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SOME GENERALIZED GROWTH PROPERTIES OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS

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Abstract. In this paper we wish to prove some results relating to the growth rates of composite entire and meromorphic functions with their corresponding left and right factors on the basis of their generalized order (α,β) and generalized lower order (α,β), where α and β are continuous non-negative functions defined on $(-\infty,+\infty)$.

1. Introduction, Definitions and Notations

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's theory of meromorphic functions which are available in [8, 11, 17]. We also use the standard notations and definitions of the theory of entire functions which are available in [16] and therefore we do not explain those in details. Let f be an entire function and $M_f(r) = \max\{|f(z)| : |z| = r\}$. When f is meromorphic, the Nevanlinna's characteristic function $T_f(r)$ (see [8, p.4]) plays the same role as $M_f(r)$. For $x \in [0,+\infty)$ and $k \in \mathbb{N}$ where $\mathbb{N}$ is the set of all positive integers, we define iterations of the exponential and logarithmic functions as $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ and $\log^{[k]} x = \log(\log^{[k-1]} x)$, with convention that $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$, and $\exp^{[-1]} x = \log x$. Further we assume that p and q always denote positive integers. Now considering this, let us recall that Juneja et al. [10] defined the (p,q)-th order and (p,q)-th lower order of an entire function, respectively, as follows:

Definition 1.1. [10] Let $p \geq q$. The (p,q)-th order $\rho^{(p,q)}(f)$ and (p,q)-th lower order $\lambda^{(p,q)}(f)$ of an entire function f are defined as:

$$\rho^{(p,q)}(f) = \limsup_{r\to+\infty}\frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \liminf_{r\to+\infty}\frac{\log^{[p]} M_f(r)}{\log^{[q]} r}.$$

If f is a meromorphic function, then

$$\rho^{(p,q)}(f) = \limsup_{r\to+\infty}\frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \liminf_{r\to+\infty}\frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r}.$$

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For any entire function f, using the inequality $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$ $\{cf.$ [8]$\}$, one can easily verify that

$$\begin{aligned} \rho^{(p,q)}(f) &= \limsup_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} = \limsup_{r\to+\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r} \\ \text{and } \lambda^{(p,q)}(f) &= \liminf_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} = \liminf_{r\to+\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r}, \end{aligned}$$

when $p \geq 2$.

Extending the notion (p,q)-th order, recently Shen et al. [9] introduced the new concept of $[p,q]$-φ order of entire and meromorphic function where $p \geq q$. Later on, combining the definition of (p,q)-order and $[p,q]$-φ order, Biswas (see, e.g., [5]) redefined the (p,q)-order of an entire and meromorphic function without restriction $p \geq q$.

However the above definition is very useful for measuring the growth of entire and meromorphic functions. If $p = l$ and $q = 1$ then we write $\rho^{(l,1)}(f) = \rho^{(l)}(f)$ and $\lambda^{(l,1)}(f) = \lambda^{(l)}(f)$ where $\rho^{(l)}(f)$ and $\lambda^{(l)}(f)$ are respectively known as generalized order and generalized lower order of entire or meromorphic function f. For details about generalized order one may see [15]. Also for $p = 2$ and $q = 1$, we respectively denote $\rho^{(2,1)}(f)$ and $\lambda^{(2,1)}(f)$ by $\rho(f)$ and $\lambda(f)$ which are classical growth indicators such as order and lower order of entire or meromorphic function f.

Now let L be a class of continuous non-negative on $(-\infty,+\infty)$ functions α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$. For any $\alpha \in L$, we say that $\alpha \in L_1^0$, if $\alpha((1+o(1))x) = (1+o(1))\alpha(x)$ as $x \to +\infty$ and $\alpha \in L_2^0$, if $\alpha(\exp((1+o(1))x)) = (1+o(1))\alpha(\exp(x))$ as $x \to +\infty$. Finally for any $\alpha \in L$, we also say that $\alpha \in L_1$, if $\alpha(cx) = (1+o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0,+\infty)$ and $\alpha \in L_2$, if $\alpha(\exp(cx)) = (1+o(1))\alpha(\exp(x))$ as $x_0 \leq x \to +\infty$ for each $c \in (0,+\infty)$. Clearly, $L_1 \subset L_1^0$, $L_2 \subset L_2^0$ and $L_2 \subset L_1$.Further we assume that throughout the present paper α_2, α_4, β, β_1, β_2, β_3 and β_4 denote the functions belonging to L_1 and $\alpha_1, \alpha_3 \in L_2$ unless otherwise specifically stated.

The value

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log M_f(r))}{\beta(\log r)} \ (\alpha \in L, \beta \in L)$$

introduced by Sheremeta [14], is called generalized order (α,β) of an entire function f. During the past decades, several authors made close investigations on the properties of entire functions related to generalized order (α,β) in some different direction. For the purpose of further applications, Biswas et al. [2, 3] rewrite the definition of the generalized order (α,β) of entire function in the following way after giving a minor modification to the original definition (e.g. see, [14]) which considerably extend the definition of φ-order of entire function introduced by Chyzhykov et al. [6]:

Definition 1.2. [2, 3] The generalized order (α,β) denoted by $\rho_{(\alpha,\beta)}[f]$ and generalized lower order (α,β) denoted by $\lambda_{(\alpha,\beta)}[f]$ of an entire function f are defined as:

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty}\frac{\alpha(M_f(r))}{\beta(r)} \text{ and}$$
$$\lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty}\frac{\alpha(M_f(r))}{\beta(r)} \text{ where } \alpha\in L_1.$$

If f is a meromorphic function, then

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty}\frac{\alpha(\exp(T_f(r)))}{\beta(r)} \text{ and}$$
$$\lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty}\frac{\alpha(\exp(T_f(r)))}{\beta(r)}, \text{ where } \alpha\in L_2.$$

Using the inequality $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$ $\{cf.$ [8]$\}$, for an entire function f, one may easily verify that

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty}\frac{\alpha(M_f(r))}{\beta(r)} = \limsup_{r\to+\infty}\frac{\alpha(\exp(T_f(r)))}{\beta(r)}$$
$$\text{and } \lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty}\frac{\alpha(M_f(r))}{\beta(r)} = \liminf_{r\to+\infty}\frac{\alpha(\exp(T_f(r)))}{\beta(r)},$$

when $\alpha\in L_2$.

