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GROWTH OF SOLUTIONS OF NON-HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS AND ITS APPLICATIONS

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Abstract. In this paper, we investigate the growth properties of solutions of the non-homogeneous linear complex differential equation $L(f) = b(z) f + c(z)$, where $L(f)$ is a linear differential polynomial and $b(z)$, $c(z)$ are entire functions and give some of its applications on sharing value problems.

1. Introduction

In this paper, we shall assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions, as found in [2,5,8,13]. The term "meromorphic function" will mean meromorphic in the whole complex plane $\mathbb{C}$. The order $\sigma(f)$ and hyper order $\sigma_1(f)$ of a meromorphic function f are defined by

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log T(r,f)}{\log r} \text{ and } \sigma_1(f) = \limsup_{r\to+\infty} \frac{\log\log T(r,f)}{\log r},$$

where $T(r,f)$ is the Nevanlinna characteristic function of f.

Clearly, if $f(z)$ is entire, then

$$\sigma(f) = \limsup_{r\to+\infty} \frac{\log\log M(r,f)}{\log r} \text{ and } \sigma_1(f) = \limsup_{r\to+\infty} \frac{\log\log\log M(r,f)}{\log r},$$

where

$$M(r,f) = \max_{|z|=r} |f(z)|$$

is the maximum modulus of f on the circle $|z| = r$.

The linear measure of a set $E \subset [0,+\infty)$ is defined as $m(E) = \int\limits_E dt$ and the logarithmic measure of a set $F \subset [1,+\infty)$ is defined by $m_l(F) = \int\limits_F \frac{dt}{t}$. The upper and lower densities of E are defined by

$$\overline{dens}E = \limsup_{r\to+\infty} \frac{m(E\cap[0,r])}{r}, \ \underline{dens}E = \liminf_{r\to+\infty} \frac{m(E\cap[0,r])}{r}.$$

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For $k \geq 1$, we consider a linear differential equation of the form

$$f^{(k)} + a_{k-1}(z) f^{(k-1)} + ... + a_1(z) f' + a_0(z) f = 0, \tag{1}$$

where $a_0(z), a_1(z), ..., a_{k-1}(z)$ are entire functions with $a_0(z) \not\equiv 0$. It is well known that all solutions of equation (1) are entire functions and if some of the coefficients of (1) are transcendental, then (1) has at least one solution with $\sigma(f) = +\infty$.

In 2002, Belaïdi [1] investigated the growth of infinite order solutions of the linear differential equation (1) and obtained the following results:

Theorem 1. [1] *Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$ and let $a_0(z), ..., a_{k-1}(z)$ be entire functions such that for some constants $0 \leq q < p$ and $\eta > 0$, we have*

$$|a_0(z)| \geq e^{p|z|^\eta}$$

and

$$|a_j(z)| \leq e^{q|z|^\eta},\ j = 1, 2, ..., k-1$$

as $z \to \infty$ for $z \in H$. Then, every solution $f \not\equiv 0$ of (1) satisfies $\sigma(f) = +\infty$ and $\sigma_1(f) \geq \eta$.

Theorem 2. [1] *Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$ and let $a_0(z), ..., a_{k-1}(z)$ be entire functions with*

$$\max\{\sigma(a_j) : j = 1, ..., k-1\} \leq \sigma(a_0) = \overline{\sigma} < +\infty$$

such that for some constants $0 \leq q < p$, we have

$$|a_0(z)| \geq e^{p|z|^{\overline{\sigma}-\varepsilon}}$$

and

$$|a_j(z)| \leq e^{q|z|^{\overline{\sigma}-\varepsilon}},\ j = 1, 2, ..., k-1$$

as $z \to \infty$ for $z \in H$. Then, every solution $f \not\equiv 0$ of (1) satisfies $\sigma(f) = +\infty$ and $\sigma_1(f) = \sigma(a_0) = \overline{\sigma}$.

