}(\breve{u})\right)$, since in closed interval $\left[0, \frac{\pi}{2}\right]$, sin is an increasing function, thus we have $\sin\left(\frac{\pi}{2}\left(1 - \breve{Z}_{\breve{I}_1}(\breve{u})\right)\right) \geq \sin\left(\frac{\pi}{2}\left(1 - \breve{Z}_{\breve{I}_2}(\breve{u})\right)\right)$, which implies that $1-\sin\left(\frac{\pi}{2}\left(1 - \breve{Z}_{\breve{I}_1}(\breve{u})\right)\right) \leq 1-\sin\left(\frac{\pi}{2}\left(1 - \breve{Z}_{\breve{I}_2}(\breve{u})\right)\right)$, hence, we get

$$\begin{pmatrix} \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_1}(\breve{u})\right)\right), \\ 1 - \sin\left(\frac{\pi}{2}\left(1 - \breve{R}_{\breve{I}_1}(\breve{u})\right)\right), \\ 1 - \sin\left(\frac{\pi}{2}\left(1 - \breve{Z}_{\breve{I}_1}(\breve{u})\right)\right) \end{pmatrix} \geq \begin{pmatrix} \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_2}(\breve{u})\right)\right), \\ 1 - \sin\left(\frac{\pi}{2}\left(1 - \breve{R}_{\breve{I}_2}(\breve{u})\right)\right), \\ 1 - \sin\left(\frac{\pi}{2}\left(1 - \breve{Z}_{\breve{I}_2}(\breve{u})\right)\right) \end{pmatrix}$$

therefore, we get the required result by using the Definition (7),

$$\sin \breve{I}_1 \geq \sin \breve{I}_2$$

□

4. Sine Trigonometric Aggregation Operators

we have described a number of aggregation operators in this portion of the article on the basis of sine trigonometric operational laws $(STOLs)$.

4.1. Sine Trigonometric Averaging Aggregation Operator.

Definition 8. Let a collection of $SFNs$ are $\breve{I}_{\hat{J}} = \left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)$, where $\hat{J} = 1, \cdots, n$. Then, the mapping $ST - SFWA : \Psi^n \to \Psi$, is known as the sine trigonometric Spherical fuzzy weighted average $(ST - SFWA)$ operator, if

$$ST - SFWA\left(\breve{I}_1, \cdots, \breve{I}_n\right) = \Theta_1. \sin \breve{I}_1 \oplus \cdots \oplus \Theta_n. \sin \breve{I}_n. \tag{4.1}$$

Where the weighted vectors of $\sin \breve{I}_{\hat{J}}\left(\hat{J} = 1, \cdots, n\right)$ are $\Theta_{\hat{J}}$, which fulfilled the criteria of $\Theta_{\hat{J}} > 0$, and $\sum_{\hat{J}=1}^{n} \Theta_{\hat{J}} = 1$.

Theorem 3. *Let a collection of $SFNs$ are $\breve{I}_{\hat{J}} = \left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)$ where $\hat{J} = 1, \cdots, n$. Then, the aggregated value is also SFN by utilizing the $ST - SFWA$ operator, and is given by*

$$ST - SFWA\left(\breve{I}_1, \cdots, \breve{I}_n\right) = \begin{pmatrix} \sqrt{1 - \prod_{\hat{J}=1}^{n}\left(1 - \sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\hat{J}}(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}}}, \\ \prod_{\hat{J}=1}^{n}\left(1 - \sin\frac{\pi}{2}\left(1 - \breve{R}_{\hat{J}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}, \\ \prod_{\hat{J}=1}^{n}\left(1 - \sin\frac{\pi}{2}\left(1 - \breve{Z}_{\hat{J}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}} \end{pmatrix} \tag{4.2}$$

Proof. By using the process of mathematical induction, we prove the said Theorem. Because, $\breve{I}_{\hat{J}} = \left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)$ is SFN for each $\hat{J}$, which implies that $\left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right) \in [0,1]$ and also $\left(\breve{E}^2_{\breve{I}_{\hat{J}}}(\breve{u}) + \breve{R}^2_{\breve{I}_{\hat{J}}}(\breve{u}) + \breve{Z}^2_{\breve{I}_{\hat{J}}}(\breve{u})\right) \leq 1$. The following mathematical induction steps were then performed.

Step 1. Now for $n = 2$, we get $ST - SFWA\left(\breve{I}_1, \breve{I}_2\right) = \Theta_1. \sin \breve{I}_1 \oplus \Theta_2. \sin \breve{I}_2$ where

$$\Theta_1 \sin \breve{I}_1 = \begin{pmatrix} \sqrt{1 - \left(1 - \sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_1}(\breve{u})\right)\right)\right)^{\Theta_1}}, \left(1 - \sin\frac{\pi}{2}\left(1 - \breve{R}_{\breve{I}_1}(\breve{u})\right)\right)^{\Theta_1}, \\ \left(1 - \sin\frac{\pi}{2}\left(1 - \breve{Z}_{\breve{I}_1}(\breve{u})\right)\right)^{\Theta_1} \end{pmatrix}$$

and

$$\Theta_2 \sin \breve{I}_2 = \begin{pmatrix} \sqrt{1 - \left(1 - \sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_2}(\breve{u})\right)\right)\right)^{\Theta_2}}, \left(1 - \sin\frac{\pi}{2}\left(1 - \breve{R}_{\breve{I}_2}(\breve{u})\right)\right)^{\Theta_2}, \\ \left(1 - \sin\frac{\pi}{2}\left(1 - \breve{Z}_{\breve{I}_2}(\breve{u})\right)\right)^{\Theta_2} \end{pmatrix}$$

and hence, by using the Definition in (7), we get

$$\Theta_1 \sin \breve{I}_1 \oplus \Theta_2 \sin \breve{I}_2 = \begin{pmatrix} \sqrt{1 - \prod_{\hat{J}=1}^{2}\left(1 - \sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\hat{J}}(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}}}, \\ \prod_{\hat{J}=1}^{2}\left(1 - \sin\frac{\pi}{2}\left(1 - \breve{R}_{\hat{J}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}, \\ \prod_{\hat{J}=1}^{2}\left(1 - \sin\frac{\pi}{2}\left(1 - \breve{Z}_{\hat{J}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}} \end{pmatrix}$$

Step 2. Now say it's true for $n = k$.

$$ST - SFWA\left(\breve{I}_1, \breve{I}_2\right) = \begin{pmatrix} \sqrt{1 - \prod_{\hat{J}=1}^{k}\left(1 - \sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\hat{J}}(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}}}, \\ \prod_{\hat{J}=1}^{k}\left(1 - \sin\frac{\pi}{2}\left(1 - \breve{R}_{\hat{J}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}, \\ \prod_{\hat{J}=1}^{k}\left(1 - \sin\frac{\pi}{2}\left(1 - \breve{Z}_{\hat{J}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}} \end{pmatrix}$$

Step 3. Now, we prove that this is true for $n = k + 1$

$$ST - SFWA\left(\breve{I}_1, \cdots, \breve{I}_{k+1}\right) = \Theta_1 \sin \breve{I}_1 \oplus \cdots \oplus \Theta_n \sin \breve{I}_n \oplus \Theta_{k+1} \sin \breve{I}_{k+1}$$

$$= \begin{pmatrix} \sqrt{1 - \prod_{\hat{J}=1}^{k}\left(1 - \sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\hat{J}}(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}}}, \\ \prod_{\hat{J}=1}^{k}\left(1 - \sin\frac{\pi}{2}\left(1 - \breve{R}_{\hat{J}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}, \\ \prod_{\hat{J}=1}^{k}\left(1 - \sin\frac{\pi}{2}\left(1 - \breve{Z}_{\hat{J}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}} \end{pmatrix} \oplus \begin{pmatrix} \sqrt{1 - \left(1 - \sin\left(\frac{\pi}{2}\left(\breve{E}^2_{k+1}(\breve{u})\right)\right)\right)^{\Theta_{k+1}}}, \\ \left(1 - \sin\frac{\pi}{2}\left(1 - \breve{R}_{k+1}(\breve{u})\right)\right)^{\Theta_{k+1}}, \\ \left(1 - \sin\frac{\pi}{2}\left(1 - \breve{Z}_{k+1}(\breve{u})\right)\right)^{\Theta_{k+1}} \end{pmatrix}$$

again, by using the Definition (7), we obtained

$$ST-SFWA\left(\breve{I}_1,\cdots,\breve{I}_{k+1}\right)=\left(\begin{array}{c}\sqrt{1-\prod_{\hat{J}=1}^{k+1}\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}_{\hat{J}}^2(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}}},\\ \prod_{\hat{J}=1}^{k+1}\left(1-\sin\frac{\pi}{2}\left(1-\breve{R}_{\hat{J}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}},\\ \prod_{\hat{J}=1}^{k+1}\left(1-\sin\frac{\pi}{2}\left(1-\breve{Z}_{\hat{J}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}\end{array}\right)$$

Hence, for the $n=k+1$ holds. Then, the statement is valid for all n through the principal of mathematical induction. □

The $ST-SFWA$ operators possess the following properties.

