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CONORMAL DERIVATIVE PROBLEM FOR ELLIPTIC EQUATIONS IN DIVERGENCE FORM WITH PARTIAL DINI MEAN OSCILLATION COEFFICIENTS

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Abstract. We provide detailed proofs for local gradient estimates for weak solutions to elliptic equations in divergence form with partial Dini mean oscillation coefficients subject to conormal derivative boundary conditions.

1. Introduction and main results

We are concerned with second-order elliptic equations in divergence form

$$\operatorname{div}(ADu) = \operatorname{div} f, \tag{1.1}$$

where the coefficient $A = (a^{ij})_{i,j=1}^d$ is a $d \times d$ matrix-valued function in $\mathbb{R}^d$ satisfying the strong ellipticity condition, i.e., there is a constant $\lambda \in (0,1]$ such that

$$a^{ij}(x)\xi_j\xi_i \ge \lambda|\xi|^2, \quad |a^{ij}(x)| \le \lambda^{-1}$$

for any $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$.

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Recently, in [1] the authors proved both interior and boundary L^p-estimates with $p \in (1, \infty]$ and weak type-$(1,1)$ estimates for derivatives of weak solutions to the elliptic equation (1.1) with the coefficient satisfying *partial Dini mean oscillation condition*. We shall say that a function is of partial Dini mean oscillation if it is merely measurable in x_1-direction and its L^1-mean oscillation with respect to $x' = (x_2, \ldots, x_d)$ satisfies the Dini condition; see Definition 1.1 for more a precise definition. As mentioned in [2], such type of coefficients with no regularity assumption in one direction can be used to model the problems of linearly elastic laminates and composite materials. We note that the boundary estimates in [1] were established under the homogeneous Dirichlet boundary condition, and as mentioned in [1, Remark 2.10], one can obtain the corresponding results under conormal derivative boundary condtions.

In this paper, we present the detailed proofs for the boundary L^p and weak type-$(1,1)$ estimates for the weak solutions to the elliptic equation (1.1) with the conormal derivative boundary conditions.

To state our main results more precisely, we first introduce some notation and definitions. We use $x = (x_1, x')$ to denote a point in $\mathbb{R}^d$ ($d \geq 2$), where $x_1 \in \mathbb{R}$ and $x' = (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}$. We also write $y = (y_1, y')$ and $x_0 = (x_{01}, x_0')$, etc. For $r > 0$, we set

$$\begin{aligned} B_r(x) &= \{y \in \mathbb{R}^d : |x - y| < r\} \\ B_r'(x') &= \{y' \in \mathbb{R}^{d-1} : |x' - y'| < 1\}, \\ B_r^+(x) &= B_r(x) \cap \mathbb{R}^d_+, \end{aligned}$$

where $\mathbb{R}^d_+ = \{x = (x_1, x') : x_1 > 0\}$. We warn the readers that $B_r^+(x)$ is not necessarily a half ball. We use the abbreviations B_r, B_r', and B_r^+ when the center is the origin. We write $D_{x'}u = (D_2 u, \ldots, D_d u)$ so that $Du = (D_1 u, D_{x'} u)$ and

$$(u)_\Omega = \frac{1}{|\Omega|} \int_\Omega u \, dx,$$

where $|\Omega|$ denotes the Lebesgue measure of a measurable set $\Omega \subset \mathbb{R}^d$.

Definition 1.1. (a) Let $f \in L^1(B_6)$. We say that f is of *partial Dini mean oscillation with respect to x' in B_4* if the function $\omega_f : (0, 1] \to$