For a meromorphic function f, the expression

$$L[f] = a_k(z) f^{(k)} + a_{k-1}(z) f^{(k-1)} + ... + a_1(z) f' + a_0(z) f,$$

is called a linear differential polynomial in f of degree k, where k is a positive integer and $a_0(z), a_1(z), ..., a_k(z)$ are entire functions with $a_k(z) \not\equiv 0$. We consider a non-homogeneous linear differential equation of the form

$$L[f] = b(z) f + c(z), \tag{2}$$

where $b(z),\ c(z)$ are entire functions. We give some growth properties of solutions of equation (2) and give some applications on growth estimates of entire function f that share one finite value with its linear differential polynomial $L[f]$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. [7,8] *Let* $f(z)=\sum_{n=0}^{\infty}a_n z^n$ *be an entire function of finite order* $\sigma(f)<+\infty$,

$$\mu(r)=\max\{|a_n|\, r^n, n=0,1,...\}$$

be the maximum term of f *and*

$$\nu(r,f)=\max\{m:\mu(r)=|a_m|\, r^m\}$$

be the central index of f. *Then*

$$\limsup_{r\to+\infty}\frac{\log\nu(r,f)}{\log r}=\sigma(f)$$

and if f *is a transcendental entire function of hyper order* $\sigma_1(f)$, *then*

$$\limsup_{r\to+\infty}\frac{\log\log\nu(r,f)}{\log r}=\sigma_1(f).$$

Lemma 2. [5, 13] *Let* f *be a non-constant meromorphic function in the finite complex plane and* k *be a positive integer. Then for* $1\le r<R<\infty$,

$$m\left(r,\frac{f^{(k)}}{f}\right)\le C\left(\log^{+}T(R,f)+\log^{+}\frac{1}{R-r}+\log R+1\right),$$

where C *is a positive constant depending only on* k *and* f.

Lemma 3. [4] *Let* f *be a transcendental meromorphic function of finite order* σ. *Let* $\Delta=\{(k_1,j_1),(k_2,j_2),...,(k_m,j_m)\}$ *denote a finite set of distinct pairs of integer satisfying* $k_i>j_i\ge 0$ *for* $i=1,...,m$ *and let* $\epsilon>0$ *be a given constant. Then there exists a set* $E\subset(1,+\infty)$ *with finite logarithmic measure such that for all* z *satisfying* $|z|\notin E\cup[0,1]$ *and for all* $(k,j)\in\Delta$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right|\le|z|^{(k-j)(\sigma-1+\epsilon)}.$$

Lemma 4. [6, 8, 12] *Let* $f(z)$ *be a transcendental entire function,* $\nu(r,f)$ *be the central index of* $f(z)$. *Then there exists a set* $E\subset(1,+\infty)$ *with finite logarithmic measure such that for* z *satisfying* $|z|=r\notin[0,1]\cup E$ *and* $|f(z)|=M(r,f)$, *we have*

$$\frac{f^{(i)}(z)}{f(z)}=\left\{\frac{\nu(r,f)}{z}\right\}^{i}(1+o(1)),\ \textit{for } i\in N.$$

Lemma 5. *Let* $g:(0,+\infty)\to\mathbb{R}$, $h:(0,+\infty)\to\mathbb{R}$ *be monotone increasing functions such that* $g(r)\le h(r)$ *outside of an exceptional set* E *of finite logarithmic measure. Then, for any* $\alpha>1$, *there exists* $r_0>0$ *such that* $g(r)\le h(\alpha r)$ *for all* $r>r_0$.

Proof. Since E has a finite logarithmic measure, $\int_E dr/r<+\infty$. Therefore, there exists $r_0>0$ such that for any $r>r_0$, the interval $[r,\alpha r]$ meets the complement of E. In fact,

$$\int_r^{\alpha r}dt/t=\log\alpha<+\infty.$$

Therefore, taking $t\in[r,\alpha r]-E$, we get

$$g(r)\le g(t)\le h(t)\le h(\alpha r).$$

This completes the proof. □

3. Growth of solutions of linear differential equations

Theorem 3. *Let H be a set of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$ and let $b(z), a_j(z)$ $(j = 0, 1, 2, ..., k)$ and $c(z)$ be entire functions such that for some constants $0 \le q < p$ and $\eta > 0$ we have $|b(z)| \ge e^{p|z|^{\eta}}$ and $|a_j(z)| \le e^{q|z|^{\eta}}$ $(j = 0, 1, 2, ..., k)$; $|c(z)| \le e^{q|z|^{\eta}}$ as $z \to \infty$ for $z \in H$. Then, every solution $f \not\equiv 0$ of (2) is of infinite order.*