Property 1. If all collection of $SFNs$ $\breve{I}_{\hat{J}}=\breve{I}$, where $\breve{I}$ is another SFN $\left(\hat{J}=1,\cdots,n\right)$, then

(4.3) $$ST-SFWA\left(\breve{I}_1,\cdots,\breve{I}_n\right)=\sin\breve{I}$$

Proof. Let $\breve{I}=\left(\breve{E}_{\breve{I}}(\breve{u}),\breve{R}_{\breve{I}}(\breve{u}),\breve{Z}_{\breve{I}}(\breve{u})\right)$ is SFN, such that $\breve{I}_{\hat{J}}=\breve{I}$. Then, we get by using Theorem (3),

$$\begin{aligned}ST-SFWA\left(\breve{I}_1,\cdots,\breve{I}_n\right) &= \left(\begin{array}{c}\sqrt{1-\prod_{\hat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}_{\hat{J}}^2(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}}},\\ \prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(1-\breve{R}_{\hat{J}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}},\\ \prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(1-\breve{Z}_{\hat{J}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}\end{array}\right)\\
&= \left(\begin{array}{c}\sqrt{1-\prod_{\hat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}}},\\ \prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(1-\breve{R}(\breve{u})\right)\right)^{\Theta_{\hat{J}}},\\ \prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(1-\breve{Z}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}\end{array}\right)\\
&= \left(\begin{array}{c}\sqrt{1-\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2(\breve{u})\right)\right)\right)^{\sum_{\hat{J}=1}^{n}\Theta_{\hat{J}}}},\\ \left(1-\sin\frac{\pi}{2}\left(1-\breve{R}(\breve{u})\right)\right)^{\sum_{\hat{J}=1}^{n}\Theta_{\hat{J}}},\\ \left(1-\sin\frac{\pi}{2}\left(1-\breve{Z}(\breve{u})\right)\right)^{\sum_{\hat{J}=1}^{n}\Theta_{\hat{J}}}\end{array}\right)\\
&= \left(\begin{array}{c}\sin\left(\frac{\pi}{2}\left(\breve{E}(\breve{u})\right)\right),\left(1-\sin\frac{\pi}{2}\left(1-\breve{R}(\breve{u})\right)\right)\\ \left(1-\sin\frac{\pi}{2}\left(1-\breve{Z}(\breve{u})\right)\right)\end{array}\right)\\
&= \sin\breve{I}\end{aligned}$$

□

Property 2. If

$$\breve{I}_{\hat{J}}=\left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}),\breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}),\breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right),$$

$$\breve{I}_{\hat{J}}^{-}=\left(\min_{\hat{J}}\left\{\breve{E}_{\hat{J}}(\breve{u})\right\},\max_{\hat{J}}\left\{\check{R}_{\hat{J}}(\breve{u})\right\},\max_{\hat{J}}\left\{\check{Z}_{\hat{J}}(\breve{u})\right\}\right)$$

and

$$\breve{I}_{\hat{J}}^{+}=\left(\max_{\hat{J}}\left\{\breve{E}_{\hat{J}}(\breve{u})\right\},\min_{\hat{J}}\left\{\check{R}_{\hat{J}}(\breve{u})\right\},\min_{\hat{J}}\left\{\check{Z}_{\hat{J}}(\breve{u})\right\}\right),$$

where $\hat{J}=1,\cdots,n$, be $SFNs$, then

$$\sin\breve{I}^{-}\leq ST-SFWA\left(\breve{I}_{1},\cdots,\breve{I}_{n}\right)\leq\sin\breve{I}^{+}. \tag{4.4}$$

Proof. Since for any $\hat{J}$, $\min_{\hat{J}}\left\{\breve{E}_{\hat{J}}(\breve{u})\right\}\leq\breve{E}_{\hat{J}}(\breve{u})\leq\max_{\hat{J}}\left\{\breve{E}_{\hat{J}}(\breve{u})\right\}$, $\min_{\hat{J}}\left\{\check{R}_{\hat{J}}(\breve{u})\right\}\leq\check{R}_{\hat{J}}(\breve{u})\leq\max_{\hat{J}}\left\{\check{R}_{\hat{J}}(\breve{u})\right\}$ and $\min_{\hat{J}}\left\{\check{Z}_{\hat{J}}(\breve{u})\right\}\leq\check{Z}_{\hat{J}}(\breve{u})\leq\max_{\hat{J}}\left\{\check{Z}_{\hat{J}}(\breve{u})\right\}$. This implies that $\breve{I}^{-}\leq\breve{I}_{\hat{J}}\leq\breve{I}^{+}$. Assume that, $ST{-}SFWA\left(\breve{I}_{1},\cdots,\breve{I}_{n}\right)=\sin\breve{I}=\left(\breve{E}_{\breve{I}}(\breve{u}),\check{R}_{\breve{I}}(\breve{u}),\check{Z}_{\breve{I}}(\breve{u})\right)$, $\sin\breve{I}^{+}=\left(\breve{E}_{\breve{I}^{+}}(\breve{u}),\check{R}_{\breve{I}^{-}}(\breve{u}),\check{Z}_{\breve{I}^{-}}(\breve{u})\right)$ and $\sin\breve{I}^{-}=\left(\breve{E}_{\breve{I}^{-}}(\breve{u}),\check{R}_{\breve{I}^{+}}(\breve{u}),\check{Z}_{\breve{I}^{+}}(\breve{u})\right)$. Then, by the monotonicity of the sine trigonometric function, we have

$$\begin{aligned}
\breve{E}_{\breve{I}}(\breve{u}) &= \sqrt{1-\prod_{\hat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}_{\hat{J}}^{2}(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}}}\geq\sqrt{1-\prod_{\hat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\min_{\hat{J}}\left\{\breve{E}_{\hat{J}}^{2}(\breve{u})\right\}\right)\right)^{\Theta_{\hat{J}}}}\\
&= \sqrt{1-\left(1-\sin\left(\frac{\pi}{2}\min\left\{\breve{E}_{\hat{J}}^{2}(\breve{u})\right\}\right)\right)^{\sum_{\hat{J}=1}^{n}\Theta_{\hat{J}}}}\\
&= \sin\left(\frac{\pi}{2}\min\left\{\breve{E}_{\hat{J}}(\breve{u})\right\}\right)\\
&= \breve{E}_{\breve{I}^{-}}(\breve{u})
\end{aligned}$$

$$\begin{aligned}
\check{R}_{\breve{I}}(\breve{u}) &= \prod_{\hat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(1-\check{R}_{\hat{J}}(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}}\geq\prod_{\hat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(1-\min_{\hat{J}}\left\{\check{R}_{\hat{J}}(\breve{u})\right\}\right)\right)\right)^{\Theta_{\hat{J}}}\\
&= \left(1-\sin\left(\frac{\pi}{2}\left(1-\min\left\{\check{R}_{\hat{J}}(\breve{u})\right\}\right)\right)\right)^{\sum_{\hat{J}=1}^{n}\Theta_{\hat{J}}}\\
&= \left(1-\sin\left(\frac{\pi}{2}\left(1-\min_{\hat{J}}\left\{\check{R}_{\hat{J}}(\breve{u})\right\}\right)\right)\right)\\
&= \check{R}_{\breve{I}^{-}}(\breve{u})
\end{aligned}$$

$$\begin{aligned}
\check{Z}_{\breve{I}}(\breve{u}) &= \prod_{\hat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(1-\check{Z}_{\hat{J}}(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}}\geq\prod_{\hat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(1-\min_{\hat{J}}\left\{\check{Z}_{\hat{J}}(\breve{u})\right\}\right)\right)\right)^{\Theta_{\hat{J}}}\\
&= \left(1-\sin\left(\frac{\pi}{2}\left(1-\min\left\{\check{Z}_{\hat{J}}(\breve{u})\right\}\right)\right)\right)^{\sum_{\hat{J}=1}^{n}\Theta_{\hat{J}}}\\
&= \left(1-\sin\left(\frac{\pi}{2}\left(1-\min_{\hat{J}}\left\{\check{Z}_{\hat{J}}(\breve{u})\right\}\right)\right)\right)\\
&= \check{Z}_{\breve{I}^{-}}(\breve{u})
\end{aligned}$$

and also

$$\begin{aligned}\breve{E}_{\breve{I}}(\breve{u}) &= \sqrt{1-\prod_{\hat{\jmath}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\hat{\jmath}}(\breve{u})\right)\right)\right)^{\Theta_{\hat{\jmath}}}} \le \sqrt{1-\prod_{\hat{\jmath}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\max_{\hat{\jmath}}\left\{\breve{E}^2_{\hat{\jmath}}(\breve{u})\right\}\right)\right)^{\Theta_{\hat{\jmath}}}} \\ &= \sqrt{1-\left(1-\sin\left(\frac{\pi}{2}\max\left\{\breve{E}^2_{\hat{\jmath}}(\breve{u})\right\}\right)\right)^{\sum_{\hat{\jmath}=1}^{n}\Theta_{\hat{\jmath}}}} \\ &= \sin\left(\frac{\pi}{2}\max_{\hat{\jmath}}\left\{\breve{E}_{\hat{\jmath}}(\breve{u})\right\}\right) \\ &= \breve{E}_{\breve{I}^+}(\breve{u})\end{aligned}$$

$$\begin{aligned}\check{R}_{\breve{I}}(\breve{u}) &= \prod_{\hat{\jmath}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(1-\check{R}_{\hat{\jmath}}(\breve{u})\right)\right)\right)^{\Theta_{\hat{\jmath}}} \le \prod_{\hat{\jmath}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(1-\max_{\hat{\jmath}}\left\{\check{R}_{\hat{\jmath}}(\breve{u})\right\}\right)\right)\right)^{\Theta_{\hat{\jmath}}} \\ &= \left(1-\sin\left(\frac{\pi}{2}\left(1-\max\left\{\check{R}_{\hat{\jmath}}(\breve{u})\right\}\right)\right)\right)^{\sum_{\hat{\jmath}=1}^{n}\Theta_{\hat{\jmath}}} \\ &= \left(1-\sin\left(\frac{\pi}{2}\left(1-\max_{\hat{\jmath}}\left\{\check{R}_{\hat{\jmath}}(\breve{u})\right\}\right)\right)\right) \\ &= \check{R}_{\breve{I}^+}(\breve{u})\end{aligned}$$

$$\begin{aligned}\check{Z}_{\breve{I}}(\breve{u}) &= \prod_{\hat{\jmath}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(1-\check{Z}_{\hat{\jmath}}(\breve{u})\right)\right)\right)^{\Theta_{\hat{\jmath}}} \ge \prod_{\hat{\jmath}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(1-\max_{\hat{\jmath}}\left\{\check{Z}_{\hat{\jmath}}(\breve{u})\right\}\right)\right)\right)^{\Theta_{\hat{\jmath}}} \\ &= \left(1-\sin\left(\frac{\pi}{2}\left(1-\max\left\{\check{Z}_{\hat{\jmath}}(\breve{u})\right\}\right)\right)\right)^{\sum_{\hat{\jmath}=1}^{n}\Theta_{\hat{\jmath}}} \\ &= \left(1-\sin\left(\frac{\pi}{2}\left(1-\max_{\hat{\jmath}}\left\{\check{Z}_{\hat{\jmath}}(\breve{u})\right\}\right)\right)\right) \\ &= \check{Z}_{\breve{I}^+}(\breve{u})\end{aligned}$$

