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CERTAIN SOLITONS ON GENERALIZED (κ, μ) CONTACT METRIC MANIFOLDS

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Abstract. The aim of the present paper is to study some solitons on three dimensional generalized (κ, μ)-contact metric manifolds. We study gradient Yamabe solitons on three dimensional generalized (κ, μ)-contact metric manifolds. It is proved that if the metric of a three dimensional generalized (κ, μ)-contact metric manifold is gradient Einstein soliton then $\mu = \frac{2\kappa}{\kappa-2}$. It is shown that if the metric of a three dimensional generalized (κ, μ)-contact metric manifold is closed m-quasi Einstein metric then $\kappa = \frac{\lambda}{m+2}$ and $\mu = 0$. We also study conformal gradient Ricci solitons on three dimensional generalized (κ, μ)-contact metric manifolds.

1. Introduction

The idea of Ricci flow was introduced by Hamilton [10] in order to solve the famous Poincare conjecture. Later Perelman [16] used the idea of Ricci flow to complete the solution of the conjecture. Since then Ricci flow has become an important topic in differential geometry and topology. A Ricci flow is a heat type parabolic partial differential equation. A self similar solution of Ricci flow is known as Ricci soliton. Ricci soliton on different manifolds have been studied by the first author

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in [20], [21], [22], [23], and [24]. A Ricci soliton is a constant solution of Ricci flow equation upto diffeomorphism and scaling. A Ricci soliton is described by an equation

$$(\mathcal{L}_X g)(U,V) + 2S(U,V) + 2\lambda g(U,V) = 0,$$

where λ is a constant and $\mathcal{L}_X$ denotes the Lie derivative operator along the vector field X. Instead of taking λ as constant, S. Pigola [17] took λ as a smooth function and introduced the notion of almost Ricci solitons. Ricci solitons and Ricci almost solitons have been studied by several authors [8], [11], [15], [18], [19], and [25]. The notion of conformal Ricci soliton was introduced in the paper [2]. A conformal Ricci soliton is given by the equation

$$(L_X g)(U,V) + 2S(U,V) = (2\lambda - (p + \frac{2}{2n+1}))g(U,V).$$

A conformal Ricci soliton is called conformal gradient Ricci soliton if it satisfies the following equation

$$\nabla\nabla f + S = [2\lambda - (p + \frac{2}{2n+1})]g. \tag{1}$$

Conformal Ricci solitons have been studied in the paper [15]. The concept of Yamabe flow was introduced by Hamilton [10]. Yamabe flow is a heat type parabolic partial differential equation of the form

$$\frac{\partial}{\partial t}g = -rg, \quad g(0) = g_0,$$

where $r(t)$ is the scalar curvature of the metric g(t). Yamabe soliton can be defined on a Riemannian manifold satisfying

$$\frac{1}{2}\mathcal{L}_X g = (r - \lambda)g, \tag{2}$$

where λ is a real number. A complete Riemannian metric g on smooth manifold M is said to be gradient Yamabe soliton if there exists a smooth function f such that its Hessian satisfies the equation

$$\nabla\nabla f = (r - \lambda)g. \tag{3}$$

The notion of (κ, μ) contact metric manifolds was introduced by Blair [3]. Taking κ and μ as smooth functions the notion of generalized (κ, μ) contact metric manifold was introduced by Koufogiorgos and Tsichlias [12]. The present paper is organised as follows:

After the introduction we give required preliminaries in Sction 2. In Section 3 we study gradient Yamabe solitons on three dimensional generalized (κ, μ) contact metric manifolds. Section 4 contains gradient Einstein solitons on three dimensional generalized (κ, μ)-contact metric manifolds. In Section 5, we study closed m-quasi Einstein metrics on three dimensional generalized (κ, μ)-contact metric manifolds. Section 6 contains conformal gradient Ricci solitons on three dimensional generalized (κ, μ) contact metric manifolds. Last Section gives supporting example.