$[0,\infty)$ defined by

$$\omega_f(r) = \sup_{x \in B_4} \fint_{B_r(x)} \left| f(y) - \fint_{B'_r(x')} f(y_1, z')\, dz' \right| dy$$

satisfies the Dini condition

$$\int_0^1 \frac{\omega_f(r)}{r}\, dr < \infty.$$

(b) Let $f \in L^1(B_6^+)$. We say that f is of *partial Dini mean oscillation with respect to x' in B_4^+* if the function $\omega_f^+ : (0,1] \to [0,\infty)$ defined by

$$\omega_f^+(r) = \sup_{x \in B_4^+} \fint_{B_r^+(x)} \left| f(y) - \fint_{B'_r(x')} f(y_1, z')\, dz' \right| dy$$

satisfies the Dini condition

$$\int_0^1 \frac{\omega_f^+(r)}{r}\, dr < \infty.$$

The main results of the paper read as follows. Throughout the paper, $u \in W^{1,1}(B_6^+)$ is said to satisfy

$$\begin{cases} \operatorname{div}(ADu) = \operatorname{div} f & \text{in } B_6^+, \\ ADu \cdot n = f \cdot n & \text{on } B_6 \cap \partial\mathbb{R}^d_+, \end{cases} \tag{1.2}$$

if

$$\int_{B_6^+} ADu \cdot D\phi\, dx = \int_{B_6^+} f \cdot D\phi\, dx$$

holds for any $\phi \in C_0^\infty(B_6)$. It is clear that, as a test function, one can also use $\phi \in C^\infty(\overline{B_6^+})$ such that $\phi = 0$ on $\partial B_6 \cap \mathbb{R}^d_+$.

Theorem 1.2. *Let $p \in (1,\infty)$ and $u \in W^{1,1}(B_6^+)$ satisfy* (1.2) *with $f = (f_1, \ldots, f_d) \in L^p(B_6^+)^d$.*

(a) *If A is of partial Dini mean oscillation with respect to x' in B_4^+, then*

$$u \in W^{1,p}(B_1^+).$$

(b) *If $f_1 \in L^\infty(B_6^+)$, and A and f are of partial Dini mean oscillation with respect to x' in B_4^+, then*

$$u \in W^{1,\infty}(B_1^+).$$

Moreover, $\hat{U} = a^{1j} D_j u - f_1$ and $D_{x'} u$ are continuous in $\overline{B_1^+}$.

Upper bounds of the L^p-norm of Du and the modulus of continuity of $U = (\hat{U}, D_{x'}u)$ can be found in Section 2.1.

There are many literature dealing with the elliptic equations with the conormal derivative boundary conditions. In [1, Remark 2.10], the $W^{1,\infty}$-regularity result as in Theorem 1.2 was obtained for $W^{1,p}$-weak solutions with $p \in (1, \infty)$. We mention recent papers [5, 6] for C^1-estimates for conormal derivative problem in divergence form and C^2-estimates for oblique derivative problem in nondivergence form with the coefficients of Dini mean oscillation in *all* directions. This class of coefficients was introduced by Dong-Kim in [4] for C^1 and C^2-regularity of solutions to elliptic equations. See [3] for the corresponding regularity results up to the boundary subject to Dirichlet boundary condition. We also refer the reader to [7] for C^1-regularity for quasilinear elliptic equations under the uniform Dini continuity condition.

Remark 1.3. One can extend the results in Theorem 1.2 to functions u satisfying

$$\begin{cases} \operatorname{div}(ADu) = \operatorname{div} f + g & \text{in } B_6^+, \\ ADu \cdot n = f \cdot n & \text{on } B_6 \cap \partial\mathbb{R}^d_+. \end{cases}$$

where, for instance, $g \in L^q(B_6^+)$ with $q > d$ for the assertion (b). To see this, we extend g to B_7^+ so that $(g)_{B_7^+} = 0$ and $\|g\|_{L^q(B_7^+)}$ is comparable to $\|g\|_{L^q(B_6^+)}$. Then by the existence of solutions to the divergence equation in a half ball, there exists $\tilde{g} \in W_0^{1,q}(B_7^+)^d$ such that $\operatorname{div} \tilde{g} = g$ in B_7^+, which implies that u satisfies

$$\begin{cases} \operatorname{div}(ADu) = \operatorname{div}(f + \tilde{g}) & \text{in } B_6^+, \\ ADu \cdot n = (f + \tilde{g}) \cdot n & \text{on } B_6 \cap \partial\mathbb{R}^d_+. \end{cases}$$

Moreover, by the Morrey inequality, we have that $\tilde{g} \in C^{\alpha}(\overline{B_7^+})^d$ with $\alpha = 1 - d/q$, and thus $\tilde{g}$ is of (partial) Dini mean oscillation.