In particular, the following definition is needed in the sequel.

Definition 1.3. Let "a" be a complex number, finite or infinite. The Nevanlinna's deficiency of "a" with respect to a meromorphic function f are defined as

$$\delta(a;f) = 1 - \limsup_{r\to+\infty}\frac{N_f(r,a)}{T_f(r)} = \liminf_{r\to+\infty}\frac{m_f(r,a)}{T_f(r)}.$$

In this paper, we intend to establish some results relating to the growth properties of composite entire and meromorphic functions on the basis of generalized order (α,β) and generalized lower order (α,β). In fact some works in this direction have already been explored in [2–4].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 2.1. [1] *If f is a meromorphic function and g is an entire function then for all sufficiently large values of r,*

$$T_{f\circ g}(r) \leqslant \{1+o(1)\}\frac{T_g(r)}{\log M_g(r)}T_f(M_g(r)).$$

Lemma 2.2. [7] *Let f and g are any two entire functions with $g(0)=0$. Also let b satisfy $0<b<1$ and $c(b)=\frac{(1-b)^2}{4b}$. Then for all sufficiently large values of r,*

$$M_f(c(b)M_g(br)) \leq M_{f\circ g}(r).$$

In addition if $b=\frac{1}{2}$, then for all sufficiently large values of r,

$$M_{f\circ g}(r) \geq M_f\left(\frac{1}{8}M_g\left(\frac{r}{2}\right)\right).$$

Lemma 2.3. [12] *Let g be an entire function with $\lambda_g < +\infty$ and assume that $a_i (i = 1, 2, ...n; n \leq +\infty)$ are entire functions satisfying $T_{a_i}(r) = \circ\{T_g(r)\}$. If $\sum_{i=1}^{n} \delta(a_i, g) = 1$, then*

$$\lim_{r\to+\infty} \frac{T_g(r)}{\log M_g(r)} = \frac{1}{\pi}.$$

Lemma 2.4. *Let g be an entire function with $\lambda_g < +\infty$ and assume that $a_i (i = 1, 2, ...n; n \leq +\infty)$ are entire functions satisfying $T_{a_i}(r) = \circ\{T_g(r)\}$. If $\sum_{i=1}^{n} \delta(a_i, g) = 1$, then for any $\alpha \in L_2$*

$$\lim_{r\to+\infty} \frac{\alpha(\exp(T_g(r)))}{\alpha(M_g(r))} = 1.$$

Proof. In view of Lemma 2.3 we get for all sufficiently large positive numbers of r that

$$\frac{1}{\pi} - \varepsilon \leq \frac{T_g(r)}{\log M_g(r)} \leq \frac{1}{\pi} + \varepsilon$$

$$i.e.,\ M_g(r) \leq \exp\left(\frac{1}{(\frac{1}{\pi} - \varepsilon)}(T_g(r))\right) \leq M_g(r)$$

$$i, e.,\ \alpha(M_g(r)) \leq (1 + o(1))\alpha(\exp(T_g(r))) \leq \alpha(M_g(r))$$

Therefore we get

$$1 \leq \liminf_{r\to+\infty} \frac{\alpha(\exp(T_g(r)))}{\alpha(M_g(r))} \leq \limsup_{r\to+\infty} \frac{\alpha(\exp(T_g(r)))}{\alpha(M_g(r))} \leq 1$$

$$i.e.,\ \lim_{r\to+\infty} \frac{\alpha(\exp(T_g(r)))}{\alpha(M_g(r))} = 1.$$

This proves the lemma. □

3. Main Results

In this section we present the main results of the paper.

Theorem 3.1. *Let f be a meromorphic function and g be an entire function such that $0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \rho_{(\alpha_1,\beta_1)}[f] < +\infty$. and $\rho_{(\alpha_2,\beta_2)}[g] < +\infty$. Also let γ be a positive continuous on $[0, +\infty)$ function increasing to $+\infty$ and $A \geq 0$ be any number.*
(i) If $\beta_1(\alpha_2^{-1}(\log r)) \leq r$ and $\lim_{r\to+\infty} \frac{\log \gamma(r)}{\log r} = +\infty$, then

$$\lim_{r\to+\infty} \frac{\left\{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))\right\}^{1+A}}{\alpha_1(\exp(T_f(\beta_1^{-1}(\gamma(r)))))} = 0 \text{ and}$$

(ii) If either $\beta_1(r) = B\alpha_2(r)$ where B is any positive constant and $\lim_{r\to+\infty} \frac{\log \gamma(r)}{\log r} = +\infty$ or $\beta_1(\alpha_2^{-1}(r)) \in L^0$ and $\lim_{r\to+\infty} \frac{\log \gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = +\infty$, then

$$\lim_{r\to+\infty} \frac{\left\{\exp(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))))\right\}^{1+A}}{\alpha_1(\exp(T_f(\beta_1^{-1}(\gamma(r)))))} = 0.$$

Proof. From the definition of $\lambda_{(\alpha_1,\beta_1)}[f]$, we get for all sufficiently large values of r that

$$\alpha_1(\exp(T_f(\beta_1^{-1}(\gamma(r))))) \geq (\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)\gamma(r)\ . \tag{1}$$

In view of Lemma 2.1 and the inequality $T_f(r) \leq \log^+ M_f(r)$ we get for all sufficiently large values of r that

$$\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))) \leqslant (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(M_g(\beta_2^{-1}(\log r))). \tag{2}$$

Now the following cases may arise :

Case I. Let $\beta_1(\alpha_2^{-1}(\log r)) \leq r$. Now we get from (2) for all sufficiently large values of r that

$$\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))) \leqslant (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(\alpha_2^{-1}(\alpha_2(M_g(\beta_2^{-1}(\log r))))) \tag{3}$$

$$i.e.,\ \alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))) \leqslant (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(\alpha_2^{-1}(\log r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)})). \tag{4}$$