Based on score function Definition (3), we get

$$Sc\left(\sin\breve{I}\right) = \breve{E}_{\breve{I}}(\breve{u}) - \check{R}_{\breve{I}}(\breve{u}) - \check{Z}_{\breve{I}}(\breve{u}) \le \breve{E}_{\breve{I}^+}(\breve{u}) - \check{R}_{\breve{I}^-}(\breve{u}) - \check{Z}_{\breve{I}^-}(\breve{u}) = Sc\left(\sin\breve{I}^+\right)$$

$$Sc\left(\sin\breve{I}\right) = \breve{E}_{\breve{I}}(\breve{u}) - \check{R}_{\breve{I}}(\breve{u}) - \check{Z}_{\breve{I}}(\breve{u}) \ge \breve{E}_{\breve{I}^-}(\breve{u}) - \check{R}_{\breve{I}^+}(\breve{u}) - \check{Z}_{\breve{I}^+}(\breve{u}) = Sc\left(\sin\breve{I}^-\right)$$

Hence, $Sc\left(\sin\breve{I}^-\right) \le Sc\left(\sin\breve{I}\right) \le Sc\left(\sin\breve{I}^+\right)$. Now, we have explain three cases:

Case 1. *If* $Sc\left(\sin\breve{I}^-\right) \le Sc\left(\sin\breve{I}\right) \le Sc\left(\sin\breve{I}^+\right)$, *then result holds.*

Case 2. *If* $Sc\left(\sin\breve{I}^+\right) = Sc\left(\sin\breve{I}\right)$, *then* $\breve{E}_{\breve{I}}(\breve{u}) - \check{R}_{\breve{I}}(\breve{u}) - \check{Z}_{\breve{I}}(\breve{u}) = \breve{E}_{\breve{I}^+}(\breve{u}) - \check{R}_{\breve{I}^+}(\breve{u}) - \check{Z}_{\breve{I}^+}(\breve{u})$, *which implies that* $\breve{E}_{\breve{I}}(\breve{u}) = \breve{E}_{\breve{I}^+}(\breve{u}), \check{R}_{\breve{I}}(\breve{u}) = \check{R}_{\breve{I}^+}(\breve{u})$, *and* $\check{Z}_{\breve{I}}(\breve{u}) = \check{Z}_{\breve{I}^+}(\breve{u})$ *and* $H\left(\sin\breve{I}^+\right) = H\left(\sin\breve{I}\right)$.

Case 3. *If* $Sc\left(\sin\breve{I}^-\right) = Sc\left(\sin\breve{I}\right)$, *then* $\breve{E}_{\breve{I}}(\breve{u}) - \check{R}_{\breve{I}}(\breve{u}) - \check{Z}_{\breve{I}}(\breve{u}) = \breve{E}_{\breve{I}^-}(\breve{u}) - \check{R}_{\breve{I}^-}(\breve{u}) - \check{Z}_{\breve{I}^-}(\breve{u})$, *which implies that* $\breve{E}_{\breve{I}}(\breve{u}) = \breve{E}_{\breve{I}^-}(\breve{u}), \check{R}_{\breve{I}}(\breve{u}) = \check{R}_{\breve{I}^-}(\breve{u})$, *and* $\check{Z}_{\breve{I}}(\breve{u}) = \check{Z}_{\breve{I}^-}(\breve{u})$ *and* $H\left(\sin\breve{I}^-\right) = H\left(\sin\breve{I}\right)$, *therefore, by combining all these cases, we get*

$$\sin\breve{I}^- \le ST - SFWA\left(\breve{I}_1, \cdots, \breve{I}_n\right) \le \sin\breve{I}^+.$$

□

Property 3. Let the collection of $SFNs$ are $\breve{I}_{\hat{J}} = \left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)$ and $\breve{I}^*_{\hat{J}} = \left(\breve{E}^*_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}^*_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{Z}^*_{\breve{I}_{\hat{J}}}(\breve{u})\right)$, where $\hat{J} = 1, \cdots, n$. If $\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}) \leq \breve{E}^*_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}) \geq \breve{R}^*_{\breve{I}_{\hat{J}}}(\breve{u})$, and $\breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u}) \geq \breve{Z}^*_{\breve{I}_{\hat{J}}}(\breve{u})$, then

$$ST - SFWA\left(\breve{I}_1, \cdots, \breve{I}_n\right) \leq ST - SFWA\left(\breve{I}^*_1, \cdots, \breve{I}^*_n\right) \tag{4.5}$$

Proof. Follow from the above, so we omit here. □

Definition 9. A sine trigonometric SF ordered weighted average operator ($ST - SFOWA$) is a mapping $ST - SFOWA : \Psi^n \to \Psi$ such that weighted vector $\Theta = (\Theta_1, \cdots, \Theta_n)^T$, which fulfilled the criteria of $\Theta_{\hat{J}} > 0$ and $\sum_{\hat{J}=1}^n \Theta_{\hat{J}} = 1$.

$$ST - SFOWA = \Theta_1 \sin \breve{I}_{\bar{O}(1)} \oplus \cdots \oplus \Theta_n \sin \breve{I}_{\bar{O}(n)}. \tag{4.6}$$

Where $(1, \cdots, n)$ is the permutation $\bar{O}$, such that $\breve{I}_{\bar{O}(\hat{J}-1)} \geq \breve{I}_{\bar{O}(\hat{J})}$ for any $\hat{J}$.

Theorem 4. *Let a collection of $SFNs$ are $\breve{I}_{\hat{J}} = \left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)$, where $\hat{J} = 1, \cdots, n$. Then, by utilized the operator i.e., $ST - SFOWA$ the aggregated value is also SFN and is given by,*

$$ST - SFOWA\left(\breve{I}_1, \cdots, \breve{I}_n\right) = \begin{pmatrix} \sqrt{1 - \prod_{\hat{J}=1}^{n}\left(1 - \sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\bar{O}(\hat{J})}(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}}}, \\ \prod_{\hat{J}=1}^{n}\left(1 - \sin\frac{\pi}{2}\left(1 - \breve{R}_{\bar{O}(\hat{J})}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}, \\ \prod_{\hat{J}=1}^{n}\left(1 - \sin\frac{\pi}{2}\left(1 - \breve{Z}_{\bar{O}(\hat{J})}(\breve{u})\right)\right)^{\Theta_{\hat{J}}} \end{pmatrix} \tag{4.7}$$

Proof. Proof is same to Theorem (3), so proof is ignore here. □

Definition 10. A sine trigonometric SF hybrid average operator ($ST - SFHA$) is a mapping $ST - SFHA : \Psi^n \to \Psi$ such that the associate vectors $\xi = (\xi_1, \xi_2, \cdots, \xi_n)^T$ which fulfilled the criteria of $\xi_{\hat{J}} > 0$ and $\sum_{\hat{J}=1}^n \xi_{\hat{J}} = 1$.

$$ST - SFHA = \xi_1 \sin \ddot{I}_{\bar{O}(1)} \oplus \cdots \oplus \xi_n \sin \ddot{I}_{\bar{O}(n)}. \tag{4.8}$$

Where $(1, \cdots, n)$ is the permutation of $\bar{O}$, as $\breve{I}_{\bar{O}(\hat{J}-1)} \geq \breve{I}_{\bar{O}(\hat{J})}$ for any $\hat{J}$ and $\ddot{I}_{\hat{J}} = n\Theta_{\hat{J}}\breve{I}_{\hat{J}}$

Theorem 5. *Let a collection of $SFNs$ are $\breve{I}_{\hat{J}} = \left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)$ where $\hat{J} = 1, \cdots, n$. Then, the aggregated value is also SFN by utilized the operator $ST -$*

SFHA and is given by,

$$ST - SFHA\left(\breve{I}_1, \cdots, \breve{I}_n\right) = \left(\begin{array}{c} \sqrt{1 - \prod_{\hat{J}=1}^{n} \left(1 - \sin\left(\frac{\pi}{2}\left(\mathring{E}^2_{\bar{O}(\hat{J})}(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}}}, \\ \prod_{\hat{J}=1}^{n} \left(1 - \sin\frac{\pi}{2}\left(1 - \mathring{R}_{\bar{O}(\hat{J})}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}, \\ \prod_{\hat{J}=1}^{n} \left(1 - \sin\frac{\pi}{2}\left(1 - \mathring{Z}_{\bar{O}(\hat{J})}(\breve{u})\right)\right)^{\Theta_{\hat{J}}} \end{array} \right) \tag{4.9}$$

Proof. Proof is same to Theorem (3), so proof is ignore here. □

4.2. Sine Trigonometric Geometric Aggregation Operator.

Definition 11. Let a collection of $SFNs$ are $\breve{I}_{\hat{J}} = \left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)$, where $\hat{J} = 1, \cdots, n$. Then, the mapping $ST - SFWG : \Psi^n \rightarrow \Psi$, is known as the sine trigonometric Spherical fuzzy weighted geometric ($ST - SFWG$) operator, if

$$ST - SFWG\left(\breve{I}_1, \cdots, \breve{I}_n\right) = \left(\sin\breve{I}_1\right)^{\Theta_1} \otimes ... \otimes \left(\sin\breve{I}_n\right)^{\Theta_1}. \tag{4.10}$$

Where the weight vectors are $\Theta = (\Theta_1, \cdots, \Theta_n)^T$ of $\sin\breve{I}_{\hat{J}}\left(\hat{J} = 1, \cdots, n\right)$, which fulfilled the criteria of $\Theta_{\hat{J}} > 0$, and $\sum_{\hat{J}=1}^{n} \Theta_{\hat{J}} = 1$.