We also prove the following weak type-$(1,1)$ estimates in a ball and a half ball. The corresponding estimates for $W^{1,2}$-weak solutions to Dirichlet problems can be found in [3, 4].

Theorem 1.4. (a) *Let T be a bounded linear operator on $L^2(B_6)^d$ defined by*

$$Tf = Du,$$

where $u \in W^{1,2}(B_6)$ *is a unique weak solution of*

$$\begin{cases} \operatorname{div}(ADu) = \operatorname{div} f & \text{in } B_6, \\ ADu \cdot n = f \cdot n & \text{on } \partial B_6 \end{cases} \tag{1.3}$$

satisfying

$$\int_{B_6} u \, dx = 0.$$

If we assume

$$\omega_A(r) \le C_0 \left(\ln \frac{r}{4} \right)^{-2} \quad \text{for } r \in (0, 1], \tag{1.4}$$

then T *has an extension on the set*

$$\{ f \in L^1(B_6)^d : \operatorname{supp} f \subset B_1 \}$$

to the weak $L^1(B_1)^d$ *space in such a way that for any* $t > 0$, *we have*

$$\left| \{ x \in B_1 : |Tf(x)| > t \} \right| \le \frac{N}{t} \int_{B_1} |f| \, dx,$$

where $N = N(d, \lambda, \omega_A, C_0)$.

(b) *The same result in the assertion* (a) *holds with* B_1^+, B_6^+, *and* ω_A^+ *in place of* B_1, B_6, *and* ω_A, *respectively.*

We end this section with a remark that our results can be extended to elliptic systems.

2. Proofs of main theorems

2.1. Proof of Theorem 1.2. The proof is based on odd/even extension technique and the following interior estimates.

Lemma 2.1 ([1, Theorem 3.2 and Remark 1.4]). *Let* $p \in (1, \infty)$ *and* $u \in W^{1,1}(B_6)$ *satisfy*

$$\operatorname{div}(ADu) = \operatorname{div} f \quad \text{in } B_6,$$

where $f = (f_1, \ldots, f_d) \in L^p(B_6)^d$.

(a) *If* A *is of partial Dini mean oscillation with respect to* x' *in* B_4, *then*

$$u \in W^{1,p}(B_1)$$

and

$$\|u\|_{W^{1,p}(B_1)} \le N \|u\|_{W^{1,1}(B_6)} + N \|f\|_{L^p(B_6)},$$

where $N = N(d, \lambda, \omega_A, p)$.

(*b*) *If* $f_1 \in L^\infty(B_6)$, *and* A *and* f *are of partial Dini mean oscillation with respect to* x' *in* B_4, *then*

$$u \in W^{1,\infty}(B_1).$$

Moreover, $\hat{U} = a^{1j}D_j u - f_1$ *and* $D_{x'}u$ *are continuous in* $\overline{B_1}$.

Proof of Theorem 1.2. We only prove the first assertion of the theorem, because the second is the same with obvious modifications.

Let $\tilde{u}$ be the even extension of u with respect to x_1-variable, i.e.,

$$\tilde{u}(x_1, x') = u(|x_1|, x').$$

Then we see that $\tilde{u} \in W^{1,1}(B_6)$ satisfies

$$\operatorname{div}(\tilde{A}D\tilde{u}) = \operatorname{div}\tilde{f} \quad \text{in } B_6, \tag{2.1}$$

where $\tilde{A}$ and $\tilde{f}$ are given as follows:

(2.2)

$$\tilde{a}^{ij}(x_1, x') = \begin{cases} a^{ij}(|x_1|, x') & \text{for } i = j = 1 \text{ or } i, j \in \{2, \dots, d\}, \\ \operatorname{sgn}(x_1)a^{ij}(|x_1|, x') & \text{otherwise,} \end{cases}$$

$$\tilde{f}_i(x_1, x') = \begin{cases} \operatorname{sgn}(x_1)f_i(|x_1|, x') & \text{for } i = 1, \\ f_i(|x_1|, x') & \text{otherwise.} \end{cases}$$