Case II. Let $\beta_1(r)=B\alpha_2(r)$ where B is any positive constant. Then we have from (2) for all sufficiently large values of r that

$$\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))) \leqslant (1+o(1))B(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\alpha_2(M_g(\beta_2^{-1}(\log r)))$$

$$i.e.,\ \alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))) \leqslant (1+o(1))B(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)\log r$$

$$i.e., \exp(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))) \leqslant r^{(1+o(1))B(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{5}$$

Case III. Let $\beta_1(\alpha_2^{-1}(r)) \in L_1$ and $\lim\limits_{r\to+\infty}\frac{\log\gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))}=+\infty$. Then we have from (3) for all sufficiently large values of r that

$$\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))) \leqslant (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(\alpha_2^{-1}(\log r))$$

$$i.e.,\ \exp(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))) \leqslant \exp((1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(\alpha_2^{-1}(\log r))). \tag{6}$$

Now when $\beta_1(\alpha_2^{-1}(\log r)) \leq r$ and $\lim\limits_{r\to+\infty}\frac{\log\gamma(r)}{\log r}=+\infty$, we obtain from (1) and (4) of Case I for all sufficiently large values of r that

$$\frac{\left\{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))\right\}^{1+A}}{\alpha_1(\exp(T_f(\beta_1^{-1}(\gamma(r)))))} \leqslant \frac{(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)^{1+A}[\beta_1(\alpha_2^{-1}(\log r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}))]^{1+A}}{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)\gamma(r)}$$

$$i.e.,\ \limsup_{r\to+\infty}\frac{\left\{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))\right\}^{1+A}}{\alpha_1(\exp(T_f(\beta_1^{-1}(\gamma(r)))))}=0,$$

This proves the first part of the theorem.

Again combining (1) and (5) of Case II, we get for all sufficiently large values of r that

$$\frac{\left\{\exp(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))))\right\}^{1+A}}{\alpha_1(\exp(T_f(\beta_1^{-1}(\gamma(r)))))} \leqslant \frac{r^{(1+o(1))B(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)(1+A)}}{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)\gamma(r)}$$

As $\lim\limits_{r\to+\infty}\frac{\log\gamma(r)}{\log r}=+\infty$, so

$$\frac{r^{(1+o(1))B(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)(1+A)}}{\gamma(r)} \to 0$$

as $r\to+\infty$. Thus it follows from above that

$$\lim_{r\to+\infty}\frac{\left\{\exp(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))))\right\}^{1+A}}{\alpha_1(\exp(T_f(\beta_1^{-1}(\gamma(r)))))}=0. \tag{7}$$

Further combining (1) and (6) of Case III it follows for all sufficiently large values of r that

$$\frac{\left\{\exp(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))))\right\}^{1+A}}{\alpha_1(\exp(T_f(\beta_1^{-1}(\gamma(r)))))} \leqslant \frac{\left[\exp((1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(\alpha_2^{-1}(\log r)))\right]^{1+A}}{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)\gamma(r)}.$$

Since $\lim\limits_{r\to+\infty}\frac{\log\gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))}=+\infty$, so

$$\frac{\left[\exp((1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(\alpha_2^{-1}(\log r)))\right]^{1+A}}{\gamma(r)} \to 0$$

as $r\to+\infty$. Thus from above we obtain that

$$\lim_{r\to+\infty}\frac{\left\{\exp(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))))\right\}^{1+A}}{\alpha_1(\exp(T_f(\beta_1^{-1}(\gamma(r)))))}=0 \tag{8}$$

Hence the second part of the theorem follows from (7) and (8).

Thus the theorem follows. □

Remark 3.2. Theorem 3.1 improves and extends Theorem 3 of [13].

Remark 3.3. In Theorem 3.1 if we take the condition $\rho_{(\alpha_1,\beta_1)}[f]>0$ instead of $0<\lambda_{(\alpha_1,\beta_1)}[f]\le\rho_{(\alpha_1,\beta_1)}[f]<+\infty$, the theorem remains true with " limit inferior" in place of "limit".

Theorem 3.4. *Let f be a meromorphic function and g be an entire function such that $0<\lambda_{(\alpha_2,\beta_2)}[g]\le\rho_{(\alpha_2,\beta_2)}[g]<+\infty$ and $\rho_{(\alpha_1,\beta_1)}[f]<+\infty$ where $\alpha_2\in L_2$. Also let γ be a positive continuous on $[0,+\infty)$ function increasing to $+\infty$ and $A\ge 0$ be any number.*

(i) If $\beta_1(\alpha_2^{-1}(\log r))\le r$ and $\lim\limits_{r\to+\infty}\frac{\log\gamma(r)}{\log r}=+\infty$, then

$$\lim_{r\to+\infty}\frac{\left\{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))\right\}^{1+A}}{\alpha_2(\exp(T_g(\beta_2^{-1}(\gamma(r)))))}=0 \textit{ and}$$

(ii) If either $\beta_1(r) = B\alpha_2(r)$ *where* B *is any positive constant and*

$$\lim_{r\to+\infty} \frac{\log\gamma(r)}{\log r} = +\infty$$

or $\beta_1(\alpha_2^{-1}(r)) \in L^0$ *and* $\lim\limits_{r\to+\infty} \frac{\log\gamma(r)}{\beta_1(\alpha_2^{-1}(\log r))} = +\infty$, *then*

$$\lim_{r\to+\infty} \frac{\left\{\exp(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))))\right\}^{1+A}}{\alpha_2(\exp(T_g(\beta_2^{-1}(\gamma(r)))))} = 0.$$

The proof of Theorem 3.4 would run parallel to that of Theorem 3.1. We omit the details.

Remark 3.5. In Theorem 3.4, if we take the condition $\rho_{(\alpha_2,\beta_2)}[g] > 0$ instead of $0 < \lambda_{(\alpha_2,\beta_2)}[g] \le \rho_{(\alpha_2,\beta_2)}[g] < +\infty$, the theorem remains true with "limit" replaced by "limit inferior".