2. Some preliminaries on contact metric manifolds

A $(2n+1)$ dimensional smooth manifold M is said to admit an almost contact metric structure (ϕ, ξ, η, g) if it admits a tensor field ϕ of type (1, 1), a vector field ξ and a 1-form η satisfying [5]:

$$\phi^2 U = -U + \eta(U)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(\phi(U)) = 0.$$

An almost contact metric structure is said to be normal if the almost complex structure J on the product manifold is defined by

$$J(X, f\frac{d}{dt}) = (\phi U - f\xi, \eta(U)\frac{d}{dt})$$

is integrable, where U is tangent to M, t is the coordinate of $\mathbb{R}$ and f is the smooth function on $M \times \mathbb{R}$. Let g be a compatible Riemannian metric with almost contact structure (ϕ, ξ, η), that is

$$g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V).$$

Then M becomes an almost contact metric structure (ϕ, ξ, η, g). From above it can be easily shown that

$$g(U, \phi V) = -g(\phi U, V), \quad g(U, \xi) = \eta(U),$$

for all $U, V \in \chi(M)$. An almost contact metric structure becomes a contact metric structure if

$$g(U, \phi V) = d\eta(U, V),$$

where $U, V \in \chi(M)$. The 1-form η is called a contact form and ξ is its chracteristic vector field. We define a (1, 1) tensor field h by $h = \frac{1}{2}\mathcal{L}_\xi \phi$, where $\mathcal{L}$ denote the Lie derivative. Then h is symmetric and satisfies the conditions $h\phi = -\phi h$, $Tr.h = Tr.\phi h = 0$ and $h\xi = 0$.

Also

$$\nabla_U \xi = -\phi U - \phi h U, \tag{4}$$

holds in a contact metric manifold. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is a Sasakian manifold if and only if

$$(\nabla_U \phi)(V) = g(U, V)\xi - \eta(V)U,$$

where $U, V \in \chi(M)$ and ∇ is the Livi-Civita connection of the Riemannian metric g. A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ for which ξ as a killing vector is said to be a K-contact metric manifold. A Sasakian manifold is K-contact but not conversely. However a 3-dimensional K-contact manifold is Sasakian. It is known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(U, V)\xi = 0$. On the other hand, on a Sasakian manifold the following relation holds

$$R(U, V)\xi = \eta(V)U - \eta(U)V,$$

where R is the Riemannian curvature tensor on M defined by

$$R(U, V)W = \nabla_U \nabla_V W - \nabla_V \nabla_U W - \nabla_{[U,V]} W. \tag{5}$$

As a generalization of both the manifolds with $R(U, V)\xi = 0$ and the Sasakian case, D. E. Blair, T. Koufogogiorgos and B. J. Papantonion [4] introduced the (κ, μ)- nullity distribution on a contact metric manifold and gave several reasons for studying it.

The (κ, μ)-nullity distribution $N(\kappa, \mu)$ [4] of a contact metric manifold M is defined by

$$\begin{aligned} N(\kappa, \mu) : p \longrightarrow N_p(\kappa, \mu) &= [W \in T_p M : R(U, V)W \\ &= (\kappa I + \mu h)(g(V, W)U - g(U, W)V)], \end{aligned}$$

for all $U,V\in T_pM$, where $(\kappa,\mu)\in\mathbb{R}^2$. Thus we have

$$R(U,V)W=(\kappa I+\mu h)R_0(U,V)\xi,$$

where $R_0(U,V)\xi=\eta(V)U-\eta(U)V$.

If $\mu=0$ the (κ,μ)-nullity distribution reduces to κ-nullity distribution [27]. The κ-nullity distribution $N(\kappa)$ of a Riemannian manifold is defined by [27].

$$\begin{aligned}N(\kappa):p\longrightarrow N_p(\kappa) &= [Z\in T_pM : R(U,V)W\\ &= \kappa(g(V,W)U-g(U,W)V)],\end{aligned}$$

κ being a constant. If the characteristic vector field $\xi\in N(\kappa)$, then we call a contact metric manifold a $N(\kappa)$-contact metric manifold. If $\kappa=1$, then the manifold is Sasakian and if $\kappa=0$, then the manifold is locally isometric to the product $E^{n+1}(0)\times S^n(4)$ for $n>1$ and flat for $n=1$.