Since $\tilde{A}$ is of partial Dini mean oscillation with respect to x' in B_4 and $\tilde{f} \in L^p(B_6)^d$, we can apply Lemma 2.1 (a) to (2.1) to obtain that

$$u \in W^{1,p}(B_1^+)$$

and

$$\|u\|_{W^{1,p}(B_1^+)} \le N\|u\|_{W^{1,1}(B_6^+)} + N\|f\|_{L^p(B_6^+)},$$

where $N = N(d, \lambda, \omega_{\tilde{A}}, p) = N(d, \lambda, \omega_A^+, p)$. The assertion ($a$) is proved.

We end this proof with a remark related to the second assertion that the upper bounds of the L^∞-norm of Du and the modulus of continuity of $\hat{U}$ and $D_{x'}u$ can be derived from the corresponding interior estimates in [1, Section 2.2]. Here, we present those upper bounds for future researches:

$$\|Du\|_{L^\infty(B_2^+)} \le N\left(\|Du\|_{L^1(B_6^+)} + \|f_1\|_{L^\infty(B_6^+)} + \int_0^1 \frac{\tilde{\omega}_f^+(t)}{t}\,dt\right)$$

and

$$
\begin{aligned}
|U(x)-U(y)| &\le N\int_0^{2|x-y|}\frac{\tilde{\omega}_f^+(t)}{t}\,dt \\
&\quad + N\left(\|Du\|_{L^1(B_6^+)}+\|f_1\|_{L^\infty(B_6^+)}+\int_0^1\frac{\tilde{\omega}_f^+(t)}{t}\right) \\
&\qquad\qquad \times\left(|x-y|^\gamma+\int_0^{2|x-y|}\frac{\tilde{\omega}_A^+(t)}{t}\,dt\right),
\end{aligned}
$$

for any $x, y \in \overline{B}_1^+$ and $\gamma \in (0,1)$, where $N = N(d,\lambda,\gamma,\omega_A^+)$, $U = (\hat{U}, D_{x'}u)$, and $\tilde{\omega}_\bullet^+$ is a function derived from $\bullet$ as formulated in [1, Section 2.3]. □

2.2. Proof of Theorem 1.4. We shall use the following lemma.

LEMMA 2.2 ([1, Lemma 2.3 (b)]). *Let T be a bounded linear operator on $L^2(B_6)^d$. Suppose that there exist constants*

$$\mu \in (0,1), \quad c \in (1,\infty), \quad C \in (0,\infty),$$

such that for any $x_0 \in B_1$, $r \in (0,\mu)$, and $g \in L^2(B_6)^d$ with

$$\operatorname{supp} g \subset B_r(x_0)\cap B_1, \qquad \int_{B_1} g\,dx = 0,$$

we have

$$\int_{B_1\setminus B_{cr}(x_0)}|Tg|\,dx \le C\int_{B_r(x_0)\cap B_1}|g|\,dx.$$

Then there exists a linear operator S from $L^1(B_1)^d$ to $L^1(B_1)^d$ such that for any $f \in L^2(B_1)^d$,

$$Sf = T(f\mathbb{I}_{B_1})$$

and that for any $t > 0$ and $f \in L^1(B_1)^d$,

$$\big|\{x\in B_1 : |Sf(x)|>t\}\big| \le \frac{N}{t}\int_{B_1}|f|\,dx,$$

where $N = N(d,\mu,c,C)$. In other words, T has an extension on the set

$$\{f\in L^1(B_6)^d : \operatorname{supp} f\subset B_1\}$$

to the weak $L^1(B_1)^d$ space in such a way that for any $t > 0$, we have

$$\big|\{x\in B_1 : |Tf(x)|>t\}\big| \le \frac{N}{t}\int_{B_1}|f|\,dx.$$

Proof of Theorem 1.4. The proof is an adaptation of that of [1, Theorem 3.1]. We first prove the assertion (a). By (1.4) we see that A is of partial Dini mean oscillation with respect to x' in B_4. The coefficients $A^\top = (a^{ji})_{i,j=1}^d$ of the adjoint operator

