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SEMIGROUP RINGS AS H-DOMAINS

Gyu Whan Chang

Abstract. Let D be an integral domain, S be a torsion-free grading monoid such that the quotient group of S is of type $(0, 0, 0, \dots)$, and $D[S]$ be the semigroup ring of S over D. We show that $D[S]$ is an H-domain if and only if D is an H-domain and each maximal t-ideal of S is a v-ideal. We also show that if $\mathbb{R}$ is the field of real numbers and if Γ is the additive group of rational numbers, then $\mathbb{R}[\Gamma]$ is not an H-domain.

1. Introduction

Let D be an integral domain with quotient field K, S be a torsion-free grading monoid, and $D[S]$ be the semigroup ring of S over D. A D-submodule F of K is called a fractional ideal of D if $dF \subseteq D$ for some nonzero $d \in D$. For a nonzero fractional ideal I of D, let $I^{-1} = \{x \in K | xI \subseteq D\}$; so I^{-1} is also a nonzero fractional ideal of D.

1.1. Motivation and Result. An integral domain D is called an *H-domain* if I is an ideal of D such that $I^{-1} = D$, then there is a finitely generated subideal J of I such that $J^{-1} = D$. In [6], Glaz and Vasconcelos introduced the notion of an H-domain, and they then proved that a completely integrally closed H-domain is a Krull domain. They also proved that if D is an H-domain, then $D[X]$, the polynomial ring over D, is also an H-domain. Let $\{X_\alpha\}$ be a nonempty set of indeterminates over D. Park showed that if D is an H-domain, then so is $D[\{X_\alpha\}]$ [10, Proposition 4.2]. Park also showed that if D is a strong Mori domain and if G is a torsion-free abelian group of type $(0, 0, 0, \cdots)$, then $D[G]$

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is an H-domain [10, Proposition 5.5]. Clearly, a strong Mori domain is an H-domain. So it is natural to ask if $D[G]$ is an H-domain when D is an H-domain.

In this paper, we study when $D[S]$ is an H-domain. Precisely, we show that if the quotient group of S is of type $(0,0,0,\dots)$, then $D[S]$ is an H-domain if and only if D is an H-domain and each maximal t-ideal of S is a v-ideal. Let $\mathbb{R}$ be the field of real numbers and let Γ be the additive group of rational numbers. We prove that $\mathbb{R}[\Gamma]$ is not an H-domain, which shows that the assumption that the quotient group of S is of type $(0,0,0,...)$ is necessary for $D[S]$ to be an H-domain.

1.2. Definition and Notation. Let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D. For each $I \in \mathbf{F}(D)$, let $I_v = (I^{-1})^{-1}$, $I_t = \cup\{J_v | J \subseteq I$ and J is a nonzero finitely generated ideal$\}$, and $I_w = \{x \in K | xJ \subseteq I$ for some nonzero finitely generated ideal J with $J^{-1} = D\}$. An $I \in \mathbf{F}(D)$ is called a $*$-*ideal* if $I_* = I$, where $* = v, t$, or w, while a $*$-ideal of D is a *maximal* $*$-*ideal* if it is maximal among proper integral $*$-ideals of D. It is well known that a prime ideal minimal over a t-ideal is a t-ideal; a maximal t-ideal is a prime ideal; and each proper integral t-ideal is contained in a maximal t-ideal. An integral domain D is a *strong Mori domain* if D satisfies the ascending chain condition on integral w-ideals. In particular, Noetherian domains are strong Mori domains.

Let S be a torsion-free grading monoid with quotient group G. It is well known that $D[S]$ is an integral domain [4, Theorem 8.1] and S admits a total order $<$ compatible with its monoid operation [4, Corollary 3.4]. Hence each $f \in D[S]$ is uniquely written in the form

$$f = a_0X^{\alpha_0} + a_1X^{\alpha_1} + \cdots + a_nX^{\alpha_n},$$

where $a_i \in D$ and $\alpha_j \in S$ with $\alpha_0 < \alpha_1 < \cdots < \alpha_n$. For any $f \in K[G]$, we denote by A_f (resp., E_f) the fractional ideal of D (resp., S) generated by the coefficients (resp., exponents) of f; hence $A_f = (a_0, a_1, \dots, a_n)$ and $E_f = (\alpha_0 + S) \cup (\alpha_1 + S) \cup \cdots \cup (\alpha_n + S)$. The torsion-free abelian group G is said to be *of type* $(0,0,0,\dots)$ if G satisfies the ascending chain condition on cyclic subgroups. As in the domain case, one can define the v- and t-operation on S; and maximal t-ideals of S.