Proof. Suppose that $f \not\equiv 0$ is a solution of equation (2) with $\sigma(f) = \sigma < +\infty$. Re-writing (2) as

$$\sum_{j=0}^{k} \frac{a_j(z)}{b(z)} \frac{f^{(j)}(z)}{f(z)} - \frac{c(z)}{b(z)} \frac{1}{f(z)} = 1,$$

$$\Rightarrow 1 \le \left|\frac{a_0(z)}{b(z)}\right| + \sum_{j=1}^{k} \left|\frac{a_j(z)}{b(z)}\right| \left|\frac{f^{(j)}(z)}{f(z)}\right| + \left|\frac{c(z)}{b(z)}\right| \frac{1}{|f|}. \tag{3}$$

By Lemma 3 there exists a set $E_1 \subset (1, +\infty)$ with finite logarithmic measure such that for all z satisfying $|z| \notin E_2 = [0, 1] \cup E_1$ and for $j = 1, 2, ..., k$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le |z|^{j(\sigma-1+\epsilon)}. \tag{4}$$

On the other hand, from the conditions of Theorem 3 there is a set H of complex numbers satisfying $\overline{dens}\{|z| : z \in H\} > 0$ such that for each $z \in H$, we have

$$|b(z)| \ge e^{p|z|^{\eta}} \text{and } |a_j(z)| \le e^{q|z|^{\eta}} \; (j = 0, 1, 2, ..., k) \; ; |c(z)| \le e^{q|z|^{\eta}} \tag{5}$$

as $z \to \infty$.

Thus from (3), (4) and (5) it follows that for all z satisfying $z \in H$, $|z| \notin E_2$,

$$\begin{aligned} 1 &\le e^{(q-p)|z|^{\eta}} + e^{(q-p)|z|^{\eta}} \sum_{j=1}^{k} |z|^{j(\sigma-1+\epsilon)} + e^{(q-p)|z|^{\eta}} \frac{1}{|f|} \\ &\le e^{(q-p)|z|^{\eta}} \left(1 + k\,|z|^{k(\sigma-1+\epsilon)} + \frac{1}{|f|}\right). \end{aligned} \tag{6}$$

Therefore from (6) there exists a set $E \subset (1, +\infty)$ with a positive upper density such that for all z satisfying $z \in H$, $r = |z| \in E$, we have

$$1 \le e^{(q-p)|z|^{\eta}} \left(1 + k\ |z|^{k(\sigma-1+\epsilon)} + \frac{1}{|f|}\right).$$

Since $e^{(q-p)|z|^{\eta}} \left(1 + k\,|z|^{k(\sigma-1+\epsilon)} + \frac{1}{|f|}\right) \to 0$ as $|z| \to +\infty$, letting $z \to \infty$, $|z| \in E$, we get a contradiction. This proves the theorem. □

Example 1. Consider the linear differential equation

$$f' + f = (e^z + 1) f.$$

Comparing it with (2), $L[f] = f' + f$, $b(z) = e^z + 1$ and $c(z) = 0$. Obviously, the conditions $|b(z)| = e^r + 1 \geq e^{p|z|^\eta}$, $|a_j(z)| = 1 \leq e^{q|z|^\eta}$ $(j = 0, 1)$ and $|c(z)| = 0 \leq e^{q|z|^\eta}$for $p = 1, q = 0$ and $\eta = 1$ hold. It is clear that $f(z) = e^{e^z}$ with $\sigma(f) = +\infty$ is a solution of the given equation.

Theorem 4. *Let $c(z)$ and $b(z)$ be entire functions with $b(z)$ transcendental. Suppose that*

$$M(r, c(z)) = O\left\{(M(r, b(z)))^\xi\right\},\ 0 < \xi < \infty.$$

Then, with at most one exception, every solution f of the differential equation (2) satisfies $\sigma_1(f) = \sigma(b)$.

Proof. Let f be a solution of equation (2). Since $b(z)$ is transcendental, f is a transcendental entire solution. From (2) we have

$$\frac{L[f]}{f} = b(z) + \frac{c(z)}{f}. \tag{7}$$

By Lemma 4 there exists a set $E \subset (1, +\infty)$ with finite logarithmic measure such that for all z satisfying $r = |z| \notin E$ and $|f(z)| = M(r, f)$, we get

$$\frac{f^{(j)}(z)}{f(z)} = \left\{\frac{\nu(r, f)}{z}\right\}^j (1 + o(1)),\ j = 1, 2, ..., k.$$