$$\mathcal{L}^* u = \operatorname{div}(A^\top Du)$$

is also of partial Dini mean oscillation with respect to x' in B_4 satisfying $\omega_{A^\top} = \omega_A$. Hence, by [4, Lemma 3.4] we have

$$\int_0^r \frac{\tilde{\omega}_{A^\top}(t)}{t}\,dt \le N\left(\ln\frac{4}{r}\right)^{-1} \quad \text{for any } r \in (0,1], \tag{2.3}$$

where $N = N(d, \lambda, C_0)$ and $\tilde{\omega}_\bullet$ is a function derived from $\bullet$ as formulated in [1, Remark 2.6].

Let $x_0 \in B_1$, $r \in (0, 1/3)$, and $f \in L^2(B_6)^d$ such that

$$\operatorname{supp} f \subset B_r(x_0) \cap B_1, \qquad \int_{B_1} f\,dx = 0.$$

By the Lax-Milgram theorem, for any $R \in [3r, 2)$ such that $B_1 \backslash B_R(x_0) \neq \emptyset$ and

$$g \in C_0^\infty((B_{2R}(x_0) \setminus B_R(x_0)) \cap B_1)^d,$$

there exists a unique $v \in W^{1,2}(B_6)$ satisfying

$$\int_{B_6} v\,dx = 0$$

and

$$\begin{cases} \operatorname{div}(A^\top Dv) = \operatorname{div} g & \text{in } B_6, \\ A^\top Dv \cdot n = g \cdot n & \text{on } \partial B_6. \end{cases} \tag{2.4}$$

Observe that

$$f \cdot Dv = \hat{f} \cdot V,$$

where

$$\hat{f}_1 = (a^{11})^{-1} f_1, \quad \hat{f}_i = f_i - a^{i1}\hat{f}_1, \quad i \in \{2, \dots, d\},$$

and $V = (a^{j1} D_j v, D_{x'} v)$. Thus, testing (1.3) with v and using the fact that

$$\operatorname{supp} \hat{f} \subset B_r(x_0) \cap B_1, \qquad \int_{B_1} \hat{f}\,dx = 0,$$

we have

$$\int_{B_6} ADu \cdot Dv\,dx = \int_{B_r(x_0)} \hat{f} \cdot \big(V - (V)_{B_r(x_0)}\big)\,dx.$$

From this and (2.4) with a test function u, we get

$$\int_{(B_{2R}(x_0)\setminus B_R(x_0))\cap B_1} g\cdot Du\,dx = \int_{B_r(x_0)} \hat{f}\cdot (V-(V)_{B_r(x_0)})\,dx. \tag{2.5}$$

Since v satisfies $\operatorname{div}(A^\top Dv)=0$ in $B_R(x_0)$, by a similar calculation that lead to [1, Eq. (2.30)] with $\gamma=1/2$, we obtain that

$$|V(x)-V(y)|\le NR^{-d/2}\|Dv\|_{L^2(B_R(x_0))}\left(\left(\frac{|x-y|}{R}\right)^{1/2}+\int_0^{2|x-y|}\frac{\tilde{\omega}_{A^\top}(t)}{t}\,dt\right)$$

for any $x,y\in B_r(x_0)\subset B_{R/6}(x_0)$, which together with (2.3) yields

$$\|V-(V)_{B_r(x_0)}\|_{L^\infty(B_r(x_0))}\le NR^{-d/2}\|Dv\|_{L^2(B_R(x_0))}\left(\left(\frac{r}{R}\right)^{1/2}+\left(\ln\frac{1}{r}\right)^{-1}\right),$$

where $N=N(d,\lambda,\omega_A,C_0)$. Thus by Hölder's inequality, (2.5), duality, and the L^2-estimate

$$\|Dv\|_{L^2(B_6)}\le N\|g\|_{L^2((B_{2R}(x_0)\setminus B_R(x_0))\cap B_1)},$$

we have

$$\int_{(B_{2R}(x_0)\setminus B_R(x_0))\cap B_1}|Du|\,dx\le N\left(\left(\frac{r}{R}\right)^{1/2}+\left(\ln\frac{1}{r}\right)^{-1}\right)\|\hat{f}\|_{L^1(B_r(x_0)\cap B_1)}. \tag{2.6}$$