The reader can refer to [3, §32 and §34] for the v- and t-operation on integral domains; to [4, §16] or [7] for the v- and t-operation on monoids; and to [4, 7] for monoids and monoid domains.

2. H-domains

Let D be an integral domain with quotient field K, S be a torsion-free grading monoid with quotient group G and $D[S]$ be the semigroup ring of S over D.

We begin this section with some equivalence conditions of an H-domain. These are already known [8, Proposition 2.4], but we give the proof for the reader's convenience.

LEMMA 1. *The following statements are equivalent.*

(1) *D is an H-domain.*
(2) *Every maximal t-ideal of D is a v-ideal.*
(3) *$I_v \subsetneq D$ for each proper integral t-ideal I of D.*

Proof. (1) $\Rightarrow$ (2) Let Q be a maximal t-ideal of D. If $Q^{-1} = D$, then there is a nonzero finitely generated ideal $J \subseteq Q$ such that $J^{-1} = D$. So $D = J_v \subseteq Q_t = Q \subsetneq D$, a contradiction. Hence $Q^{-1} \supsetneq D$, and since Q is a maximal t-ideal, $Q = Q_v$. (2) $\Rightarrow$ (3) If $I = I_t \subsetneq D$, then there is a maximal t-ideal Q of D such that $I \subseteq Q$. Hence $I_v \subseteq Q_v = Q \subsetneq D$. (3) $\Rightarrow$ (1) Let I be a nozero ideal of D with $I^{-1} = D$. Then $I_v = D$, and hence $I_t = D$ by (3). So there is a finitely generated ideal $J \subseteq I$ such that $J_v = D$ or $J^{-1} = D$. □

LEMMA 2. *Let S be a torsion-free grading monoid with quotient group G, K be a field, Q be a maximal t-ideal of $K[S]$, and $N = \{X^\alpha | \alpha \in S\}$.*

(1) *If $Q \cap N \neq \emptyset$, then $J = \{\alpha \in S | X^\alpha \in Q\}$ is a maximal t-ideal of S and $Q = K[J]$.*
(2) *If $Q \cap N = \emptyset$, then $QK[G]$ is a maximal t-ideal of $K[G]$.*
(3) *If G is of type $(0, 0, 0, ...)$ and if $Q \cap N = \emptyset$, then Q is a height-one t-invertible prime t-ideal.*

Proof. (1) [1, Corollary 1.3].

(2) Suppose that $(QK[G])_t = K[G]$. Note that $K[S]_N = K[G]$. Then there is a finitely generated subideal A of Q such that $A^{-1} \subseteq A^{-1}K[G] = (AK[G])^{-1} = K[G]$ [11, Lemma 1.4]. Note that $Q \subsetneq K[\cup_{f \in Q} E_f]$ and Q is a maximal t-ideal of $K[S]$. So $(\cup_{f \in Q} E_f)_t = S$ (cf. [2, Lemma 2.3]), and hence $(E_{f_1} \cup \cdots \cup E_{f_k})_v = S$ for some $f_1, \dots, f_k \in Q$. Let $I = (A, f_1, \dots, f_k)$. Then I is a finitely generated subideal of Q such that $I^{-1} \subseteq I^{-1}K[G] = (IK[G])^{-1} \subseteq (AK[G])^{-1} = K[G]$. Let $0 \neq g \in I^{-1}$. Then $gf_i \in K[S]$ for $i = 1, \dots, k$ and $K[S] = K[((m_1 + 1)E_{f_1} \cup \cdots \cup$

$(m_k+1)E_{f_k})_t] = (K[(m_1+1)E_{f_1} \cup \cdots \cup (m_k+1)E_{f_k}])_t$ [2, Lemma 2.3], and hence $K[E_g] = (K[(m_1+1)E_{f_1} \cup \cdots \cup (m_k+1)E_{f_k}]K[E_g])_t = (K[(m_1E_{f_1}+E_{f_1g}) \cup \cdots \cup (m_kE_{f_k}+E_{f_kg})])_t \subseteq K[S]$ for some positive integers m_i [5, Proposition 6.2]; so $g \in K[S]$. Thus $I^{-1} = K[S]$ and $I_v \subseteq Q \subsetneq K[S]$, a contradiction. Thus $(QK[G])_t \subsetneq K[G]$. If Q_0 is a prime ideal of $K[S]$ such that $QK[G] \subseteq Q_0K[G]$ and $Q_0K[G]$ is a maximal t-ideal of $K[G]$. Then $Q \subseteq Q_0K[G] \cap K[S] = Q_0$ and Q_0 is also a prime t-ideal [9, Lemma 3.17]. Thus, $Q = Q_0$ and $QK[G] = Q_0K[G]$.