Theorem 3.6. *Let* f *be a meromorphic function and* g, h, k *be three entire functions such that* $\lambda_{(\alpha_3,\beta_3)}[h] > 0$, $\lambda_{(\alpha_4,\beta_4)}[k] > 0$ *and* $\rho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_4,\beta_4)}[k]$. *Also let* C *and* D *be any two positive constants.*
(i) Any one of the following four conditions are assumed to be satisfied:
(a) $\beta_1(r) = C(\exp(\alpha_2(r)))$ *and* $\beta_3(r) = D\exp(\alpha_4(r))$;
(b) $\beta_1(r) = C(\exp(\alpha_2(r)))$ *and* $\beta_3(r) > \exp(\alpha_4(r))$;
(c) $\exp(\alpha_2(r)) > \beta_1(r)$ *and* $\beta_3(r) = D\exp(\alpha_4(r))$;
(d) $\exp(\alpha_2(r)) > \beta_1(r)$ *and* $\beta_3(r) > \exp(\alpha_4(r))$; *then*

$$\lim_{r\to+\infty} \frac{\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))} = +\infty.$$

(ii) Any one of the following two conditions are assumed to be satisfied:
(a) $\beta_1(r) = C(\exp(\alpha_2(r)))$ *and* $\alpha_4(\beta_3^{-1}(r)) \in L_1$;
(b) $\beta_3(r) > \exp(\alpha_4(r))$ *and* $\alpha_4(\beta_3^{-1}(r)) \in L_1$; *then*

$$\lim_{r\to+\infty} \frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r)))))))}{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))} = +\infty.$$

(iii) Any one of the following two conditions are assumed to be satisfied:
(a) $\beta_3(r) = D\exp(\alpha_4(r))$ *and* $\alpha_2(\beta_1^{-1}(r)) \in L_1$;
(b) $\beta_3(r) > \exp(\alpha_4(r))$ *and* $\alpha_2(\beta_1^{-1}(r)) \in L_1$; *then*

$$\lim_{r\to+\infty} \frac{\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))))))} = +\infty.$$

(iv) If $\alpha_2(\beta_1^{-1}(r)) \in L_1$ *and* $\alpha_4(\beta_3^{-1}(r)) \in L_1$, *then*

$$\lim_{r\to+\infty} \frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r)))))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))))))} = +\infty.$$

Proof. **Case I.** Let $\beta_1(r) = C(\exp(\alpha_2(r)))$. Then we have from (2) for all sufficiently large values of r that

$$\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))) \leqslant C(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f] + \varepsilon)r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}. \tag{9}$$

Case II. Let $\exp(\alpha_2(r)) > \beta_1(r)$. Then we have from (2) for all sufficiently large values of r that

$$(10) \qquad \alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))) < (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}.$$

Case III. Let $\alpha_2(\beta_1^{-1}(r)) \in L_1$. Then we get from(2) for all sufficiently large values of r that

$$(11) \qquad \exp(\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))))) \leqslant r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}.$$

Further using the inequality $T_f(r) \le \log M_f(r) \le 3T_f(2r)$ $\{cf.$ [8]$\}$ for an entire function f, it follows from Lemma 2.2 and for all sufficiently large values r that

$$\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r)))) \ge \alpha_3(\exp(\frac{1}{3}T_h(\frac{1}{8}M_k(\frac{\beta_4^{-1}(\log r)}{4}))))$$

$$i.e., \ \alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r)))) \ge (1+o(1))(\alpha_3(\exp(T_h(\frac{1}{8}M_k(\frac{\beta_4^{-1}(\log r)}{4})))))$$

$$(12) \qquad i.e., \ \alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r)))) \ge (1+o(1))(\lambda_{(\alpha_3,\beta_3)}[h]-\varepsilon)\beta_3(M_k(\frac{\beta_4^{-1}(\log r)}{4})).$$

Case IV. Let $\beta_3(r) = D\exp(\alpha_4(r))$ Then from (12) it follows for all sufficiently large values of r that

$$(13) \qquad \alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r)))) \ge D(1+o(1))(\lambda_{(\alpha_3,\beta_3)}[h]-\varepsilon)r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}.$$

Case V. Let $\beta_3(r) > \exp(\alpha_4(r))$. Now from (12) it follows for all sufficiently large values of r that

$$(14) \qquad \alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r)))) > (1+o(1))(\lambda_{(\alpha_3,\beta_3)}[h]-\varepsilon)r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}.$$

Case VI. Let $\alpha_4(\beta_3^{-1}(r)) \in L_1$. Then from (12) we obtain for all sufficiently large values of r that

$$(15) \qquad \exp(\alpha_4(\beta_3^{-1}(\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r))))))) \geqslant r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}.$$

Since $\rho_{(\alpha_2,\beta_2)}[g] < \lambda_{(\alpha_4,\beta_4)}[k]$ we can choose $\varepsilon(>0)$ in such a way that

$$(16) \qquad \rho_{(\alpha_2,\beta_2)}[g]+\varepsilon < \lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon.$$

Now combining (9) of Case I and (13) of Case IV it follows for all sufficiently large values of r that

$$\frac{\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))} \ge \frac{D(1+o(1))(\lambda_{(\alpha_3,\beta_3)}[h]-\varepsilon)r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{C(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}.$$

So from (16) and above we obtain that

$$(17) \qquad \liminf_{r\to+\infty}\frac{\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))} = +\infty.$$

Further combining (9) of Case I and (14) of Case V it follows for all sufficiently large values of r that

$$\frac{\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))} > \frac{(1+o(1))(\lambda_{(\alpha_3,\beta_3)}[h]-\varepsilon)r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{C(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}.$$

Hence from (16) and above we get that

$$\liminf_{r\to+\infty}\frac{\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))} = +\infty. \tag{18}$$

Similarly combining (10) of Case II and (13) of Case IV, we obtain that

$$\liminf_{r\to+\infty}\frac{\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))} = +\infty. \tag{19}$$

Likewise combining (10) of Case II and (14) of Case V it follows that

$$\liminf_{r\to+\infty}\frac{\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r))))}{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))} = +\infty. \tag{20}$$

Hence the first part of the theorem follows from (17), (18), (19) and (20).