Theorem 6. *Let a collection of $SFNs$ are $\breve{I}_{\hat{J}} = \left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)$, where $\hat{J} = 1, \cdots, n$. Then, the aggregated value is also SFN by using the $ST - SFWG$ operator, and is given by,*

$$ST - SFWG\left(\breve{I}_1, \cdots, \breve{I}_n\right) = \left(\begin{array}{c} \prod_{\hat{J}=1}^{n} \left(\sin\left(\frac{\pi}{2}\left(\breve{E}_{\hat{J}}(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}}, \\ \sqrt{1 - \prod_{\hat{J}=1}^{n} \left(\sin\frac{\pi}{2}\left(1 - \breve{R}^2_{\hat{J}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}}, \\ \sqrt{1 - \prod_{\hat{J}=1}^{n} \left(\sin\frac{\pi}{2}\left(1 - \breve{Z}^2_{\hat{J}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}} \end{array} \right) \tag{4.11}$$

Proof. Proof is similar to Theorem (3), so procedure is ignore here. □

The $ST - SFWG$ operators possess the following properties.

Property 1. If all collection of $SFNs$ $\breve{I}_{\hat{J}} = \breve{I}$, where $\breve{I}$ is another SFN $\left(\hat{J} = 1, \cdots, n\right)$, then

$$ST - SFWG\left(\breve{I}_1, \cdots, \breve{I}_n\right) = \sin\breve{I} \tag{4.12}$$

Property 2. If $\breve{I}_{\hat{J}} = \left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)$, where $\hat{J} = 1, \cdots, n$, $\breve{I}^-_{\hat{J}} = \left(\min_{\hat{J}}\left\{\breve{E}_{\hat{J}}(\breve{u})\right\}, \max_{\hat{J}}\left\{\breve{R}_{\hat{J}}(\breve{u})\right\}, \max_{\hat{J}}\left\{\breve{Z}_{\hat{J}}(\breve{u})\right\}\right)$ and $\breve{I}^+_{\hat{J}} = \left(\max_{\hat{J}}\left\{\breve{E}_{\hat{J}}(\breve{u})\right\}, \min_{\hat{J}}\left\{\breve{R}_{\hat{J}}(\breve{u})\right\}, \min_{\hat{J}}\left\{\breve{Z}_{\hat{J}}(\breve{u})\right\}\right)$ be $SFNs$, then

$$\sin\breve{I}^- \leq ST - SFWG\left(\breve{I}_1, \cdots, \breve{I}_n\right) \leq \sin\breve{I}^+. \tag{4.13}$$

Property 3. Let the collection of $SFNs$ are $\breve{I}_{\hat{J}} = \left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)$ and $\breve{I}^{*}_{\hat{J}} = \left(\breve{E}^{*}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}^{*}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{Z}^{*}_{\breve{I}_{\hat{J}}}(\breve{u})\right)$, where $\hat{J} = 1, \cdots, n$. If $\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}) \leq \breve{E}^{*}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}) \geq \breve{R}^{*}_{\breve{I}_{\hat{J}}}(\breve{u})$, and $\breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u}) \geq \breve{Z}^{*}_{\breve{I}_{\hat{J}}}(\breve{u})$. Then,

$$ST - SFWG\left(\breve{I}_1, \cdots, \breve{I}_n\right) \leq ST - SFWA\left(\breve{I}^{*}_1, \cdots, \breve{I}^{*}_n\right) \tag{4.14}$$

Definition 12. A $ST - SFOWG$ is a mapping from Ψ^n to Ψ such that the weighted vector $\Theta = (\Theta_1, \cdots, \Theta_n)^T$ which fulfilled the criteria of $\Theta_{\hat{J}} > 0$ and $\sum_{\hat{J}=1}^{n} \Theta_{\hat{J}} = 1$.

$$ST - SFOWG = \left(\sin \breve{I}_{\bar{O}(1)}\right)^{\Theta_1} \oplus \cdots \oplus \left(\sin \breve{I}_{\bar{O}(n)}\right)^{\Theta_n}. \tag{4.15}$$

Where $\bar{O}$ is the permutation of $(1, \cdots, n)$ as $\breve{I}_{\bar{O}(\hat{J}-1)} \geq \breve{I}_{\bar{O}(\hat{J})}$ for any $\hat{J}$.

Theorem 7. *Let a family of $SFNs$ are $\breve{I}_{\hat{J}} = \left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)$, where $\hat{J} = 1, \cdots, n$. Then, the aggregated value is also SFN by using the $ST - SFOWG$ operator, and is given by*

$$ST - SFOWG\left(\breve{I}_1, \cdots, \breve{I}_n\right) = \begin{pmatrix} \prod_{\hat{J}=1}^{n} \left(\sin\left(\frac{\pi}{2}\left(\breve{E}_{\bar{O}(\hat{J})}(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}}, \\ \sqrt{1 - \prod_{\hat{J}=1}^{n} \left(\sin\frac{\pi}{2}\left(1 - \breve{R}^2_{\bar{O}(\hat{J})}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}}, \\ \sqrt{1 - \prod_{\hat{J}=1}^{n} \left(\sin\frac{\pi}{2}\left(1 - \breve{Z}^2_{\bar{O}(\hat{J})}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}} \end{pmatrix} \tag{4.16}$$

Proof. Similar to Theorem (3) □

Definition 13. A sine trigonometric Spherical fuzzy hybrid geometric operator $(ST - SFHG)$ is a mapping $ST - SFHG : \Psi^n \to \Psi$, such that the associate vectors are $\xi = (\xi_1, \xi_2, \cdots, \xi_n)^T$, which fulfilled the condition $\xi_{\hat{J}} > 0$ and $\sum_{\hat{J}=1}^{n} \xi_{\hat{J}} = 1$.

$$ST - SFHG = \left(\sin \ddot{I}_{\bar{O}(1)}\right)^{\xi_1} \otimes \cdots \otimes \left(\sin \ddot{I}_{\bar{O}(n)}\right)^{\xi_n}. \tag{4.17}$$

Where $\bar{O}$ is the permutation of $(1, \cdots, n)$ as $\breve{I}_{\bar{O}(\hat{J}-1)} \geq \breve{I}_{\bar{O}(\hat{J})}$ for any $\hat{J}$ and $\ddot{I}_{\hat{J}} = n\Theta_{\hat{J}}\breve{I}_{\hat{J}}$

Theorem 8. *Let a family of $SFNs$ are $\breve{I}_{\hat{J}} = \left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)$, where $\hat{J} = 1, \cdots, n$. Then, by utilized the operator i.e., $ST - SFHG$ the aggregated value is also SFN and is given by*

$$ST - SFHG\left(\breve{I}_1, \cdots, \breve{I}_n\right) = \begin{pmatrix} \prod_{\hat{J}=1}^{n} \left(\sin\left(\frac{\pi}{2}\left(\mathring{E}_{\bar{O}(\hat{J})}(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}}, \\ \sqrt{1 - \prod_{\hat{J}=1}^{n} \left(\sin\frac{\pi}{2}\left(1 - \mathring{R}^2_{\bar{O}(\hat{J})}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}}, \\ \sqrt{1 - \prod_{\hat{J}=1}^{n} \left(\sin\frac{\pi}{2}\left(1 - \mathring{Z}^2_{\bar{O}(\hat{J})}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}} \end{pmatrix} \tag{4.18}$$

Proof. Proof is same to Theorem:(3), so proof is ignore here. □

As similar to $ST-SFWA, ST-SFOWA$, $ST-SFHA, ST-SFWG$, $ST-SFOWG$ and $ST-SFHG$ operators satisfy the properties such as Idempotency, Boundedness, Monotonicity.