and so

$$\begin{aligned}
\left|\frac{L[f]}{f}\right| &\leq \sum_{j=1}^{k} |a_j| \left|\frac{\nu(r, f)}{z}\right|^j (1 + o(1)) + |a_0| \\
&\leq \sum_{j=1}^{k} |a_j| \left|\frac{\nu(r, f)}{z}\right|^j (1 + o(1)) + |a_0| (1 + o(1)) \\
&\leq \left(\left|\frac{\nu(r, f)}{z}\right|^\mu \left(\sum_{j=1}^{k} |a_j|\right) + |a_0|\right) (1 + o(1))
\end{aligned}$$

$$\Rightarrow \left|\frac{L[f]}{f}\right| = \left(\left|\frac{\nu(r, f)}{z}\right|^\mu \left(\sum_{j=1}^{k} |a_j|\right) + |a_0|\right) (1 + o(1)), \tag{8}$$

where $\mu = 1$ or k according as $\nu(r, f) < r$ or $\nu(r, f) > r$.

Thus, from (7) and (8) we have

$$\left(\left|\frac{\nu(r, f)}{z}\right|^\mu \left(\sum_{j=1}^{k} |a_j|\right) + |a_0|\right) (1 + o(1)) = \left|b(z) + \frac{c(z)}{f}\right|$$

$$\Rightarrow \left|\frac{\nu(r, f)}{z}\right|^\mu \left(\sum_{j=1}^{k} |a_j|\right) \leq \left(|b(z)| + \frac{|c(z)|}{|f|}\right).$$

and so

$$\{\nu(r, f)\}^\mu \leq r^\mu \left(\sum_{j=1}^{k} |a_j|\right)^{-1} \left(M(r, b) + \frac{M(r, c)}{|f|}\right),\ r \to \infty,\ r \notin E.$$

Since $M(r,c(z)) = O\left\{(M(r,b(z)))^{\xi}\right\}$, $0<\xi<\infty$ and $f(z)$ is transcendental, we get

$$\{\nu(r,f)\}^{\mu} \le 2r^{\mu}\left(\sum_{j=1}^{k}|a_j|\right)^{-1}\{M(r,b)\}^{s}$$

where $s=\max\{1,\xi-1\}$ is a constant. Thus, from Lemma 5 there exists a positive number r_0 such that for all $r>r_0$

$$\{\nu(r,f)\}^{\mu} \le 2^{\mu+1}r^{\mu}\left(\sum_{j=1}^{k}|a_j|\right)^{-1}\{M(2r,b)\}^{s}.$$

Hence, we obtain

$$\sigma_1(f) \le \sigma(b). \tag{9}$$

On the other hand, let g and h be any two distinct solutions of equation (2). Then

$$L[g]-b(z)\,g = c(z),\ \ L[h]-b(z)\,h = c(z).$$

It follows that

$$b = \frac{L[g]-L[h]}{g-h} = \frac{L[g-h]}{g-h} \tag{10}$$

From (10) and Lemma 2 we have

$$\begin{aligned} T(r,b) &= m(r,b) \\ &\le \sum_{j=0}^{k} m\left(r,\frac{(g-h)^{j}}{g-h}\right)+\sum_{j=0}^{k} m(r,a_j)+\log(k+1) \\ &\le C\left(\log^{+}T(2r,g-h)+\log^{+}r+1\right)+\log(k+1) \end{aligned}$$

Hence,

$$\log^{+}T(r,b) \le \log^{+}\log^{+}T(2r,g-h)+K,$$

where K is a constant. From the above inequality, we get

$$\sigma(b) \le \sigma_1(g-h) \tag{11}$$

Since

$$\sigma_1(g-h) \le \max\{\sigma_1(g),\sigma_1(h)\},$$

we know from (11) that there is at most one solution of equation (2) that does not satisfy $\sigma(b) \le \sigma_1(f)$. Together with (9), we complete the proof of Theorem 4. □

Corollary 1. *Let $\phi(z)$ be a non-constant entire function. Then, with at most one exception, every solution f of the differential equation*

$$L[f] = e^{\phi(z)}f+1 \tag{12}$$

satisfies $\sigma_1(f) = \sigma\left(e^{\phi}\right)$.

Corollary 2. *Let $\phi(z)$ be a non-constant entire function and let $p(z)$, $q(z)$ are nonzero polynomials. Then, every solution f of the differential equation*

$$L[f] = pe^{\phi(z)}f+q \tag{13}$$

satisfies $\sigma_1(f) = \sigma\left(e^{\phi}\right)$ with at most one exception.