(3) Note that $K[G]$ is a UFD [4, Theorem 14.15], because G is of type $(0,0,0,...)$. Since $QK[G]$ is a t-ideal of $K[G]$ by (2), we have $QK[G] = hK[G]$ for some $h \in Q$ and $\text{ht}Q = \text{ht}(QK[G]) = 1$. Let $f_1, \ldots, f_k \in Q$ such that $(E_{f_1} \cup \cdots \cup E_{f_k})_v = S$ (see the proof of (2)). Then $(f_1, \ldots, f_k, h)_v = Q$ [9, Proposition 2.8]. Also, since Q is a maximal t-ideal, Q is t-locally principal. Thus, Q is t-invertible [9, Corollary 2.7]. □

LEMMA 3. *Let S be a torsion-free grading monoid, and let Q be a maximal t-ideal of $D[S]$.*

(1) *If $Q \cap D \neq 0$, then $Q = (Q \cap D)D[S]$ and $Q \cap D$ is a maximal t-ideal of D.*
(2) *If $Q \cap D = 0$, then $QK[S]$ is a maximal t-ideal of $K[S]$.*

Proof. (1) [1, Corollary 1.3].

(2) Note that $D[S]_{D\setminus\{0\}} = K[S]$. If $(QK[S])_t = K[S]$, then there exists a finitely generated ideal $B \subseteq Q$ such that $B^{-1} \subseteq B^{-1}K[S] = (BK[S])^{-1} = K[S]$ [11, Lemma 1.4]. Since Q is a maximal t-ideal of $D[S]$ with $Q \cap D = 0$, $(\sum_{g\in Q} A_g)_t = D$, and hence $(A_{g_1}+\cdots+A_{g_m})_v = D$ for some $g_1, \ldots, g_m \in Q$. Let $J = (B, g_1, \ldots, g_m)$. Then J is a finitely generated subideal of Q such that $J^{-1} \subseteq J^{-1}K[S] = (JK[S])^{-1} \subseteq (BK[S])^{-1} = K[S]$. Let $0 \neq h \in J^{-1}$. Then $hg_i \in D[S]$ for $i = 1, \ldots, m$, and hence $A_h[S] = ((A_{g_1}^{k_1+1} + \cdots + A_{g_m}^{k_m+1})[S])(A_h[S]))_t = ((A_{g_1}^{k_1}A_{g_1h}) + \cdots + (A_{g_m}^{k_m}A_{g_mh})[S])_t \subseteq D[S]$ for some positive integers k_i ([5, Theorem 4.3] and [2, Lemma 2.3]); so $h \in K[S]$. Thus $J^{-1} = D[S]$ and $J_v \subseteq Q \subsetneq D[S]$, a contradiction. Thus $(QK[S])_t \subsetneq K[S]$. If Q' is a prime ideal of $K[S]$ such that $QK[S] \subseteq Q'K[S]$ and $Q'K[S]$ is a maximal t-ideal of $K[S]$. Then $Q \subseteq Q'K[S] \cap D[S] = Q'$ and Q' is also a prime t-ideal [9, Lemma 3.17]. Thus, $Q = Q'$ and $QK[S]$ is a maximal t-ideal. □

Lemma 4. *Let S be a torsion-free grading monoid. If $D[S]$ is an H-domain, then D is an H-domain and every maximal t-ideal of S is a v-ideal.*

Proof. Let P be a maximal t-ideal of D. Then $PD[S]$ is a prime t-ideal of $D[S]$, and hence $P_v \subseteq (P_vD[S])_v = (PD[S])_v \subsetneq D[S]$ [2, Lemma 2.3]; so $P_v \subsetneq D$. Thus D is an H-domain by Lemma 1. Let J be a maximal t-ideal of S. Then $D[J]$ is a t-ideal of $D[S]$ [2, Corollary 2.4], and hence $D[J_v] = (D[J])_v \subsetneq D[S]$ [2, Lemma 2.3]. Hence $J_v \subsetneq S$, and thus $J = J_v$. □

Theorem 5. *Let S be a torsion-free grading monoid with quotient group G such that G is of type $(0, 0, 0, \dots)$. Then $D[S]$ is an H-domain if and only if D is an H-domain and every maximal t-ideal of S is a v-ideal.*

Proof. ($\Rightarrow$) Lemma 4.

($\Leftarrow$) Let $N = \{X^\alpha | \alpha \in S\}$ and let Q be a maximal t-ideal of $D[S]$. By Lemma 1, it suffices to show that $Q_v = Q$.