5. Fundamental properties of the proposed Aggregation Operators

In this section of the paper, we discuss many relation between the proposed AOs and also studied their fundamental properties as given,

THEOREM 9. *For any two $SFNs$ i.e., $\breve{I}_1$ and $\breve{I}_2$ we have* $\sin\breve{I}_1 \oplus \sin\breve{I}_2 \geq \sin\breve{I}_1 \otimes \sin\breve{I}_2$

Proof. Let the two $SFNs$ are $\breve{I}_1 = \left(\breve{E}_{\breve{I}_1}(\breve{u}), \breve{R}_{\breve{I}_1}(\breve{u}), \breve{Z}_{\breve{I}_1}(\breve{u})\right)$ and $\breve{I}_2 = \left(\breve{E}_{\breve{I}_2}(\breve{u}), \breve{R}_{\breve{I}_2}(\breve{u}), \breve{Z}_{\breve{I}_2}(\breve{u})\right)$. Then, by using definition (6), (7), we get

$$\begin{aligned}
&\sin\breve{I}_1 \oplus \sin\breve{I}_2 \\
&= \left(\begin{array}{c}
\sqrt{1-\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_1}(\breve{u})\right)\right)\right).\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_2}(\breve{u})\right)\right)\right)}, \\
\left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_1}(\breve{u})\right)\right)\right).\left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_2}(\breve{u})\right)\right)\right), \\
\left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_1}(\breve{u})\right)\right)\right).\left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_2}(\breve{u})\right)\right)\right)
\end{array}\right) \\
&= \left(\begin{array}{c}
\sqrt{1-\prod_{\hat{J}=1}^{2}\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)\right)}, \prod_{\hat{J}=1}^{2}\left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)\right), \\
\prod_{\hat{J}=1}^{2}\left(1-\sin\left(\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)\right)
\end{array}\right)
\end{aligned}$$

and also

$$\begin{aligned}
\sin\breve{I}_1 \otimes \sin\breve{I}_2 &= \left(\begin{array}{c}
\left(\sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_1}(\breve{u})\right)\right)\right).\left(\sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_2}(\breve{u})\right)\right)\right), \\
\sqrt{1-\left(\sin\left(\frac{\pi}{2}\left(1-\breve{R}^2_{\breve{I}_1}(\breve{u})\right)\right)\right).\left(\sin\left(\frac{\pi}{2}\left(1-\breve{R}^2_{\breve{I}_2}(\breve{u})\right)\right)\right)}, \\
\sqrt{1-\left(\sin\left(\frac{\pi}{2}\left(1-\breve{Z}^2_{\breve{I}_1}(\breve{u})\right)\right)\right).\left(\sin\left(\frac{\pi}{2}\left(1-\breve{Z}^2_{\breve{I}_2}(\breve{u})\right)\right)\right)}
\end{array}\right) \\
&= \left(\begin{array}{c}
\prod_{\hat{J}=1}^{2}\left(\sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)\right), \sqrt{1-\prod_{\hat{J}=1}^{2}\left(\sin\left(\frac{\pi}{2}\left(1-\breve{R}^2_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)\right)}, \\
\sqrt{1-\prod_{\hat{J}=1}^{2}\left(\sin\left(\frac{\pi}{2}\left(1-\breve{Z}^2_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)\right)}
\end{array}\right)
\end{aligned}$$

since for any two non-negative real number a and b, their arithmetic mean is greater than or equal to their geometric mean therefore, $\frac{a+b}{2} \geq ab$ which follows that $a+b-ab \geq ab$. Thus by taking $a = \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_1}(\breve{u})\right)\right)$ and $b = \sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_2}(\breve{u})\right)\right)$, we have $1-\left(1-\sin\frac{\pi}{2}\left(\breve{E}_{\breve{I}_1}(\breve{u})\right)\right).\left(1-\sin\frac{\pi}{2}\left(\breve{E}_2(\breve{u})\right)\right) \geq \left(\sin\frac{\pi}{2}\left(\breve{E}_{\breve{I}_1}(\breve{u})\right)\right).\left(\sin\frac{\pi}{2}\left(\breve{E}_{\breve{I}_2}(\breve{u})\right)\right)$ which further gives that $1-\prod_{\hat{J}=1}^{2}\left(1-\sin\frac{\pi}{2}\left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right) \geq \prod_{\hat{J}=1}^{2}\left(\sin\frac{\pi}{2}\left(\breve{E}_{\breve{I}_1}(\breve{u})\right)\right)$.

Similarly, we have obtained the other two as

$$\prod_{\hat{J}=1}^{2}\left(1-\sin\left(\frac{\pi}{2}\left(1-\check{R}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)\right)\leq 1-\prod_{\hat{J}=1}^{2}\left(\sin\left(\frac{\pi}{2}\left(1-\check{R}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)\right)$$

and

$$\prod_{\hat{J}=1}^{2}\left(1-\sin\left(\frac{\pi}{2}\left(1-\check{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)\right)\leq 1-\prod_{\hat{J}=1}^{2}\left(\sin\left(\frac{\pi}{2}\left(1-\check{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)\right).$$

Hence, by using Definition (7), we get

$$\sin\breve{I}_1\oplus\sin\breve{I}_2\geq\sin\breve{I}_1\otimes\sin\breve{I}_2$$

□

THEOREM 10. *For any SFNs i.e.,* $\breve{I}$ *and positive real number* $\dot{\tau}>0$, $\dot{\tau}\sin\breve{I}\geq\left(\sin\breve{I}\right)^{\dot{\tau}}\Longleftrightarrow\dot{\tau}\geq 1$ *and* $\dot{\tau}\sin\breve{I}\leq\left(\sin\breve{I}\right)^{\dot{\tau}}\Longleftrightarrow 0<\dot{\tau}\leq 1$.

Proof. Proof is same to Theorem (9) □

LEMMA 1. *For* $a_{\hat{J}}\geq 0$ *and* $b_{\hat{J}}\geq 0$, *then we have* $\prod_{\hat{J}=1}^{n}a_{\hat{J}}^{b_{\hat{J}}}\leq\sum_{\hat{J}=1}^{n}b_{\hat{J}}.a_{\hat{J}}$ *and the equality holds iff* $a_1=a_2=\cdots=a_n$.

LEMMA 2. *Let* $0\leq a,b\leq 1$, *and* $0\leq x\leq 1$, *then* $0\leq ax+b(1-x)\leq 1$.

LEMMA 3. *Let* $0\leq a,b\leq 1$, *and* $\sqrt{1-(1-a^2)(1-b^2)}\geq ab$.

THEOREM 11. *For any SFNs* $\breve{I}_{\hat{J}}=\left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}),\check{R}_{\breve{I}_{\hat{J}}}(\breve{u}),\check{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)$, *the operators* $ST-SFWA$ *and* $ST-SFWG$ *satisfy the inequality*

$$ST-SFWA\left(\breve{I}_1,\cdots,\breve{I}_n\right)\geq ST-SFWG\left(\breve{I}_1,\cdots,\breve{I}_n\right). \tag{5.1}$$

Where equality holds iff $\breve{I}_1=\breve{I}_2=\cdots=\breve{I}_n$.

Proof. For n, $SFNs$ $\breve{I}_{\hat{J}}=\left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}),\check{R}_{\breve{I}_{\hat{J}}}(\breve{u}),\check{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)$ and normalized weight vector $\Theta_{\hat{J}}>0$, we have

$$ST-SFWA\left(\breve{I}_1,\cdots,\breve{I}_n\right)=\begin{pmatrix}\sqrt{1-\prod_{\hat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}}},\\ \prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(1-\check{R}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}},\\ \prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(1-\check{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}\end{pmatrix}$$

and

$$ST-SFWG\left(\breve{I}_1,\cdots,\breve{I}_n\right)=\begin{pmatrix}\prod_{\hat{J}=1}^{n}\left(\sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}},\\ \sqrt{1-\prod_{\hat{J}=1}^{n}\left(\sin\frac{\pi}{2}\left(1-\check{R}^2_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}},\\ \sqrt{1-\prod_{\hat{J}=1}^{n}\left(\sin\frac{\pi}{2}\left(1-\check{Z}^2_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}}\end{pmatrix}$$

For $\Theta_{\hat{J}} > 0$, $\sin\left(\frac{\pi}{2}\left(\breve{E}_{\hat{J}}\right)\right) \in [0,1]$ and by the Lemma (1), we get

$$1-\prod_{\hat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}} \geq 1-\sum_{\hat{J}=1}^{n}\Theta_{\hat{J}}\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)\right)$$

$$\geq 1-1+\sum_{\hat{J}=1}^{n}\Theta_{\hat{J}}\left(\sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)\right) \geq \prod_{\hat{J}=1}^{n}\left(\sin\left(\frac{\pi}{2}\left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}}$$

which implies that

$$\sqrt{1-\prod_{\hat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(\breve{E}^{2}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}}} \geq \sqrt{\prod_{\hat{J}=1}^{n}\left(\sin\left(\frac{\pi}{2}\left(\breve{E}^{2}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)\right)^{\Theta_{\hat{J}}}}.$$

For neutral and negative membership component, we have

$$\prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}} \leq \sum_{\hat{J}=1}^{n}\Theta_{\hat{J}}.\left(1-\sin\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)$$

$$\leq 1-\sum_{\hat{J}=1}^{n}\Theta_{\hat{J}}.\left(\sin\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right) \leq 1-\prod_{\hat{J}=1}^{n}\left(\sin\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}$$

which implies that

$$\prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}} \leq 1-\prod_{\hat{J}=1}^{n}\left(\sin\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}.$$

similar the negative grade, as

$$\prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}} \leq \sum_{\hat{J}=1}^{n}\Theta_{\hat{J}}.\left(1-\sin\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)$$

$$\leq 1-\sum_{\hat{J}=1}^{n}\Theta_{\hat{J}}.\left(\sin\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right) \leq 1-\prod_{\hat{J}=1}^{n}\left(\sin\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}$$

which implies that

$$\prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}} \leq 1-\prod_{\hat{J}=1}^{n}\left(\sin\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}.$$