Futhermore, in a three dimensional generalized (κ,μ) contact metric manifold the following relations hold [28]:

$$h^2=(\kappa-1)\phi^2,\ \ \kappa\leqslant 1.$$

$$\begin{aligned}R(U,V)W &= -(\kappa+\mu)[g(V,W)U-g(U,W)V]\\ &+ (2\kappa+\mu)[g(V,W)\eta(U)\xi-g(U,W)\eta(V)\xi\\ &+ \eta(V)\eta(W)U-\eta(U)\eta(W)V]\\ &+ \mu[g(V,W)hU-g(U,W)hV+g(hV,W)U-g(hU,W)V].\end{aligned}\tag{6}$$

$$S(U,V)=-\mu g(U,V)+\mu g(hU,V)+(2\kappa+\mu)\eta(U)\eta(V).\tag{7}$$

$$QU=(-U+hU)\mu+(2\kappa+\mu)\eta(U)\xi.\tag{8}$$

$$r=2(\kappa-\mu).\tag{9}$$

$$(\nabla_U\eta)V=g(U,\phi V)-g(U,\phi hV).\tag{10}$$

$$(\nabla_U h)V = [(1-\kappa)g(U,\phi V)\xi + g(U,h\phi V)]\xi - \eta(V)[(1-\kappa)\phi U + \phi hU)] - \mu\eta(U)\phi hV. \tag{11}$$

$$(\nabla_U \phi)V = [g(U,V) + g(U,hV)]\xi - \eta(V)(U + hU). \tag{12}$$

$$R(U,V)\xi = \kappa[\eta(V)U - \eta(U)V] + \mu[\eta(V)hU - \eta(U)hV]. \tag{13}$$

A (κ,μ)-contact metric manifold $M^3(\phi,\xi,\eta,g)$ is a generalized (κ,μ)-contact metric manifold in which κ, μ are smooth functions. In a generalized (κ,μ) contact metric manifold $M^3(\phi,\xi,\eta,g)$, besides, the following relations also hold [2]:

$$\xi\kappa = 0. \tag{14}$$

$$hgrad\ \mu = grad\ \kappa. \tag{15}$$

Generalized (κ,μ)-contact manifolds have been studied by several authors such as Gouli-Andreou [9], Yildiz et al. [28], De et al. [7] and many others.

LEMMA 2.1. *In a three-dimensional generalized (κ,μ)-contact metric manifold, $\xi r = 0$.*

Proof. Covariant differentiation of (8) is taken along the vector field V and using (10), (11) we get

$$\begin{aligned}(\nabla_V Q)U &= \mu[((1-\kappa)g(V,\phi U) - g(V,\phi hU))\xi \\ &- \eta(U)((1-\kappa)\phi U + \phi hU) - \mu\eta(V)\phi hU] \\ &+ V(\mu)(-U + hU) + (2\kappa+\mu)\eta(U)\nabla_V\xi \\ &+ (2\kappa+\mu)[g(V,\phi U) - g(V,\phi hU)]\xi + (2V(\kappa) + V(\mu))\eta(U)\xi.\end{aligned} \tag{16}$$

Replacing U by ξ in (16) we have

$$(\nabla_V Q)\xi = 2V(\kappa)\xi + (2\kappa+\mu)(-\phi V - \phi hV).$$

Contracting the above equation along the vector field V and using (14) and $\text{div}Q = \frac{1}{2}dr$, we get

$$\xi r = 0.$$

This completes the proof. □

3. Gradient Yamabe solitons on three dimensional generalized (κ, μ) contact metric manifolds

THEOREM 3.1. *If a three dimensional generalized (κ, μ) contact metric manifold admits gradient Yamabe soliton, then $\kappa = 0$.*

Proof. From (3) we obtain

$$\nabla_V Df = (r - \lambda)V. \tag{17}$$

Differentiating covariantly along the vector gield U of (17) and applying (5) we get

$$R(U, V)Df = dr(U)V - dr(V)U. \tag{18}$$

Contracting the equation (18) along U we obtain

$$S(V, Df) = -2dr(V). \tag{19}$$

Substituting U by Df in (7) and using (19) we have

$$-2dr(V) = -\mu g(Df, V) + \mu g(hDf, V) + (2\kappa + \mu)\eta(Df)\eta(V).$$

Putting $V = \xi$ in the above equation and using Lemma 2.1. we get

$$\kappa = 0 \quad \text{or} \quad \xi f = 0.$$

This completes the proof. □

THEOREM 3.2. *If a three dimensional generalized (κ, μ) contact metric manifold admits gradient Yamabe soliton, then $\mu = \frac{grad\ r}{hgrad\ f}$.*