Again combining (9) of Case I and (15) of Case VI we obtain for all sufficiently large values of r that

$$\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r)))))))}{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))} \geq \frac{r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{C(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}.$$

So from (16) and above we obtain that

$$\lim_{r\to+\infty}\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r)))))))}{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))} = +\infty. \tag{21}$$

Now in view of (10) of Case II and (15) of Case VI we get for all sufficiently large values of r that

$$\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r)))))))}{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))} > \frac{r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}.$$

So from (16) and above we obtain that

$$\liminf_{r\to+\infty}\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r)))))))}{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))} = +\infty, \tag{22}$$

Therefore the second part of the theorem follows from (21) and (22).

Further combining (11) of Case III and (13) of Case IV it follows for all sufficiently large values of r that

$$\frac{\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))))))} \geq \frac{D(1+o(1))(\lambda_{(\alpha_3,\beta_3)}[h]-\varepsilon)r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}. \tag{23}$$

Now in view of (16) we obtain from (23) that

$$\lim_{r\to+\infty}\frac{\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))))))}=+\infty. \tag{24}$$

Similarly combining (11) of Case III and (14) of Case V we get that

$$\lim_{r\to+\infty}\frac{\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))))))}=+\infty. \tag{25}$$

Hence the third part of the theorem follows from (24) and (25).

Again combining (11) of Case III and (15) of Case VI we obtain for all sufficiently large values of r that

$$\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r)))))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))))))}\geq\frac{r^{(1+o(1))(\lambda_{(\alpha_4,\beta_4)}[k]-\varepsilon)}}{r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}.$$

Now in view of (16) we obtain from above that

$$\lim_{r\to+\infty}\frac{\exp(\alpha_4(\beta_3^{-1}(\alpha_3(\exp(T_{h\circ k}(\beta_4^{-1}(\log r)))))))}{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))))))}=+\infty.$$

This proves the fourth part of the theorem.

Thus the theorem follows. □

Theorem 3.7. *Let f be a meromorphic function and g be an entire function such that $0<\lambda_{(\alpha_1,\beta_1)}[f]\leq\rho_{(\alpha_1,\beta_1)}[f]<+\infty$. and $\rho_{(\alpha_2,\beta_2)}[g]<+\infty$. If $\alpha_2(\beta_1^{-1}(r))\in L_1$, then*

$$\limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f\circ g}(r)))))}{\alpha_1(\exp(T_f(\beta_1^{-1}(\beta_2(r)))))}\leq\frac{\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}.$$

Proof. In view of (1) it follows for all sufficiently large values of r that

$$\alpha_1(\exp(T_f(\beta_1^{-1}(\beta_2(r)))))\geq(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)\beta_2(r). \tag{26}$$

Again in view of (2), we get for all sufficiently large values of r that

$$\alpha_1(\exp(T_{f\circ g}(r)))\leqslant(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(M_g(r)).$$

Since $\alpha_2(\beta_1^{-1}(r))\in L_1$, we obtain from above for all sufficiently large values of r that

$$\begin{aligned}\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f\circ g}(r)))))&\leq(1+o(1))\alpha_2(M_g(r))\\ \textit{i.e.},\ \alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f\circ g}(r)))))&\leq(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)\beta_2(r).\end{aligned}$$

Now combining (26) and above we get that

$$\limsup_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f\circ g}(r)))))}{\alpha_1(\exp(T_f(\beta_1^{-1}(\beta_2(r)))))}\leq\frac{\rho_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}.$$

Hence the theorem follows. □

Theorem 3.8. *Let f be a meromorphic function and g be an entire function such that $0<\lambda_{(\alpha_1,\beta_1)}[f]\leq\rho_{(\alpha_1,\beta_1)}[f]<+\infty$. and $\lambda_{(\alpha_2,\beta_2)}[g]<+\infty$. If $\alpha_2(\beta_1^{-1}(r))\in L_1$, then*

$$\liminf_{r\to+\infty}\frac{\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f\circ g}(r)))))}{\alpha_1(\exp(T_f(\beta_1^{-1}(\beta_2(r)))))}\leq\frac{\lambda_{(\alpha_2,\beta_2)}[g]}{\lambda_{(\alpha_1,\beta_1)}[f]}.$$

The proof of Theorem 3.8 would run parallel to that of Theorem 3.7. We omit the details.

Theorem 3.9. *Let f be a meromorphic function and g be an entire function such that $\lambda_{(\alpha_1,\beta_1)}[f]$ and λ_g are both finite where $\beta_1 \in L_2$. Also suppose that there exist entire functions $a_i (i = 1, 2, ...n; n \leq +\infty)$ satisfying*
(A) $T_{a_i}(r) = \circ\{T_g(r)\}$ as $r \to +\infty$ and
(B) $\sum_{i=1}^{n} \delta(a_i, g) = 1$. Then

$$\liminf_{r\to+\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\beta_1(\exp(T_g(r)))} \leq \lambda_{(\alpha_1,\beta_1)}[f].$$

Proof. In view of Lemma 2.1 and the inequality $T_f(r) \leq \log^+ M_f(r)$ we get for a sequence of values of r tending to infinity that

$$\frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\beta_1(\exp(T_g(r)))} \leqslant \frac{(1 + o(1))(\lambda_{(\alpha_1,\beta_1)}[f] + \varepsilon)\beta_1(M_g(r))}{\beta_1(\exp(T_g(r)))}.$$

In view of Lemma 2.4 and as $\varepsilon(> 0)$ is arbitrary we obtain from above that

$$\liminf_{r\to+\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\beta_1(\exp(T_g(r)))} \leq \lambda_{(\alpha_1,\beta_1)}[f].$$

Thus the theorem follows. □

The following theorem can be proved in the line of Theorem 3.9 and so its proof is omitted.