Example 2. Consider the linear differential equation

$$f^{''}+f^{'}=e^{-z}f+2e^{z}-1.$$

Here, $L[f]=f^{''}+f^{'}, b(z)=e^{-z}$ and $c(z)=2e^{z}-1$. Therefore, $M(r,c(z))\leq 2e^{r}+1$, $M(r,b(z))=e^{r}$ and $\sigma(b)=\sigma(e^{-z})=1$. Since, the conditions $M(r,c(z))=O\left\{(M(r,b(z)))^{\xi}\right\}$ for $\xi=2$ hold, every solution f of the given differential equation satisfies $\sigma_1(f)=\sigma(b)$. But, we see that $f(z)=e^{z}$ is an exceptional solution of the given equation as $\sigma_1(e^{z})=0\neq\sigma(b)$.

Now, let us verify corollaries of the Theorem 4.

Example 3. Consider the linear differential equation

$$f^{''}-f^{'}=e^{-z}f+1.$$

Comparing it with (12), $L[f]=f^{''}-f^{'}$, $\phi(z)=-z$ and $\sigma\left(e^{\phi}\right)=\sigma(e^{-z})=1$. Clearly, all the conditions of Corollary 1 hold. Therefore, every solution f of the given differential equation satisfies $\sigma_1(f)=\sigma\left(e^{\phi}\right)$. But, we see that $f(z)=-e^{z}$ is an exceptional solution of the given equation as $\sigma_1(-e^{z})=0\neq\sigma\left(e^{\phi}\right)$.

Example 4. Consider the linear differential equation

$$f^{''}+f^{'}=ze^{z}f-z.$$

Here, $L[f]=f^{''}+f^{'}, \phi(z)=z, p(z)=z, q(z)=-z$ and $\sigma\left(e^{\phi}\right)=\sigma(e^{z})=1$. Since, all the conditions of Corollary 2 hold, every solution f of the given differential equation satisfies $\sigma_1(f)=\sigma\left(e^{\phi}\right)$. But, we see that $f(z)=e^{-z}$ is an exceptional solution of the given equation as $\sigma_1(e^{-z})=0\neq\sigma\left(e^{\phi}\right)$.

4. Applications

For two functions f and g meromorphic in the finite complex plane $\mathbb{C}$, we say that f and g share a finite value a IM (ignoring multiplicities) when $f-a$ and $g-a$ have the same zeros. If $f-a$ and $g-a$ have the same zeros with the same multiplicities, then we say that f and g share the value a CM (counting multiplicities).

The shared value problems related to a meromorphic function f and its derivatives have been a more widely studied subtopic of the uniqueness theory of entire and meromorphic functions in the field of complex analysis (see Chen et al. 2014; Li and Yi 2007; Liao 2015; Mues and Steinmetz 1986; Zhang and Yang 2009; Zhang 2005; Zhao 2012).

In this section, we give some growth estimates of entire functions f that share one finite value with its linear differential polynomial $L[f]$.

Theorem 5. *Let $a\neq 0$ be a finite value and $S(f)=\{f: f$ shares a CM with $L[f]$, f is an entire and of infinite order$\}$. Then, we have for each $f\in S(f)$, $\sigma_1(f)$ is an integer, or infinite with at most one exception.*

Proof. Let $f \in S(f)$. We know that f is an entire function of infinite order, since f and $L[f]$ share the value a CM. Then

$$\frac{L[f]-a}{f-a} = e^{\alpha},$$

where $\alpha(z)$ is a non-constant entire function. Setting, $g = \frac{f}{a} - 1$. Then g satisfies the following differential equation

$$L[g] = e^{\alpha} g + 1.$$

Therefore, the conclusion follows from Corollary 1 of Theorem 4. □

We say that two entire functions f and g have the same fixed points, if $f - z$ and $g - z$ have the same zeros with the same multiplicities. We have the following results.

THEOREM 6. *Let $S_1(f)$ be the set of entire functions of infinite order, in which each f and $L(f)$ have the same fixed points. Then for each $f \in S_1(f)$, $\sigma_1(f)$ is an integer, or infinite with at most one exception.*

Proof. Let $f \in S_1(f)$. We know that f is an entire function of infinite order. Since f and $L[f]$ have the same fixed points, then we have

$$\frac{L[f]-z}{f-z} = e^{\alpha},$$

where $\alpha(z)$ is a non-constant entire function. Setting, $g = f - z$. Then, g satisfies the following differential equation

$$L[g] = e^{\alpha} g - a_1 - a_0\, z.$$

Therefore, the conclusion follows from Corollary 2 of Theorem 4. □

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