Hence, from all the above Equations, we get

$$ST-SFWA\left(\breve{I}_1,\cdots,\breve{I}_n\right) \geq ST-SFWG\left(\breve{I}_1,\cdots,\breve{I}_n\right).$$

□

THEOREM 12. *Let* $\breve{I}_{\hat{J}} = \left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\left(\hat{J}=1,\cdots,n\right)$ *and* $\breve{I} = \left(\breve{E}_{\breve{I}}(\breve{u}), \breve{R}_{\breve{I}}(\breve{u}), \breve{Z}_{\breve{I}}(\breve{u})\right)$ *are SFNs. Then,*

$$\begin{aligned} ST-SFWA\left(\breve{I}_1\oplus\cdots\oplus\breve{I}_n\right) &\geq ST-SFWA\left(\breve{I}_1\otimes\cdots\otimes\breve{I}_n\right) \\ ST-SFWG\left(\breve{I}_1\oplus\cdots\oplus\breve{I}_n\right) &\geq ST-SFWG\left(\breve{I}_1\otimes\cdots\otimes\breve{I}_n\right) \end{aligned} \tag{5.2}$$

Proof. Here, we prove only the first part, while the other parts can be deduced similarly, for this, let $\breve{I}_{\hat{J}} = \left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right)$ and $\breve{I} = \left(\breve{E}_{\breve{I}}(\breve{u}), \breve{R}_{\breve{I}}(\breve{u}), \breve{Z}_{\breve{I}}(\breve{u})\right)$, since both $\breve{I}_{\hat{J}}$ and $\breve{I}$ are $SFNs$.

$$ST-SFWA\left(\breve{I}_1 \oplus \cdots \oplus \breve{I}_n \oplus \breve{I}\right)$$
$$= \begin{pmatrix} \sqrt{1-\prod_{\hat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(1-\left(1-\breve{E}^2_{\breve{I}_{\hat{J}}}(\breve{u})\right)\left(1-\breve{E}^2_{\breve{I}}(\breve{u})\right)\right)\right)\right)^{\Theta_{\hat{J}}}}, \\ \prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_{\hat{J}}}(\breve{u}).\breve{R}_{\breve{I}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}, \\ \prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u}).\breve{Z}_{\breve{I}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}} \end{pmatrix}$$

and

$$ST-SFWA\left(\breve{I}_1 \otimes \cdots \otimes \breve{I}_n \otimes \breve{I}\right)$$
$$= \begin{pmatrix} 1-\prod_{\hat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}).\breve{E}_{\breve{I}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}, \\ \prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(1-\breve{R}_{\breve{I}_{\hat{J}}}(\breve{u})\right).\left(1-\breve{R}_{\breve{I}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}, \\ \prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(1-\breve{Z}_{\breve{I}_{\hat{J}}}(\breve{u})\right).\left(1-\breve{Z}_{\breve{I}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}} \end{pmatrix}$$

For $\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}), \breve{E}_{\breve{I}}(\breve{u}) \in [0,1]$ and Lemma (3), we have $\sqrt{1-\left(1-\breve{E}^2_{\breve{I}_{\hat{J}}}(\breve{u})\right)\left(1-\breve{E}^2_{\breve{I}}(\breve{u})\right)} \geq \breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}).\breve{E}_{\breve{I}}(\breve{u})$. Since "sine" is an increasing function, we get

$$\sin\left(\frac{\pi}{2}\left(1-\left(1-\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u})\right)\left(1-\breve{E}_{\breve{I}}(\breve{u})\right)\right)\right) \geq \sin\frac{\pi}{2}\left(\breve{E}_{\breve{I}_{\hat{J}}}(\breve{u}).\breve{E}_{\breve{I}}(\breve{u})\right),$$

which gives that

$$\Rightarrow \sqrt{1-\sin\left(\frac{\pi}{2}\left(1-\left(1-\breve{E}^2_{\breve{I}_{\hat{J}}}(\breve{u})\right)\left(1-\breve{E}^2_{\breve{I}}(\breve{u})\right)\right)\right)} \leq \sqrt{1-\sin\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_{\hat{J}}}(\breve{u}).\breve{E}^2_{\breve{I}}(\breve{u})\right)}$$

$$\Rightarrow \sqrt{\prod_{\hat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(1-\left(1-\breve{E}^2_{\breve{I}_{\hat{J}}}(\breve{u})\right)\left(1-\breve{E}^2_{\breve{I}}(\breve{u})\right)\right)\right)\right)^{\Theta_{\hat{J}}}}$$
$$\leq \sqrt{\prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_{\hat{J}}}(\breve{u}).\breve{E}^2_{\breve{I}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}}$$

$$\Rightarrow \sqrt{1-\prod_{\hat{J}=1}^{n}\left(1-\sin\left(\frac{\pi}{2}\left(1-\left(1-\breve{E}^2_{\breve{I}_{\hat{J}}}(\breve{u})\right)\left(1-\breve{E}^2_{\breve{I}}(\breve{u})\right)\right)\right)\right)^{\Theta_{\hat{J}}}}$$
$$\geq \sqrt{1-\prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(\breve{E}^2_{\breve{I}_{\hat{J}}}(\breve{u}).\breve{E}^2_{\breve{I}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}}$$

Similarly the neutral and negative grade, we get

$$\prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(1-\check{R}_{\check{I}_{\hat{J}}}(\breve{u}).\check{R}_{\check{I}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}} \leq \prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(1-\check{R}_{\check{I}_{\hat{J}}}(\breve{u})\right).\left(1-\check{R}_{\check{I}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}},$$

$$\prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(1-\check{Z}_{\check{I}_{\hat{J}}}(\breve{u}).\check{Z}_{\check{I}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}} \leq \prod_{\hat{J}=1}^{n}\left(1-\sin\frac{\pi}{2}\left(1-\check{Z}_{\check{I}_{\hat{J}}}(\breve{u})\right).\left(1-\check{Z}_{\check{I}}(\breve{u})\right)\right)^{\Theta_{\hat{J}}}$$

Therefore, from the above equation we get

$$ST-SFWA\left(\breve{I}_1\oplus\cdots\oplus\breve{I}_n\right)\geq ST-SFWA\left(\breve{I}_1\otimes\cdots\otimes\breve{I}_n\right)$$

□

6. Decision Making Approach

This section provides a strategy, preceded by an example, to solve the DM problem. For this reason, let it be the m alternative $(\hat{\gamma}_1,\cdots,\hat{\gamma}_m)$ that is evaluated by a group of experts under the n different attribute $\left(\breve{G}_1,\cdots,\breve{G}_n\right)$. That expert test $\hat{\gamma}_i$ and $\breve{G}_j$ and gives their preferences in terms of $SFNs$ $\alpha_{ij}^{(\kappa)}=\left(\breve{E}_{ij}^{(\kappa)},\check{R}_{ij}^{(\kappa)},\check{Z}_{ij}^{(\kappa)}\right)$, where $i=1\,(1)\,m$; $j=1\,(1)\,n$; $\kappa=1\,(1)\,d$. Then, the value of every alternative $\hat{\gamma}_i$ with the $\breve{G}_j$ is shown as;

$$\hat{\gamma}_i=\left[\left(\breve{G}_1,\alpha_{i1}\right),\left(\breve{G}_2,\alpha_{i2}\right),\cdots,\left(\breve{G}_n,\alpha_{in}\right)\right],$$

let $\Theta_j>0$ be the normalized weights of attribute $\breve{G}_j$. The following steps are taken to calculate the best choice.

Step 1: In terms of decision matrix, summarize the values of each alternative $\hat{D}^{(\kappa)}=\alpha_{ij}^{(\kappa)}$ with SFS information.

Step 2: Aggregate the different preferences $\alpha_{ij}^{(\kappa)},\kappa=1,\cdots,d$ into $\alpha_{ij}=\left(\breve{E}_{ij},\check{R}_{ij},\check{Z}_{ij}\right)$ utilizing either $ST-SFWA$ or $ST-SFWG$ by operators.

Step 3: Establish the normalized decision matrix $R=(r_{ij})$ from $\hat{D}=(\alpha_{ij})$, where r_{ij} is computed as

$$r_{ij}=\begin{cases}\left(\breve{E}_{ij},\check{R}_{ij},\check{Z}_{ij}\right) & \text{if benefit type attributes}\\ \left(\check{Z}_{ij},\check{R}_{ij},\breve{E}_{ij}\right) & \text{if cost type attributes}\end{cases}$$

Step 4: If the weights of the attributes are known as before, then use them. Otherwise, we measure these by using the entropy principle. For this, the information entropy of attribute $\breve{G}_j$ is given as;

(6.1)

$$\Xi_j=\frac{1}{\left(\sqrt{2}-1\right)m}\sum_{j=1}^{m}\left[\sin\left(\frac{\pi}{4}\left(1+\breve{E}_{ij}-\check{R}_{ij}-\check{Z}_{ij}\right)\right)+\sin\left(\frac{\pi}{4}\left(1-\breve{E}_{ij}+\check{R}_{ij}+\check{Z}_{ij}\right)\right)-1\right],$$

where $\frac{1}{\left(\sqrt{2}-1\right)m}$ is a constant for assuring $0\leq\Xi_j\leq 1$.

Based on it, the weights of the attributes are obtained as $\omega = (\omega_1, \cdots, \omega_n)$, where

$$\omega_j = \frac{1 - \Xi_j}{n - \sum_{j=1}^{n} \Xi_j} \tag{6.2}$$

Step 5: With weight vector ω and using the proposed ST-SFOWA or ST-SFOWG aggregation operators, the collective values r_i for each alternative $\hat{\gamma}_i$ are calculated.