Theorem 3.10. *Let f be a meromorphic function and g be an entire function such that $\rho_{(\alpha_1,\beta_1)}[f]$ and λ_g are both finite where $\beta_1 \in L_2$ .Also suppose that there exist entire functions $a_i (i = 1, 2, ...n; n \leq +\infty)$ satisfying*
(A) $T_{a_i}(r) = \circ\{T_g(r)\}$ as $r \to +\infty$ and
(B) $\sum_{i=1}^{n} \delta(a_i, g) = 1$. Then

$$\limsup_{r\to+\infty} \frac{\alpha_1(\exp(T_{f\circ g}(r)))}{\beta_1(\exp(T_g(r)))} \leq \rho_{(\alpha_1,\beta_1)}[f].$$

Theorem 3.11. *Let f be meromorphic and g be entire such that $\rho_{(\alpha_1,\beta_1)}[f \circ g] < +\infty$ and $\lambda_{(\alpha_3,\beta_3)}[g] > 0$. Then*

$$\lim_{r\to+\infty} \frac{\left\{\alpha_1(\exp(T_{f\circ g}(\beta_1^{-1}(\log r))))\right\}^2}{\alpha_3(\exp(T_g(\beta_3^{-1}(\log r)))) \cdot \alpha_3(\exp(T_g(\beta_3^{-1}(r))))} = 0.$$

Proof. For arbitrary positive ε we have for all sufficiently large values of r that

$$\alpha_1(\exp(T_{f\circ g}(\beta_1^{-1}(\log r)))) \leq (\rho_{(\alpha_1,\beta_1)}[f \circ g] + \varepsilon) \log r. \tag{27}$$

Again for all sufficiently large values of r we get

$$\alpha_3(\exp(T_g(\beta_3^{-1}(\log r)))) \geq (\lambda_{(\alpha_3,\beta_3)}[g] - \varepsilon) \log r. \tag{28}$$

Similarly for all sufficiently large values of r we have

$$\alpha_3(\exp(T_g(\beta_3^{-1}(r)))) \geq (\lambda_{(\alpha_3,\beta_3)}[g] - \varepsilon)r. \tag{29}$$

From (27) and (28) we have for all sufficiently large values of r that

$$\frac{\alpha_1(\exp(T_{f\circ g}(\beta_1^{-1}(\log r))))}{\alpha_3(\exp(T_g(\beta_3^{-1}(\log r))))} \leq \frac{(\rho_{(\alpha_1,\beta_1)}[f \circ g] + \varepsilon) \log r}{(\lambda_{(\alpha_3,\beta_3)}[g] - \varepsilon) \log r}.$$

As $\varepsilon(>0)$ is arbitrary we obtain from above that

$$\limsup_{r\to+\infty}\frac{\alpha_1(\exp(T_{f\circ g}(\beta_1^{-1}(\log r))))}{\alpha_3(\exp(T_g(\beta_3^{-1}(\log r))))}\leq\frac{\rho_{(\alpha_1,\beta_1)}[f\circ g]}{\lambda_{(\alpha_3,\beta_3)}[g]}. \tag{30}$$

Again from (27) and (29) we get for all sufficiently large values of r that

$$\frac{\alpha_1(\exp(T_{f\circ g}(\beta_1^{-1}(\log r))))}{\alpha_3(\exp(T_g(\beta_3^{-1}(r))))}\leq\frac{(\rho_{(\alpha_1,\beta_1)}[f\circ g]+\varepsilon)\log r}{(\lambda_{(\alpha_3,\beta_3)}[g]-\varepsilon)r}.$$

Since $\varepsilon(>0)$ is arbitrary it follows from above that

$$\lim_{r\to+\infty}\frac{\alpha_1(\exp(T_{f\circ g}(\beta_1^{-1}(\log r))))}{\alpha_3(\exp(T_g(\beta_3^{-1}(r))))}=0. \tag{31}$$

Thus the theorem follows from (30) and (31). □

THEOREM 3.12. *Let f be meromorphic and g be entire such that $\rho_{(\alpha_2,\beta_2)}[g]<\lambda_{(\alpha_1,\beta_1)}[f]\leq\rho_{(\alpha_1,\beta_1)}[f]$. Also let C be any positive constant and $\beta_1\in L_2$.*
(i) Any one of the following two conditions are assumed to be satisfied:
(a) $\beta_1(r)=C(\exp(\alpha_2(r)))$;
(b) $\exp(\alpha_2(r))>\beta_1(r)$; then

$$\limsup_{r\to+\infty}\frac{\left\{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))\right\}^2}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\log r)))))\cdot\beta_1(\exp(T_g(2(\beta_2^{-1}(\log r)))))}=0.$$

(ii) If $\alpha_2(\beta_1^{-1}(r))\in L_1$, then

$$\lim_{r\to+\infty}\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))))))\cdot\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\log r)))))\cdot\beta_1(\exp(T_g(2(\beta_2^{-1}(\log r)))))}=0.$$

Proof. From the definition of generalized lower order (α_1,β_1) of f we have for arbitrary positive ε and for all sufficiently large values of r that

$$\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\log r)))))\geq r^{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)}. \tag{32}$$

As $\rho_{(\alpha_2,\beta_2)}[g]<\lambda_{(\alpha_1,\beta_1)}[f]$ we can choose $\varepsilon(>0)$ in such a way that

$$\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon<\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon. \tag{33}$$

Now combining (9) of Case I and (32) we have for all large positive numbers of r,

$$\frac{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\log r)))))}\leq\frac{C(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)}}.$$

In view of (33) we get from above that

$$\lim_{r\to+\infty}\frac{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\log r)))))}=0. \tag{34}$$

Again combining (10) of Case II and (32) it follows for all sufficiently large positive numbers of r that

$$\frac{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\log r)))))}\leq\frac{(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)r^{(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)}}.$$

Now in view of (33) we obtain from above that

$$\lim_{r\to+\infty}\frac{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\log r)))))}=0. \tag{35}$$

Further combining (11) of Case III and (32) it follows for all sufficiently large positive numbers of r that

$$\frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\log r)))))} \leq \frac{r^{(1+o(1))(\rho_{(\alpha_2,\beta_2)}[g]+\varepsilon)}}{r^{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)}}.$$

So in view of (33) we obtain from above that

$$\lim_{r\to+\infty} \frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\log r)))))} = 0. \tag{36}$$