Proof. Taking inner product with ξ of the equation (18) we lead

$$g(R(U, V)\xi, Df) = dr(U)\eta(V) - dr(V)\eta(U). \tag{20}$$

Using (13) in (20) we have

$$\mu\eta(V)df(hU) - \mu\eta(U)df(hV) = dr(U)\eta(V) - dr(V)\eta(U).$$

Setting $U=\xi$ in the above and using Lemma 2.1. we obtain

$$\mu=\frac{grad\, r}{hgrad\, f}.$$

This completes the proof. $\square$

4. Three dimensional generalized (κ,μ) contact metric manifolds admitting gradient Einstein solitons

DEFINITION. Let (M,g) be a Riemannian manifold. Then the metric g is said to be gradient Einstein soliton if there is a function $f: M\to\mathbb{R}$ and a constant $\lambda\in\mathbb{R}$ satisfying

$$S-\frac{1}{2}rg+\nabla^2 f=\lambda g. \tag{21}$$

For details about gradient Einstain solitons see [6].

THEOREM 4.1. *If a three-dimensional generalized (κ,μ) contact metric manifold admits gradient Einstein soliton, then $\mu=\frac{2\kappa}{\kappa-2}$.*

Proof. From (21) we obtain

$$QU-\frac{1}{2}rU+\nabla_U Df=\lambda U. \tag{22}$$

Equations (22) and (8) together implies that

$$\nabla_U Df=(\lambda+\kappa)U-\mu hU-(2\kappa+\mu)\eta(U)\xi. \tag{23}$$

Differentiating covariantly along the vector field V of (23) we have

$$\begin{aligned}\nabla_V\nabla_U Df &= V(\kappa)U+(\lambda+\kappa)\nabla_V U-V(\mu)hU-\mu\nabla_V hU\\ &- (2V(\kappa)+V(\mu))\eta(U)\xi-(2\kappa+\mu)(\nabla_V\eta(U))\xi\\ &- (2\kappa+\mu)(\nabla_V\xi)\eta(U).\end{aligned} \tag{24}$$

Interchanging U by V and V by U in the above equation we get

$$\begin{aligned}\nabla_U\nabla_V Df &= U(\kappa)V+(\lambda+\kappa)\nabla_U V-U(\mu)hV-\mu\nabla_U hV\\ &- (2U(\kappa)+U(\mu))\eta(V)\xi-(2\kappa+\mu)(\nabla_U\eta(V))\xi\\ &- (2\kappa+\mu)(\nabla_U\xi)\eta(V).\end{aligned} \tag{25}$$

Putting the values of (24) and (25) in (5) and using (10), (11) we obtain

$$\begin{aligned} R(U,V)Df &= U(\kappa)V - V(\kappa)U - U(\mu)hV + V(\mu)hU \\ &+ 2\mu(1-\kappa)g(V,\phi U)\xi + \mu\eta(V)[(1-\kappa)\phi U + \phi hU] \\ &+ \mu^2\eta(U)\phi hV - \mu\eta(U)[(1-\kappa)\phi V + \phi hV] - \mu^2\eta(V)\phi hU \\ &- (2U(\kappa) + U(\mu))\eta(V)\xi + (2V(\kappa) + V(\mu))\eta(U)\xi \\ &+ 2(2\kappa+\mu)g(V,\phi U)\xi - (2\kappa+\mu)\eta(V)(-\phi U - \phi hU) \\ &+ (2\kappa+\mu)\eta(U)(-\phi V - \phi hV). \end{aligned}$$

Taking inner product with ξ of the above equation and using (13) we get

$$\begin{aligned} \kappa\eta(U)g(V,Df) &= \mu\eta(V)g(hU,Df) - \mu\eta(U)g(hV,Df) \\ &+ U(\kappa)\eta(V) - V(\kappa)\eta(U) + 2\mu(1-\kappa)g(V,\phi U) \\ &- (2U(\kappa) + U(\mu))\eta(V) + (2V(\kappa) + V(\mu))\eta(U) \\ &+ 2(2\kappa+\mu)g(V,\phi U) + \kappa\eta(V)g(U,Df). \end{aligned}$$

Replacing U by ϕU and V by ϕV in the above equation we have

$$\mu = \frac{2\kappa}{\kappa - 2}.$$

This completes the proof. □

5. Closed m-quasi Einstein metrics on three dimensional generalized (κ, μ)-contact metric manifolds

Definition. Ricci tensor S of a Riemannian manifold (M, g) is called η-parallel if $(\nabla_U S)(\phi V, \phi W) = 0$ for all vector fields U, V, W tangent to M and orthogonal to ξ,

where ∇ denotes the Riemannian connection [26]. Besides η-parallel Ricci tensor has been studied in the paper [13].