Step 6: Find the score values of $r_i\,(i = 1, \cdots, m)$

Step 7: Grade all the possible alternative $\hat{\gamma}_i\,(i = 1, \cdots, m)$ and select the most desirable alternative(s).

7. Illustrative Example

In this section , the results of the established MAGDM approach are reviewed with the example and their results are compared with those of the current MAGDM approaches.

7.1. Application of the proposed MAGDM method. Consider a decision-making problem adapted from [27] about the selection of the best option. To apply the developed approach effectively, a decision making problem with customers' choice to purchase a laptop from four different options alternative $(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3, \hat{\gamma}_4, \hat{\gamma}_5)$ on the basis of the parameters $\left(\breve{G}_1, \breve{G}_2, \breve{G}_3\right)$, where $\breve{G}_1$ stands for processor, $\breve{G}_2$ stands for system memory, and $\breve{G}_3$ stands for hard size, respectively. Let, four decision makers/experts (E_1, E_2, E_3) with weight vector $\Theta = (0.33, 0.37, 0.30)$ provide their individual assessment in the form of SFNs for each option and the corresponding assessments are presented in Table 1-3, respectively. Then, the steps of the presented approach are implemented to find the best suitable option and are demonstrated as follows:

Step 1: The evaluation of each expert is summarized in Table 1, 2 and 3.

Table 1. Decision matrix given by expert $E_{(1)}$

	$\breve{G}_1$	$\breve{G}_2$	$\breve{G}_3$
$\hat{\gamma}_1$	$(0.29, 0.54, 0.61)$	$(0.44, 0.59, 0.56)$	$(0.60, 0.31, 0.33)$
$\hat{\gamma}_2$	$(0.54, 0.44, 0.63)$	$(0.61, 0.48, 0.54)$	$(0.55, 0.34, 0.36)$
$\hat{\gamma}_3$	$(0.27, 0.65, 0.68)$	$(0.73, 0.43, 0.42)$	$(0.51, 0.55, 0.27)$
$\hat{\gamma}_4$	$(0.30, 0.22, 0.63)$	$(0.60, 0.47, 0.63)$	$(0.46, 0.47, 0.37)$
$\hat{\gamma}_5$	$(0.54, 0.55, 0.49)$	$(0.71, 0.54, 0.42)$	$(0.41, 0.53, 0.46)$

Table 2. Decision matrix given by expert $E_{(2)}$

	$\breve{G}_1$	$\breve{G}_2$	$\breve{G}_3$
$\hat{\gamma}_1$	$(0.42, 0.34, 0.68)$	$(0.56, 0.47, 0.37)$	$(0.74, 0.26, 0.30)$
$\hat{\gamma}_2$	$(0.78, 0.42, 0.44)$	$(0.67, 0.24, 0.49)$	$(0.81, 0.20, 0.29)$
$\hat{\gamma}_3$	$(0.59, 0.37, 0.51)$	$(0.44, 0.62, 0.34)$	$(0.46, 0.44, 0.53)$
$\hat{\gamma}_4$	$(0.47, 0.39, 0.54)$	$(0.49, 0.58, 0.42)$	$(0.34, 0.66, 0.40)$
$\hat{\gamma}_5$	$(0.56, 0.36, 0.48)$	$(0.50, 0.25, 0.55)$	$(0.52, 0.35, 0.53)$

Table 3. Decision matrix given by expert $E_{(3)}$

	$\breve{G}_1$	$\breve{G}_2$	$\breve{G}_3$
$\hat{\gamma}_1$	$(0.53, 0.27, 0.36)$	$(0.51, 0.31, 0.48)$	$(0.55, 0.33, 0.42)$
$\hat{\gamma}_2$	$(0.61, 0.38, 0.51)$	$(0.54, 0.47, 0.29)$	$(0.65, 0.29, 0.55)$
$\hat{\gamma}_3$	$(0.58, 0.45, 0.27)$	$(0.59, 0.33, 0.68)$	$(0.61, 0.42, 0.38)$
$\hat{\gamma}_4$	$(0.42, 0.39, 0.57)$	$(0.58, 0.26, 0.52)$	$(0.81, 0.23, 0.49)$
$\hat{\gamma}_5$	$(0.26, 0.64, 0.50)$	$(0.27, 0.59, 0.44)$	$(0.44, 0.39, 0.57)$

Step 2. By taking the weight of the experts i.e., $\Theta = (0.33, 0.37, 0.30)^T$ and utilize the $ST - SFWA$ operator to achieve the collective data on each alternative of the expert. The result are shown in Table 4.

Table 4. Aggregated values of experts by using ST-SFWA operator

	$\breve{G}_1$	$\breve{G}_2$	$\breve{G}_3$
$\hat{\gamma}_1$	$(0.561, 0.274, 0.371)$	$(0.642, 0.242, 0.362)$	$(0.428, 0.514, 0.283)$
$\hat{\gamma}_2$	$(0.472, 0.383, 0.481)$	$(0.361, 0.462, 0.391)$	$(0.615, 0.326, 0.527)$
$\hat{\gamma}_3$	$(0.421, 0.281, 0.634)$	$(0.526, 0.373, 0.473)$	$(0.426, 0.512, 0.249)$
$\hat{\gamma}_4$	$(0.631, 0.193, 0.263)$	$(0.438, 0.254, 0.187)$	$(0.369, 0.417, 0.532)$
$\hat{\gamma}_5$	$(0.386, 0.562, 0.362)$	$(0.297, 0.393, 0.541)$	$(0.335, 0.318, 0.436)$

Step 3: Almost all of the three attributes are just to be the benefit types, then normalization are not needed.

Step 4: Since the attributes weight are completly unknown. Thus, by utilizing the data of the Table 2 and idea of the entropy Eq. (6.1). We obtain the values;

$$\Xi_1 = 0.9498, \Xi_2 = 0.9194, \Xi_3 = 0.8750.$$

By the help of this we find the attribute weights $\omega = (0.3461, 0.3351, 0.3188)^T$. In the figure 1, we show graphicaly the weight vector of the attributes as;

Step 5: Based on $\omega = (0.3461, 0.3351, 0.3188)^T$ and utilizing the $ST - SFWA$ operator, the collective values of each alternatives are gain as;

$$\begin{aligned} \gamma_1 &= (0.452, 0.474, 0.383), \\ \gamma_2 &= (0.552, 0.361, 0.284), \\ \gamma_3 &= (0.621, 0.243, 0.264), \\ \gamma_4 &= (0.398, 0.528, 0.424), \\ \gamma_5 &= (0.628, 0.343, 0.297). \end{aligned}$$

Step 6: We can get the scores of each alternative by using the Eq. (2.3);

$$\bar{S}c(\gamma_1) = 0.543, \bar{S}c(\gamma_2) = 0.629, \bar{S}c(\gamma_3) = 0.600, \bar{S}c(\gamma_4) = 0.502, \bar{S}c(\gamma_5) = 0.674.$$

Step 7: According to the score values as $\bar{S}c(\gamma_5) > \bar{S}c(\gamma_2) > \bar{S}c(\gamma_3) > \bar{S}c(\gamma_1) > \bar{S}c(\gamma_4)$. Thus, the ranking order is $\hat{\gamma}_5 > \hat{\gamma}_2 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_4$. Hence, $\hat{\gamma}_5$ is the best alternative.

During Step 5 of the established method, the complete analysis by changing aggregation operators is analyzed and their results are shown in Table 5.

Table 5. Impact of different AOs and their ranking

Operators	Score values					Ranking
	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\gamma}_3$	$\hat{\gamma}_4$	$\hat{\gamma}_5$	
ST-SFOWA	0.543	0.629	0.600	0.502	0.674	$\hat{\gamma}_5 > \hat{\gamma}_2 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_4$
ST-SFHA	0.484	0.573	0.542	0.425	0.591	$\hat{\gamma}_5 > \hat{\gamma}_2 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_4$
ST-SFOWG	0.534	0.716	0.642	0.587	0.753	$\hat{\gamma}_5 > \hat{\gamma}_2 > \hat{\gamma}_3 > \hat{\gamma}_4 > \hat{\gamma}_1$
ST-SFHG	0.663	0.753	0.711	0.638	0.776	$\hat{\gamma}_5 > \hat{\gamma}_2 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_4$

We can therefore conclude from all the above-mentioned computational process that the alternative $\hat{\gamma}_2$ is really the best option amongst the other options and therefore it is strongly recommended that the appropriate option is $\hat{\gamma}_2$. In the figure 2, we show graphicaly the ranking of the alternatives by using their score values.

7.2. Comparative Analysis. In this subsection, we give some brief discussion on the comparison of the proposed method with some well know related methods [4, 5, 28, 36, 43, 60].

7.3. Comparison Analysis. The comparison of the developed approach with the existing approaches to examine the reliability and effectiveness of the explored method. The established method are compared with the some other methods based on SFS was established by Ashraf et al. [4], Spherical fuzzy Dombi aggregation operators and their application in group DM problems, Ashraf et al. [5], Spherical fuzzy sets and its representation of Spherical fuzzy t-norms and t-conorms, Kutlu et al. [28], Spherical fuzzy sets and Spherical fuzzy TOPSIS method, Mahmood et al. [36], An approach toward DM and medical diagnosis problems using the concept of Spherical fuzzy sets, Rafiq et al. [43], The cosine similarity measures of Spherical fuzzy sets and their applications in DM, Zeng et al. [60], Covering based Spherical fuzzy rough set model hybrid with TOPSIS for multiple attribute DM.