Let n be the smallest positive integer such that $B_1\subset B_{2^n\cdot 3\cdot r}(x_0)$, and observe that

$$2^n\cdot 3\cdot r<4.$$

For $i\in\{0,\dots,n-1\}$, by taking $R=2^i\cdot 3\cdot r\in[3r,2)$ in (2.6), and using the fact that $n-1\le N\ln(1/r)$, we get

$$\begin{aligned}\int_{B_1\setminus B_{3r}(x_0)}|Du|\,dx&\le N\sum_{i=0}^{n-1}\left(2^{-i/2}+\left(\ln\frac{1}{r}\right)^{-1}\right)\|\hat{f}\|_{L^1(B_r(x_0)\cap B_1)}\\&\le N\|f\|_{L^1(B_r(x_0)\cap B_1)}\end{aligned}$$

This implies that T satisfies the hypothesis of Lemma 2.2 with $\mu=1/3$, $c=3$, and $C=C(d,\lambda,\omega_A,C_0)>0$. The assertion (a) is proved.

Next, we prove the assertion (b). Let T_1 be a bounded linear operator on $L^2(B_6)^d$ given by

$$T_1g=Dv,$$

where $v \in W^{1,2}(B_6)$ is a unique weak solution of

$$\begin{cases} \operatorname{div}(\tilde{A}Dv) = \operatorname{div} g & \text{in } B_6, \\ \tilde{A}Dv \cdot n = g \cdot n & \text{on } \partial B_6 \end{cases} \tag{2.7}$$

satisfying

$$\int_{B_6} v\,dx = 0.$$

Then for any $f \in L^2(B_6^+)^d$, we have

$$Tf = T_1\tilde{f} \quad \text{in } B_6^+. \tag{2.8}$$

In the above, $\tilde{A}$ and $\tilde{f}$ are as in (2.2). Indeed, one can check (2.8) as follows. Let $Tf = Du$ and $\phi \in C^\infty(\overline{B_6})$, and set

$$\tilde{\phi}(x_1, x') = \phi(-|x_1|, x').$$

Since $u \in W^{1,2}(B_6^+)$ satisfies

$$\begin{cases} \operatorname{div}(ADu) = \operatorname{div} f & \text{in } B_6^+, \\ ADu \cdot n = f \cdot n & \text{on } \partial B_6^+, \end{cases}$$

we see that

$$\int_{B_6^+} ADu \cdot D\phi\,dx = \int_{B_6^+} f \cdot D\phi\,dx \tag{2.9}$$

and

$$\int_{B_6^+} ADu \cdot D\tilde{\phi}\,dx = \int_{B_6^+} f \cdot D\tilde{\phi}\,dx. \tag{2.10}$$

Let $\tilde{u}(x_1, x') = u(|x_1|, x')$, and observe that

$$\int_{B_6} \tilde{u}\,dx = 0, \quad \tilde{u} \in W^{1,2}(B_6).$$

By (2.10), we have

$$\int_{B_6 \setminus B_6^+} \tilde{A}D\tilde{u} \cdot D\phi\,dx = \int_{B_6 \setminus B_6^+} \tilde{f} \cdot D\phi\,dx,$$

which together with (2.9) implies

$$\int_{B_6} \tilde{A}D\tilde{u} \cdot D\phi\,dx = \int_{B_6} \tilde{f} \cdot D\phi\,dx.$$

Since the above identity holds for any $\phi \in C^\infty(\overline{B_6})$, $\tilde{u}$ is a unique weak solution of (2.7) with $\tilde{f}$ in place of g. Therefore,

$$D\tilde{u} = T_1 \tilde{f} \quad \text{in } B_6,$$

which proves (2.8). Since $\tilde{A}$ also satisfies (1.4), by applying the result in (a) to the operator T_1, we see that the assertion (b) holds. The theorem is proved. $\square$

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