Now from(2)we have for all sufficiently large values of r that

$$\frac{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))}{\beta_1(\exp(T_g(2(\beta_2^{-1}(\log r)))))} \leq \frac{(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)\beta_1(M_g(\beta_2^{-1}(\log r)))}{\beta_1(\exp(T_g(2(\beta_2^{-1}(\log r)))))}.$$

Since $\varepsilon(>0)$ is arbitrary, in view of $\log^+ M_f(r) \leq 3T_f(2r)$ $\{cf.$ [8]$\}$, we get from above that

$$\limsup_{r\to+\infty} \frac{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))}{\beta_1(\exp(T_g(2(\beta_2^{-1}(\log r)))))} \leq \rho_{(\alpha_1,\beta_1)}[f]. \tag{37}$$

From (34) and (37) we obtain for all sufficiently large values of r that

$$\begin{aligned}
&\limsup_{r\to+\infty} \frac{\left\{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))\right\}^2}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\log r))))) \cdot \beta_1(\exp(T_g(2(\beta_2^{-1}(\log r)))))} \\
&= \lim_{r\to+\infty} \frac{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\log r)))))} \cdot \limsup_{r\to+\infty} \frac{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))}{\beta_1(\exp(T_g(2(\beta_2^{-1}(\log r)))))} \\
&\leq 0.\rho_{(\alpha_1,\beta_1)}[f]
\end{aligned}$$

$$i.e., \limsup_{r\to+\infty} \frac{\left\{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))\right\}^2}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\log r))))) \cdot \beta_1(\exp(T_g(2(\beta_2^{-1}(\log r)))))} \leq 0. \tag{38}$$

Similarly from (35) and (37) we obtain that

$$\limsup_{r\to+\infty} \frac{\left\{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))\right\}^2}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\log r))))) \cdot \beta_1(\exp(T_g(2(\beta_2^{-1}(\log r)))))} = 0.$$

Therefore the first part of the theorem follows from (38) and above.

Again from (36) and (37) we get for all large values of r that

$$\begin{aligned}
&\limsup_{r\to+\infty} \frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))))) \cdot \alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\log r))))) \cdot \beta_1(\exp(T_g(2(\beta_2^{-1}(\log r)))))} \\
&= \lim_{r\to+\infty} \frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))))))}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\log r)))))} \\
&\quad \cdot \limsup_{r\to+\infty} \frac{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))}{\beta_1(\exp(T_g(2(\beta_2^{-1}(\log r)))))} \\
&\leq 0 \cdot \rho_{(\alpha_1,\beta_1)}[f] = 0.
\end{aligned}$$

$$i.e., \lim_{r\to+\infty} \frac{\exp(\alpha_2(\beta_1^{-1}(\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))))) \cdot \alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\log r))))) \cdot \beta_1(\exp(T_g(2(\beta_2^{-1}(\log r)))))} = 0.$$

Thus the second part of the theorem is established. □

THEOREM 3.13. *Let f be meromorphic and g be entire such that $\lambda_{(\alpha_1,\beta_1)}[f] < +\infty$ and $\rho_{(\alpha_3,\beta_3)}[f \circ g] < +\infty$ where $\alpha_2, \beta_1 \in L_2$. Then*

$$\limsup_{r\to+\infty} \frac{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))) \cdot \alpha_3(\exp(T_{f\circ g}(\beta_3^{-1}(r))))}{\beta_1(\exp(T_g(2(\beta_2^{-1}(\log r))))) \cdot \alpha_2(\exp(T_g(\beta_2^{-1}(r))))}$$
$$\leq \frac{\rho_{(\alpha_3,\beta_3)}[f \circ g] \cdot \rho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_2,\beta_2)}[g]}.$$

Proof. For all sufficiently large values of r we have

$$\alpha_3(\exp(T_{f\circ g}(\beta_3^{-1}(r)))) \leq (\rho_{(\alpha_3,\beta_3)}[f \circ g] + \varepsilon)r. \tag{39}$$

Again for all sufficiently large values of r it follows that

$$\alpha_2(\exp(T_g(\beta_2^{-1}(r)))) \geq (\lambda_{(\alpha_2,\beta_2)}[g] - \varepsilon)r. \tag{40}$$

Now combining (39) and (40) we have for all sufficiently large values of r that

$$\frac{\alpha_3(\exp(T_{f\circ g}(\beta_3^{-1}(r))))}{\alpha_2(\exp(T_g(\beta_2^{-1}(r))))} \leq \frac{\rho_{(\alpha_3,\beta_3)}[f \circ g] + \varepsilon}{\lambda_{(\alpha_2,\beta_2)}[g] - \varepsilon}.$$

As $\varepsilon(> 0)$ is arbitrary we get from above that

$$\limsup_{r\to+\infty} \frac{\alpha_3(\exp(T_{f\circ g}(\beta_3^{-1}(r))))}{\alpha_2(\exp(T_g(\beta_2^{-1}(r))))} \leq \frac{\rho_{(\alpha_3,\beta_3)}[f \circ g]}{\lambda_{(\alpha_2,\beta_2)}[g]}. \tag{41}$$

Now from (37) and (41) we obtain that

$$\limsup_{r\to+\infty} \frac{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r)))) \cdot \alpha_3(\exp(T_{f\circ g}(\beta_3^{-1}(r))))}{\beta_1(\exp(T_g(2(\beta_2^{-1}(\log r))))) \cdot \alpha_2(\exp(T_g(\beta_2^{-1}(r))))}$$

$$\leq \limsup_{r\to+\infty} \frac{\alpha_1(\exp(T_{f\circ g}(\beta_2^{-1}(\log r))))}{\beta_1(\exp(T_g(2(\beta_2^{-1}(\log r)))))} \cdot \limsup_{r\to+\infty} \frac{\alpha_3(\exp(T_{f\circ g}(\beta_3^{-1}(r))))}{\alpha_2(\exp(T_g(\beta_2^{-1}(r))))}$$
$$\leq \frac{\rho_{(\alpha_3,\beta_3)}[f \circ g] \cdot \rho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_2,\beta_2)}[g]}.$$