Definition. We say that a Riemannian manifold (M, g) is a m-quasi Einstein manifold if there exists a function $f : M \to \mathbb{R}$ satisfying

$$S + \nabla^2 f - \frac{1}{m} df \otimes df = \lambda g, \tag{26}$$

where $0 < m \leq \infty$ is an integer. In the above equation, Barros-Ribeiro Jr [1] and Limoncu [13] have taken a 1-form X^b instead of df and the generalization of this equation, which is defind as follows

$$S + \frac{1}{2}\mathcal{L}_X g - \frac{1}{m}X^b \otimes X^b = \lambda g, \tag{27}$$

where X is a potential vector field and X^b is associated to the vector field X. When the 1-form X^b is closed that is $dX^b = 0$, then the metric g is called closed m-quasi Einstein metric.

THEOREM 5.1. *If a three-dimensional generalized (κ, μ) contact metric manifold admits closed m-quasi Einstein metric, then $\kappa = \frac{\lambda}{m+2} =$ constant.*

Proof. For a closed m-quasi Einstein metric we have

$$g(\nabla_U X, V) = g(\nabla_V X, U). \tag{28}$$

We know that $(\mathcal{L}_X g)(U, V) = g(\nabla_U X, V) + g(\nabla_V X, U)$. From (27) and (28) we get

$$QU + \nabla_U X - \frac{1}{m}X^b(U)X = \lambda U. \tag{29}$$

Using (8) in (29) we obtain

$$\nabla_U X = (\lambda + \mu)U - \mu hU - (2\kappa + \mu)\eta(U)\xi + \frac{1}{m}g(U, X)X. \tag{30}$$

Differentiating covariantly along the vector field V we get

$$\begin{aligned}
\nabla_V \nabla_U X &= V(\mu)U + (\lambda + \mu)\nabla_V U - V(\mu)hU - \mu\nabla_V hU \\
&- (2V(\kappa) + V(\mu))\eta(U)\xi - (2\kappa + \mu)\nabla_V \eta(U)\xi \\
&- (2\kappa + \mu)\eta(U)\nabla_V \xi + \frac{1}{m}g(\nabla_V U, X)X + [\frac{\lambda + \mu}{m}g(U, V) \\
&- \frac{\mu}{m}g(U, hV) - \frac{2\kappa + \mu}{m}\eta(U)\eta(V) + \frac{1}{m^2}X^b(V)X^b(U)]X \\
&+ \frac{1}{m}X^b(U)(\lambda + \mu)V - \frac{\mu}{m}X^b(U)hV \\
&- \frac{(2\kappa + \mu)}{m}X^b(U)\eta(V)\xi + \frac{1}{m^2}X^b(U)X^b(V)X.
\end{aligned} \tag{31}$$

Interchanging U and V in the above equation we get

$$\begin{aligned}
\nabla_U\nabla_V X &= U(\mu)V+(\lambda+\mu)\nabla_U V-U(\mu)hV-\mu\nabla_U hV\\
&- (2U(\kappa)+U(\mu))\eta(V)\xi-(2\kappa+\mu)\nabla_U\eta(V)\xi\\
&- (2\kappa+\mu)\eta(V)\nabla_U\xi+\frac{1}{m}g(\nabla_U V,X)X+[\frac{\lambda+\mu}{m}g(V,U)\\
&- \frac{\mu}{m}g(V,hU)-\frac{2\kappa+\mu}{m}\eta(V)\eta(U)+\frac{1}{m^2}X^b(U)X^b(V)]X\\
&+ \frac{1}{m}X^b(V)(\lambda+\mu)U-\frac{\mu}{m}X^b(V)hU\\
&- \frac{(2\kappa+\mu)}{m}X^b(V)\eta(U)\xi+\frac{1}{m^2}X^b(U)X^b(V)X. \qquad (32)
\end{aligned}$$