Table 6. Comparison of ranking with different AOs

Authors	Score values					Ranking
	$\hat{\gamma}_1$	$\hat{\gamma}_2$	$\hat{\gamma}_3$	$\hat{\gamma}_4$	$\hat{\gamma}_5$	
Ashraf et al. [4]	0.761	0.832	0.804	0.746	0.855	$\hat{\gamma}_5 > \hat{\gamma}_2 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_4$
Ashraf et al. [5]	0.529	0.638	0.593	0.514	0.664	$\hat{\gamma}_5 > \hat{\gamma}_2 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_4$
Kutlu et al. [28]	0.669	0.746	0.707	0.653	0.773	$\hat{\gamma}_5 > \hat{\gamma}_2 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_4$
Mahmood et al. [36]	0.227	0.311	0.274	0.158	0.352	$\hat{\gamma}_5 > \hat{\gamma}_2 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_4$
Rafiq et al. [43]	0.478	0.539	0.512	0.464	0.587	$\hat{\gamma}_5 > \hat{\gamma}_2 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_4$
Zeng et al. [60]	0.421	0.515	0.465	0.388	0.532	$\hat{\gamma}_5 > \hat{\gamma}_2 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_4$

From the outcomes of the proposed operators and the other existing methods, we conclude that ranking lists obtained from both the defined method and the compared methods are slightly different, but the best alternative from all the approaches is same. Thus the sine trigonometric aggeregation operators with the Spherical fuzzy set environment is a good idea to solve DM problems, and there are many hindrances which can be solved by using our proposed theory. The sine trigonometric aggeregation operators with the Spherical fuzzy set environment are more flexible and easy approach and the best alternative can be obtained by a short process. Thus, the result obtained from the defined method are more accurate and closest. In the figure 3, we show graphicaly the ranking of the alternatives based on their score values by using different methods.

7.4. Verification. In this portion, the results given by the proposed aggregation operators are verified by TOPSIS and VIKOR methods.

7.4.1. *By TOPSIS Method.* Here we verify the numerical problem given in Section 7 by TOPSIS method. We have the aggregated information by ST-SFWA operator in Table 4, We will apply TOPSIS method on the data given in Table 4.

To solve the mentioned problem in Section 7, we follow the steps of TOPSIS method as follows.

Step 1: Normalize the decision matrix given in Table 4. Here is no need of normalization as all the measure values are of same type, i.e., benefit type.

Step 2: Identifying the PIS $R^+$and NIS R^-, which are defined as,

$$R^+ = (\zeta_1^+, ..., \zeta_5^+), R^- = (\zeta_1^-, ..., \zeta_5^-),$$

where

$$\zeta_j^+ = \max\{\zeta_{ij}/1 \leq i \leq 5\} \text{ and } \zeta_j^- = \min\{\zeta_{ij}/1 \leq i \leq 5\},$$

which are calculated by using the score function

$$\bar{S}c\left(\breve{I}\right) = \breve{E}_{\breve{I}}(\breve{u}) + \breve{R}_{\breve{I}}(\breve{u}) - \breve{Z}_{\breve{I}}(\breve{u}).$$

Step 3: Calculate the distance for each alternative to R^+ and R^- using the proposed distance measures with criteria weight vector $\omega = (0.3461, 0.3351, 0.3188)$ .i.e.,

$$d_i^+ = \sqrt{\Sigma_{j=1}^m w_j(\zeta_j^+ - \zeta_{ij})^2}, \text{ and } d_i^- = \sqrt{\Sigma_{j=1}^m w_j(\zeta_j^- - \zeta_{ij})^2}.$$

Step 4: Calculate the closeness coefficients to the ideal solution by each alternative by applying the equation,

$$cc_i = d_i^-/(d_i^- + d_i^+)(i = 1, .., 5),$$

the overall closeness coefficients are obtained.

Step 5: Ranking the alternatives by using the score function of SFNs and select the best one. We have the ranking result as

$$\hat{\gamma}_5 > \hat{\gamma}_2 > \hat{\gamma}_3 > \hat{\gamma}_1 > \hat{\gamma}_4.$$

All the calculation results and the alternatives ranking is given in Table 7. According to the calculations of overall coefficients the best one with largest closeness coefficient is $\hat{\gamma}_5$.

Hence by TOPSIS method it is again verified that $\hat{\gamma}_5$ is the most suitable robot to be selected by the manufacturing unit.

Table 7. Calculation Results and Ranking of the Alternatives

Alternatives	Distance for alternative to PIS (d_i^+)	Distance for alternative to NIS (d_i^-)	Closeness coefficients to the ideal solution of alternative (cc_i)	Rank
$\hat{\gamma}_1$	0.172	0.111	0.392	4
$\hat{\gamma}_2$	0.201	0.154	0.434	2
$\hat{\gamma}_3$	0.213	0.149	0.412	3
$\hat{\gamma}_4$	0.372	0.131	0.261	5
$\hat{\gamma}_5$	0.156	0.163	0.511	1

7.4.2. *By VIKOR Method.* Here we solve the numerical problem given in Section 7 by VIKOR method. The aggregated values of all the individual experts evaluation information based on ST-SFWA operator is given in Table 4. For this purpose using $\omega = (0.3461, 0.3351, 0.3188)^T$ as the attribute weight vector, we will apply VIKOR method on the information given in Table 4.

Now, to solve the problem using the VIKOR method, the following steps are utilized.

Step 1: Normalize the decision matrix given in Table 4. Here is no need of normalization as all the measure values are of same type, i.e., benefit type.

Step 2: Identifying the PIS $R^+$and NIS R^-. The PIS $R^+$and NIS R^- are defined as follows:

$$R^+ = (\zeta_1^+, ..., \zeta_5^+), R^- = (\zeta_1^-, ..., \zeta_5^-),$$

where

$$\zeta_j^+ = \max\{\zeta_{ij}/1 \leq i \leq 5\} \text{ and } \zeta_j^- = \min\{\zeta_{ij}/1 \leq i \leq 5\},$$

which are calculated by using the score function $\bar{S}c\left(\breve{I}\right) = \breve{E}_{\breve{I}}(\breve{u}) + \breve{R}_{\breve{I}}(\breve{u}) - \breve{Z}_{\breve{I}}(\breve{u})$.

Step 3: Calculate the values S_i, R_i and Q_i can be obtained by using equations,

$$S_i = \sum_{j=1}^{m} \frac{w_j d(\zeta_{ij}, \zeta_j^+)}{d(\zeta_j^+, \zeta_j^-)},$$

$$R_i = \max_{i \leq j \leq m} \frac{w_j d(\zeta_{ij}, \zeta_j^+)}{d(\zeta_j^+, \zeta_j^-)},$$

and

$$Q_i = \frac{v(S_i - S^*)}{(S^- - S^*)} + \frac{(1-v)(R_i - R^*)}{(R^- - R^*)}.$$

Assume $v = 0.5$, then the calculated results are shown in the Table 8. Also,

$$S^* = 0.42, S^- = 0.76, R^* = 0.33, R^- = 0.421.$$

Step 4: Rank the alternatives by sorting each S_i, R_i, and Q_i values in an decreasing order. The values of Q_i are ranked as

$$Q_4 > Q_1 > Q_3 > Q_2 > Q_5.$$

Step 5: Propose a compromise solution, from the ranking results it can be seen that $\hat{\gamma}_5$, which is ranked the best by measure Q_5, is the compromise solution.

Table 8. Ranking of the Alternatives

Alternatives	S_i	R_i	Q_i	Rank
$\hat{\gamma}_1$	0.731	0.253	0.798	4
$\hat{\gamma}_2$	0.637	0.163	0.431	2
$\hat{\gamma}_3$	0.698	0.234	0.728	3
$\hat{\gamma}_4$	0.912	0.320	0.891	5
$\hat{\gamma}_5$	0.401	0.172	0.00	1

Thus, both the methods (TOPSIS, VIKOR) have been successfully applied for the verification of the results given by the proposed ST-SFWA aggregation operators for the bset alternative selection. Alternative $\hat{\gamma}_5$ is the highest ranked. Hence verified that $\hat{\gamma}_5$ is the best alternative among all.

8. Conclusion

A research relating to aggregation operators was investigated in this study by establishing some new sine trigonometric operation laws for SFSs. During decision making problems, the well defined operation laws play a major role. On the other hand, the sine trigonometric function has both the characteristics of the periodicity and the symmetrical nature of the origin and therefore the most likely to satisfy the experts preference over a multi-time period. Therefore, we describe some sine trigonometric operation laws for SFNs and study their properties in order to take these advantages and make a decision smoother and more important. We have defined various average and geometric AOs on the basis of these operators to club decision makers preference. The different elementary relations between the aggregation operators are discussed and explained in detail. We developed a new MAGDM approach for group DM problems in which goals are classify in terms of SFNs to enforce the proposed laws on decision making problems. Further, we compute the weight of the attribute by combining the subjective and the objective data in terms of measure. The functionality of the proposed method is applied on an example of laptope selection, and superiority and feasibility of the approach are investigated in detail. A comparative study is often carried out with current works to verify its performance.

In the future, we will use the framework built on new multiple attribute assessment models to tackle fuzziness and ambiguity in a variety of DM parameters, such as design choices, building options, site selection and DM problems.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

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Muhammad Qiyas
Department of Mathematics, Abdul Wali Khan University,
Mardan, Pakistan
E-mail: muhammadqiyas@awkum.edu.pk

Saleem Abdullah
Department of Mathematics, Abdul Wali Khan University,
Mardan, Pakistan
E-mail: saleemabdullah@awkum.edu.pk