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ON DEFERRED CESÀRO MEAN IN PARANORMED SPACES

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Abstract. The aim of the present study is to introduce the concepts of deferred statistical convergence, deferred statistical Cauchy sequence and deferred Cesàro summability in paranormed spaces. We investigate some properties of these concepts and some inclusion relations with examples.

1. Introduction

Zygmund [26] introduced the idea of statistical convergence in 1935. Fast [8] and Steinhaus [18] introduced statistical convergence to assign a limit to sequences which are not convergent in the usual sense independently in the same year.

We begin by recalling the notion of asymptotic (or natural) density of a set $A \subset \mathbb{N}$ such that

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : k \in A\}\right|,$$

whenever the limit exists. $|\{.\}|$ indicate the cardinality of the enclosed set. A sequence (x_k) of numbers is called statistically convergent to a number L provided that for $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \le n : |x_k - L| \ge \varepsilon\}\right| = 0.$$

In this case, it is written by $S - \lim_{k\to\infty} x_k = L$. We note that throughout the paper $\mathbb{N} := \{1, 2, ...\}$. The notion of statistical convergence is used an effective tool to resolve many problems in Ergodic theory, Fuzzy set theory, Trigonometric series and Banach spaces in the past years. Also various researchers studied applications and generalizations of this notion (see [17]- [9]).

Agnew [1] defined the deferred Cesàro mean of real (or complex) valued sequences by

$$(D_{p,q}x)_n = \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x_k,$$

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where $p = \{p(n) : n \in \mathbb{N}\}$ and $q = \{q(n) : n \in \mathbb{N}\}$ are the sequences of nonnegative integers satisfying

$$p(n) < q(n) \text{ and } \lim_{n\to\infty} q(n) = \infty. \tag{1.1}$$

Recently, Küçükaslan and Yılmaztürk [13] introduced deferred statistical convergence. For some more relevant study, we refer to ([4]- [24]).

A paranorm $g : X \to \mathbb{R}$ is defined on a linear space X provided that for all x, y, $z \in X$

(i) $g(x) = 0$ if $x = \theta$,

(ii) $g(-x) = g(x)$,

(iii) $g(x+y) \leq g(x) + g(y)$,

(iv) if (α_n) is a sequence of scalars with $\alpha_n \to \alpha_0$ $(n \to \infty)$ and x_n, $a \in X$ with $x_n \to a$ $(n \to \infty)$ in the sense that $g(x_n - a) \to 0$ $(n \to \infty)$, then $\alpha_n x_n \to \alpha_0 a$ $(n \to \infty)$, in the sense that $g(\alpha_n x_n - \alpha_0 a) \to 0$ $(n \to \infty)$.

A paranorm g for which $g(x) = 0$ implies that $x = \theta$ is said to be a total paranorm on X. (X, g) is said to be a total paranormed space. We recall that each seminorm (norm) g on X is a paranorm (total). However, the converse is not true.

Definition 1.1 A sequence (x_k) in a paranormed space (X, g) is said to be convergent (or g-convergent) to a number L in (X, g) if for every $\varepsilon > 0$ there exists a positive integer k_0 such that $g(x_k - L) < \varepsilon$ whenever $k > k_0$. In this case we write $g - \lim_{k\to\infty} x_k = L$ and L is called the g-limit of (x_k) [2].

Recently, Alotaibi and Alroqi [2] studied strongly Cesàro summability, statistical convergence, statistical Cauchy in paranormed spaces. Then Alghamdi and Mursaleen [15] introduced λ-statistical convergence in paranormed spaces.

2. Main Results

In this section, we introduce the concepts of deferred statistical convergence, deferred statistical Cauchy and deferred Cesàro summability in paranormed space. Furthermore, we establish certain interesting results related to these concepts.

The deferred density of $K \subset \mathbb{N}$ is given by

$$\delta_{p,q}(K) = \lim_{n\to\infty} \frac{1}{q(n) - p(n)} \left|\{p(n) < k \leq q(n), k \in K\}\right|$$

whenver the limit exists. We note that the deferred density coincides with the natural density whenever $q(n) = n$ and $p(n) = 0$.

Now, we begin with the following definitions.

Definition 2.1 A sequence (x_k) is said to be deferred statistically convergent (or $g(DS[p, q])$-convergent) to L in (X, g) if for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \left|\{p(n) < k \leq q(n) : g(x_k - L) \geq \varepsilon\}\right| = 0.$$

It is denoted by $g\left(DS[p,q]\right)-\lim\limits_{k\to\infty} x_k = L$. The set of all such sequences is denoted by $g\left(DS[p,q]\right)$.

(i) If we choose $q\left(n\right)=n$ and $p\left(n\right)=0$, then Definition 2.1 is reduced to statistical convergence in paranormed spaces (cf. [2]).

(ii) If we choose $q\left(n\right)=\lambda_n$ and $p\left(n\right)=0$, then Definition 2.1 is reduced to λ-statistical convergence in paranormed spaces (cf. [15]).