Using (31), (32) and (5) we obtain

$$\begin{aligned}
R(U,V)X &= U(\mu)V-V(\mu)U-U(\mu)hV-V(\mu)hU-\mu(\nabla_U h)V\\
&+ \mu(\nabla_V h)U-(2U(\kappa)+U(\mu))\eta(V)\xi+(2V(\kappa)+V(\mu))\eta(U)\xi\\
&+ \frac{(\lambda+\mu)}{m}[X^b(V)U-X^b(U)V]-\frac{\mu}{m}[X^b(V)hU-X^b(U)hV]\\
&- (2\kappa+\mu)(\nabla_U\eta)(V)\xi+(2\kappa+\mu)(\nabla_V\eta)(U)\xi\\
&+ (2\kappa+\mu)\eta(U)\nabla_V\xi-(2\kappa+\mu)\eta(V)\nabla_U\xi\\
&- \frac{(2\kappa+\mu)}{m}[X^b(V)\eta(U)-X^b(U)\eta(V)]\xi. \qquad (33)
\end{aligned}$$

Taking inner product with respect to ξ of (33) and using (10), (11) and (13) we obtain

$$\begin{aligned}
\kappa\eta(U)X^b(V) &= \mu\eta(V)X^b(hU)-\mu\eta(U)X^b(hV)+U(\mu)\eta(V)\\
&- (2U(\kappa)+U(\mu))\eta(V)+2\mu(1-\kappa)g(V,\phi U)\\
&- V(\mu)\eta(U)+2(2\kappa+\mu)g(V,\phi U)+V(\mu))\eta(U)\\
&+ (2V(\kappa)+\frac{(\lambda+\mu)}{m}[X^b(V)\eta(U)-X^b(U)\eta(V)]\\
&- \frac{(2\kappa+\mu)}{m}[X^b(V)\eta(U)-X^b(U)\eta(V)]+\kappa\eta(V)X^b(U). \qquad (34)
\end{aligned}$$

Putting $U=\xi$ in (34) we get

$$(\frac{\lambda}{m}-\frac{2\kappa}{m}-\kappa)g(X,\phi V)-\mu g(X,h\phi V)=0. \qquad (35)$$

Antisymmetrizing the foregoing equation we obtain

$$\kappa = \frac{\lambda}{m+2}.$$

This completes the proof. □

THEOREM 5.2. *If a three-dimensional generalized* (κ, μ) *contact metric manifold admits closed* m*-quasi Einstein metric, then* $\mu = 0$.

Proof. Using the value of κ in (35) we get

$$\mu = 0.$$

This completes the proof. □

From the above two results we get the following:

COROLLARY 5.3. *If a three-dimensional generalized* (κ, μ) *contact metric manifold admits closed* m*-quasi Einstein metric, then it is* (κ, μ)*-contact metric manifold.*

Putting the values of κ, μ, in (7) and (9) we have $S(U, V) = \frac{\lambda}{m+2}\eta(U)\eta(V)$ and $r = \frac{2\lambda}{m+2} =$ constant. Hence we state the following:

COROLLARY 5.4. *If a three-dimensional generalized* (κ, μ) *contact metric manifold admits closed* m*-quasi Einstein metric, then its scalar curvature is constant.*

Since κ, μ are constants, we get from (7)

$$\begin{aligned} (\nabla_W S)(U, V) &= \frac{\lambda}{m+2}[\eta(V)(g(W, \phi U) - g(W, \phi h U)) \\ &+ \eta(U)(g(W, \phi V) - g(W, \phi h V))]. \end{aligned}$$

If we take U, V orthogonal to ξ then from the above

$$(\nabla_W S)(\phi U, \phi V) = 0.$$

Which implies the following:

COROLLARY 5.5. *If a three-dimensional generalized* (κ, μ) *contact metric manifold admits closed* m*-quasi Einstein metric, then its Ricci tensor is* η*-parallel.*

6. Conformal gradient Ricci solitons on three dimensional generalized (κ, μ) contact metric manifolds

THEOREM 6.1. *If a three-dimensional generalized (κ, μ) contact metric manifold admits conformal gradient Ricci soliton, then $\mu = \frac{2\kappa}{\kappa - 2}$.*