Case 1. $Q \cap D \neq 0$. Then $Q \cap D$ is a maximal t-ideal of D and $Q = (Q \cap D)D[S]$ by Lemma 3(1). Thus $Q = (Q \cap D)D[S] = (Q \cap D)_vD[S] = ((Q \cap D)D[S])_v = Q_v$.

Case 2. $Q \cap D = 0$. Then $QK[S]$ is a maximal t-ideal of $K[S]$ by Lemma 3(2). If $QK[S] \cap N = \emptyset$, then $QK[S]$ is a height-one t-invertible prime ideal of $K[S]$ by Lemma 2(3); so $Q_v \subseteq Q_vK[S] \subseteq (Q_vK[S])_v = (QK[S])_v = QK[S]$. Thus $Q_v \subseteq QK[S] \cap D[S] = Q$, and hence $Q_v = Q$. If $QK[S] \cap N \neq \emptyset$, then $QK[S] = K[J]$ for some maximal t-ideal J of S by Lemma 2(1). Thus $Q_v \subseteq Q_vK[S] \subseteq (Q_vK[S])_v = (QK[S])_v = (K[J])_v = K[J_v] = K[J] = Q$; so $Q_v = Q$. □

In [10, Proposition 5.5], M.H. Park shows that if D is a strong Mori domain and if G is a torsion-free abelian group of type $(0, 0, 0, \dots)$, then $D[G]$ is an H-domain. It is well-known that a strong Mori domain is an H-domain. Thus the following corollary is a generalization of Park's results [10, Propositions 4.2 and 5.5].

Corollary 6. *Let D be an integral domain and G be a torsion-free abelian group of type $(0, 0, 0, \dots)$. Then $D[G]$ is an H-domain if and only if D is an H-domain.*

Corollary 7. ([10, Proposition 4.2]) *Let $\{X_\alpha\}$ be a nonempty set of indeterminates over D. Then D is an H-domain if and only if $D[\{X_\alpha\}]$ is an H-domain.*

Proof. Let $S = \sum_\alpha (\mathbb{Z}_+)_\alpha$, where $(\mathbb{Z}_+)_\alpha$ is the additive semigroup of nonnegative integers. Then $D[S] = D[\{X_\alpha\}]$ and S is a torsion-free grading monoid whose quotient group is of type $(0, 0, 0, ...)$. Thus, D is an H-domain if and only if $D[\{X_\alpha\}]$ is an H-domain by Theorem 5. □

We end this paper with an example which shows that in Theorem 5, the assumption that G is of type $(0, 0, 0, ...)$ is necessary.

Example 8. *Let $\mathbb{R}$ be the field of real numbers and Γ be the additive group of rational numbers.*

(1) *Γ is a torsion-free abelian group.*
(2) *Γ is not of type $(0, 0, 0, ...)$.*
(3) *$\mathbb{R}[\Gamma]$ is a GCD-domain, but not a UFD.*
(4) *Each maximal t-ideal of Γ is a v-ideal.*
(5) *$\mathbb{R}[\Gamma]$ has the (Krull) dimension one.*
(6) *$\mathbb{R}[\Gamma]$ is not an H-domain.*

Proof. (1) Clear. (2) This follows, because we have an infinite sequence of cyclic subgroups of Γ, say, $(\frac{1}{2}) \subsetneq (\frac{1}{2^2}) \subsetneq (\frac{1}{2^3}) \subsetneq \cdots$. (3) This is an immediate consequence of (1), (2) and [4, Theorems 14.5 and 14.16]. (4) Clear. (5) This follows from [4, Theorem 17.1]. (6) Let Q be a maximal t-ideal of $\mathbb{R}[\Gamma]$ such that $Q^{-1} \supsetneq \mathbb{R}[\Gamma]$. Choose $u \in Q^{-1} \setminus \mathbb{R}[\Gamma]$. Then $\mathbb{R}[\Gamma] \subsetneq (1, u) \subseteq Q^{-1}$, and since Q is a maximal t-ideal, we have $Q = Q_v = (1, u)^{-1}$. Since $\mathbb{R}[\Gamma]$ is a GCD-domain, $Q = (1, u)^{-1}$ must be principal. So by (3) and (5), there is a maximal t-ideal Q of $\mathbb{R}[\Gamma]$ with $Q^{-1} = \mathbb{R}[\Gamma]$. Thus, $\mathbb{R}[\Gamma]$ is not an H-domain. □

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Department of Mathematics
University of Incheon
Incheon 406-772, Korea
E-mail: whan@incheon.ac.kr