Hence the theorem follows. □

THEOREM 3.14. *Let f be meromorphic and g be entire such that $\rho_{(\alpha_1,\beta_1)}[f] < +\infty$ and $\lambda_{(\alpha_3,\beta_3)}[f \circ g] = +\infty$. Then*

$$\lim_{r\to+\infty} \frac{\alpha_3(\exp(T_{f\circ g}(r)))}{\alpha_1(\exp(T_f(\beta_1^{-1}(\beta_3(r)))))} = +\infty.$$

Proof. Let us suppose that the conclusion of the theorem do not hold. Then we can find a constant $\Delta > 0$ such that for a sequence of values of r tending to infinity

$$\alpha_3(\exp(T_{f\circ g}(r))) \leq \Delta \cdot \alpha_1(\exp(T_f(\beta_1^{-1}(\beta_3(r))))). \tag{42}$$

Again from the definition of $\rho_{(\alpha_1,\beta_1)}[f]$, it follows for all sufficiently large values of r that

$$\alpha_1(\exp(T_f(\beta_1^{-1}(\beta_3(r))))) \leq (\rho_{(\alpha_1,\beta_1)}[f] + \epsilon)\beta_3(r). \tag{43}$$

Thus from (42) and (43), we have for a sequence of values of r tending to infinity that

$$\alpha_3(\exp(T_{f\circ g}(r))) \leq \Delta(\rho_{(\alpha_1,\beta_1)}[f] + \epsilon)\beta_3(r)$$

$$i.e., \ \frac{\alpha_3(\exp(T_{f\circ g}(r)))}{\beta_3(r)} \leq \frac{\Delta(\rho_{(\alpha_1,\beta_1)}[f]+\epsilon)\beta_3(r)}{\beta_3(r)}$$

$$i.e., \ \liminf_{r+\infty} \frac{\alpha_3(\exp(T_{f\circ g}(r)))}{\beta_3(r)} = \lambda_{(\alpha_3,\beta_3)}[f\circ g] < +\infty.$$

This is a contradiction.

Thus the theorem follows. □

Remark 3.15. Theorem 3.14 is also valid with "limit superior" instead of "limit" if $\lambda_{(\alpha_3,\beta_3)}[f\circ g] = +\infty$ is replaced by $\rho_{(\alpha_3,\beta_3)}[f\circ g] = +\infty$ and the other conditions remain the same.

Analogously one may also state the following theorem without its proof as it may be carried out in the line of Theorem 3.14.

Theorem 3.16. *Let f be meromorphic and g be entire such that $\rho_{(\alpha_1,\beta_1)}[g] < +\infty$ and $\rho_{(\alpha_3,\beta_3)}[f\circ g] = +\infty$. Then*

$$\limsup_{r\to+\infty} \frac{\alpha_3(\exp(T_{f\circ g}(r)))}{\alpha_1(\exp(T_g(\beta_1^{-1}(\beta_3(r)))))} = +\infty.$$

Remark 3.17. Theorem 3.16 is also valid with "limit" instead of "limit superior" if $\rho_{(\alpha_3,\beta_3)}[f\circ g] = +\infty$ is replaced by $\lambda_{(\alpha_3,\beta_3)}[f\circ g] = +\infty$ and the other conditions remain the same.

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References

[1] W. Bergweiler, *On the Nevanlinna Characteristic of a composite function*, Complex Variables **10** (1988), 225–236.

[2] T. Biswas and C. Biswas, *Some results on generalized relative order (α,β) and generalized relative type (α,β) of meromorphic function with respect to an entire function*, Ganita **70** (2) (2020), 239–252.

[3] T. Biswas, C. Biswas and R. Biswas, *A note on generalized growth analysis of composite entire functions*, Poincare J. Anal. Appl. **7** (2) (2020), 277–286.

[4] T. Biswas and C. Biswas, *Generalized order (α,β) orientied some growth properties of composite entire functions*, Ural Math. J. **6** (2) (2020), 25–37.

[5] T. Biswas, *On some inequalities concerning relative (p,q)-φ type and relative (p,q)-φ weak type of entire or meromorphic functions with respect to an entire function*, J. Class. Anal. **13** (2) (2018), 107–122.

[6] I. Chyzhykov and N. Semochko, *Fast growing entire solutions of linear differential equations,* Math. Bull. Shevchenko Sci. Soc. **13** (2016), 68–83.

[7] J. Clunie, *The composition of entire and meromorphic functions*, Mathematical Essays dedicated to A. J. Macintyre, Ohio University Press (1970), 75–92.

[8] W.K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.

[9] X. Shen, J. Tu and H. Y. Xu, *Complex oscillation of a second-order linear differential equation with entire coefficients of $[p,q]$-φ order*, Adv. Difference Equ. **2014** (1): 200, (2014), 14 pages, http://www.advancesindifferenceequations.com/content/2014/1/200.

[10] O. P. Juneja, G. P. Kapoor and S. K. Bajpai, *On the (p,q)-order and lower (p,q)-order of an entire function*, J. Reine Angew. Math. **282** (1976), 53–67.

[11] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, De Gruyter, Berlin, 1993.

[12] Q. Lin and C. Dai, *On a conjecture of Shah concerning small functions*, Kexue Tong bao (English Ed.) **31** (4) (1986), 220–224.
[13] M. N. Sheremeta, *On the growth of a composition of entire functions*, Carpathian Math. Publ. **9** (2) (2017), 181–187.
[14] M. N. Sheremeta, *Connection between the growth of the maximum of the modulus of an entire function and the moduli of the coefficients of its power series expansion*, Izv. Vyssh. Uchebn. Zaved Mat. **2** (1967), 100–108. (in Russian).
[15] D. Sato, *On the rate of growth of entire functions of fast growth*, Bull. Amer. Math. Soc. **69** (1963), 411–414.
[16] G. Valiron, *Lectures on the general theory of integral functions*, Chelsea Publishing Company, New York (NY), USA, 1949.
[17] L. Yang, *Value distribution theory*, Springer-Verlag, Berlin, 1993.

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