Proof. Using (1) in the following

$$R(U,V)Df = \nabla_U \nabla_V Df - \nabla_V \nabla_U Df - \nabla_{[U,V]} Df,$$

where D is the gradient operator, we get

$$R(U,V)Df = (\nabla_V Q)U - (\nabla_U Q)V. \tag{36}$$

From (8) we obtain

$$\begin{aligned} (\nabla_V Q)U &= (-U + hU)V(\mu) + \mu(\nabla_V h)U + (2\kappa + \mu)(\nabla_V \eta)U\xi \\ &+ (2V(\kappa) + V(\mu))\eta(U)\xi + (2\kappa + \mu)\eta(U)\nabla_V \xi. \end{aligned}$$

Interchanging U and V in the foregoing equation we have

$$\begin{aligned} (\nabla_U Q)V &= (-V + hV)U(\mu) + \mu(\nabla_U h)V + (2\kappa + \mu)(\nabla_U \eta)V\xi \\ &+ (2U(\kappa) + U(\mu))\eta(V)\xi + (2\kappa + \mu)\eta(V)\nabla_U \xi. \end{aligned}$$

Using above two equations in (36) we get

$$\begin{aligned} R(U,V)Df &= (-U + hU)V(\mu) + \mu(\nabla_V h)U + (2\kappa + \mu)(\nabla_V \eta)U\xi \\ &+ (2V(\kappa) + V(\mu))\eta(U)\xi + (2\kappa + \mu)\eta(U)\nabla_V \xi \\ &- (-V + hV)U(\mu) - \mu(\nabla_U h)V - (2\kappa + \mu)(\nabla_U \eta)V\xi \\ &- (2U(\kappa) + U(\mu))\eta(V)\xi - (2\kappa + \mu)\eta(V)\nabla_U \xi. \end{aligned} \tag{37}$$

Putting $U = \xi$ and using (11), (10) in (37) we have

$$\begin{aligned} R(\xi,V)Df &= -\mu[(1-\kappa)\phi V + \phi hV] + 2V(\kappa)\xi \\ &- (2\kappa + \mu)(\phi V + \phi hV) + \mu^2 \phi hV \\ &- (-V + hV)\xi(\mu) - \xi(\mu)\eta(V)\xi. \end{aligned} \tag{38}$$

Taking inner product with respect to X with (38) and using (6) we get

$$- \quad (\kappa + \mu)g(V,Df)\eta(X) + (\kappa + \mu)\eta(Df)g(V,X)$$

$$
\begin{aligned}
&+ \quad (2\kappa+\mu)\eta(X)g(V,Df) - (2\kappa+\mu)\eta(Df)g(V,X)\\
&- \quad \mu\eta(Df)g(hV,X) + \mu g(hV,Df)\eta(X)\\
&= \quad -\mu[(1-\kappa)g(\phi V,X) + g(\phi hV,X)] + 2V(\kappa)\eta(X)\\
&- \quad (2\kappa+\mu)(g(\phi V,X) + g(\phi hV,X)) + \mu^2 g(\phi hV,X)\\
&- \quad \xi(\mu)(g(-V,X) + g(hV,X)) - \xi(\mu)\eta(X)\eta(V).
\end{aligned}
$$

Antisymmetrizing the above equation we obtain

$$
\begin{aligned}
&- \quad (\kappa+\mu)[g(V,Df)\eta(X) - g(X,Df)\eta(V)]\\
&+ \quad (2\kappa+\mu)[g(V,Df)\eta(X) - g(X,Df)\eta(V)]\\
&+ \quad \mu[g(hV,Df)\eta(X) - g(hX,Df)\eta(V)]\\
&= \quad -2\mu(1-\kappa)g(\phi V,X) - 2(2\kappa+\mu)g(\phi V,X)\\
&+ \quad 2(V(\kappa)\eta(X) - X(\kappa)\eta(V)).
\end{aligned}
\tag{39}
$$

Replacing X by ϕX and V by ϕV in the above equation we get

$$\mu = \frac{2\kappa}{\kappa-2}.$$

This completes the proof. □

7. Example

EXAMPLE 7.1. We consider the 3-dimensional manifold $M = \{(u,v,w) \in \mathbb{R}^3 |\ w \neq 0\}$, where (u,v,w) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$e_1 = \frac{\partial}{\partial u}, \quad e_2 = -2vw\frac{\partial}{\partial u} + \frac{2u}{w^3}\frac{\partial}{\partial v} - \frac{1}{w^2}\frac{\partial}{\partial w}, \quad e_3 = \frac{1}{w}\frac{\partial}{\partial v}$$

are linerarly independent at each of M. Let g be the Riemannian metric defined by $g(e_i,e_j) = \delta_{ij}$, i, j = 1, 2, 3. Let ∇ be the Riemannian connection and R the curvature tensor of g. We easily get

$$[e_1,e_2] = \frac{2}{w^2}e_3, \quad [e_2,e_3] = 2e_1 + \frac{1}{w^3}e_3, \quad [e_3,e_1] = 0.$$

Let η be the 1-form defined by $\eta(V) = g(V,e_1)$ for any $V \in \chi(M)$. Because $\eta \wedge d\eta \neq 0$ everywhere on M, η is a contact form. Let ϕ be the (1, 1)-tensor field, defined by $\phi e_1 = 0$, $\phi e_2 = e_3$, $\phi e_3 = -e_2$. Using the linearity of ϕ, $d\eta$, and g we define $\eta(e_1) = 1$, $\phi^2 V = -V + \eta(V)e_1$, $d\eta(V, W) = g(V, \phi W)$ and $g(\phi V, \phi W) = g(V, W) - \eta(V)\eta(W)$ for any $V, W \in \chi(M)$. Hence (ϕ, e_1, η, g) defines a contact metric structure on M and so M together with this structure is a contact metric manifold. Putting $\xi = e_1$, $U = e_2$, $\phi U = e_3$ and using Koszul formula

we calculate

$$\nabla_U \xi = -(1 + \frac{1}{w^2})\phi U, \qquad \nabla_{\phi U}\xi = (1 - \frac{1}{w^2})U$$

$$\nabla_\xi U = (-1 + \frac{1}{w^2})\phi U, \quad \nabla_\xi \phi U = (1 - \frac{1}{w^2})U, \quad \nabla_U U = 0$$

$$\nabla_U \phi U = (1 + \frac{1}{w^2})\xi, \;\; \nabla_{\phi U} U = (-1 + \frac{1}{w^2})\xi - \frac{1}{w^3}\phi U, \;\; \nabla_{\phi U}\phi U = \frac{1}{w_3^3}U.$$

Therefore for the tensor field h we get $h\xi = 0$, $hU = \lambda U$, $h\phi U = -\lambda \phi U$ where $\lambda = \frac{1}{w^2}$. Now, putting $\mu = 2(1 - \frac{1}{w^2})$ and $\kappa = \frac{w^4 - 1}{w^4}$ we finally get

$$R(U, \xi)\xi = \kappa(\eta(\xi)U - \eta(U)\xi) + \mu(\eta(\xi)hU - \eta(U)h\xi),$$

$$R(\phi U, \xi)\xi = \kappa(\eta(\xi)\phi U - \eta(\phi U)\xi) + \mu(\eta(\xi)h\phi U - \eta(\phi U)h\xi),$$

$$R(U, \phi U)\xi = \kappa(\eta(\phi U)U - \eta(U)\phi U) + \mu(\eta(\phi U)hU - \eta(U)h\phi U).$$

These relations yield the following, by straightforward calculation

$$R(Z, W)\xi = \kappa(\eta(W)Z - \eta(Z)W) + \mu(\eta(W)hZ - \eta(Z)hW),$$

where κ and μ are non-constant smooth functions. Hence M is a generalized (κ, μ)-contact metric manifold. For more detalis about this example see [12].

In this example, if we choose $w = -1$ everywhere on the manifold, then $\kappa = 0$ and $\mu = 0$. For $\lambda = -1$ and $f = d(u + v + w) + e$, where d, e are real constants that refers the Riemannian metric g is a gradient Einstein soliton, which verifies Theorems 3.1 and 4.1. For $\lambda = -1$ and $f =$ constant, the Riemannian metric g is m-quasi Einstein metric, which verifies Theorems 5.1, 5.2 and 6.1 and Corollary 5.3. Again for any real number of λ the Ricci tensor is η- parallel. From the components of the Ricci tensor of the manifold it follows that the scalar curvature $r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) = 0$, which is a constant. Therefore Corollaries 5.4 and 5.5 are verified.

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