Definition 2.2 A sequence (x_k) is said to be deferred statistically Cauchy sequence in (X,g) (or $g\left(DS[p,q]\right)$-Cauchy) if for every $\varepsilon>0$ there exists a number $N=N\left(\varepsilon\right)$ such that

$$\lim_{n\to\infty}\frac{1}{q\left(n\right)-p\left(n\right)}\left|\left\{p\left(n\right)<k\leq q\left(n\right): g\left(x_k-x_N\right)\geq\varepsilon\right\}\right|=0.$$

Definition 2.3 A sequence (x_k) is said to be deferred strongly $D_{p,q}$-convergent to L in (X,g) if

$$\lim_{n\to\infty}\frac{1}{q\left(n\right)-p\left(n\right)}\sum_{k=p(n)+1}^{q(n)}\left|g\left(x_k-L\right)\right|=0$$

and we write it as $g\left(D[p,q]\right)-\lim\limits_{k\to\infty} x_k = L$.

Theorem 2.4 *If a sequence (x_k) is deferred statistically convergent in (X,g) then $g\left(DS[p,q]\right)-\lim$ is unique.*

Proof. Assume that $g\left(DS[p,q]\right)-\lim x=L_1$ and $g\left(DS[p,q]\right)-\lim x=L_2$. For given $\varepsilon>0$, define the sets $K_1\left(\varepsilon\right)$ and $K_2\left(\varepsilon\right)$ by

$$\begin{aligned} K_1\left(\varepsilon\right) &= \left\{p\left(n\right)<k\leq q\left(n\right): g\left(x_k-L_1\right)\geq\varepsilon/2\right\}, \\ K_2\left(\varepsilon\right) &= \left\{p\left(n\right)<k\leq q\left(n\right): g\left(x_k-L_2\right)\geq\varepsilon/2\right\}. \end{aligned}$$

Since $g\left(D[p,q]\right)-\lim x=L_1$, we have $\delta\left(K_1\left(\varepsilon\right)\right)=0$. Similarly, $g\left(DS[p,q]\right)-\lim x=L_2$ implies that $\delta\left(K_2\left(\varepsilon\right)\right)=0$. Now, let $K\left(\varepsilon\right)=K_1\left(\varepsilon\right)\cup K_2\left(\varepsilon\right)$. Then $\delta\left(K\left(\varepsilon\right)\right)=0$ and $\delta\left(K^c\left(\varepsilon\right)\right)=1$. Now, if $k\in\mathbb{N}\backslash K\left(\varepsilon\right)$, then we have $g\left(L_1-L_2\right)\leq g\left(x_k-L_1\right)+g\left(x_k-L_2\right)<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$. Since $\varepsilon>0$ is arbitrary, we get $g\left(L_1-L_2\right)=0$ and hence $L_1=L_2$. □

Theorem 2.5 *If $g-\lim x=L$. Then $g\left(DS[p,q]\right)-\lim x=L$ but the converse is not true.*

Proof. Assume that $g{-}\lim x=L$. Then there exists $N\in\mathbb{N}^+$ such that $g\left(x_n-L\right)<\varepsilon$ for all $n\geq N$ and $\varepsilon>0$. Since $A\left(\varepsilon\right)=\left\{k\in\mathbb{N}: g\left(x_k-L\right)\geq\varepsilon\right\}\subset\left\{1,2,3,...\right\}$, $\delta\left(A\left(\varepsilon\right)\right)=0$. Hence $g[DS[p,q])-\lim x=L$. □

The following example shows that the converse need not be true.

Example 2.6 Let $X = \ell(1/k) = \left\{ x = (x_k) : \sum_{k=1}^{\infty} |x_k|^{1/k} < \infty \right\}$ with the paranorm $g(x) = \left(\sum_{k=1}^{\infty} |x_k|^{1/k} \right)$. Define the sequence

$$x_k = \begin{cases} k, & if\ k = n^2,\ n \in \mathbb{N}; \\ 0, & otherwise; \end{cases}$$

and write

$$K(\varepsilon) = \{ k \le n : g(x_k) \ge \varepsilon \},\ 0 < \varepsilon < 1.$$

We see that

$$g(x_k) = \begin{cases} k^{1/k}, & if\ k = n^2, n \in \mathbb{N}; \\ 0, & otherwise; \end{cases}$$

and hence

$$\lim_{k\to\infty} g(x_k) = \begin{cases} 1, & if\ k = n^2, n \in \mathbb{N}; \\ 0, & otherwise; \end{cases}$$

Therefore $g - \lim_{k\to\infty} x_k$ does not exist. On the other hand $\delta_{p,q}(K(\varepsilon)) = 0$, that is, $g(DS[p,q]) - \lim_{k\to\infty} x_k = 0$.

Theorem 2.7 *Let* $g(DS[p,q]) - \lim x = L_1$ *and* $g(DS[p,q]) - \lim y = L_2$. *Then*
(*i*) $g(DS[p,q]) - \lim(x \pm y) = L_1 \pm L_2$.
(*ii*) $g(DS[p,q]) - \lim \alpha x = \alpha L, \alpha \in \mathbb{R}$.

Theorem 2.8 *A sequence* (x_k) *in* (X, g) *is statistically convergent to* L *if and only if there exists a set* $K = \{k_1 < k_2 < k_3 < ... < k_n < ...\} \subseteq \mathbb{N}$ *with* $\delta_{p,q}(K) = 1$ *such that* $g(x_{k_n} - L) \to 0\ \ (n \to \infty)$.

Proof. Assume that (x_k) is deferred statistically convergent to L, that is, $g(DS[p,q]) - \lim_{k\to\infty} x_k = L$. Now, write $K_r = \left\{ n \in \mathbb{N} : g(x_{k_n} - L) \ge \frac{1}{r} \right\}$, $M_r = \left\{ n \in \mathbb{N} : g(x_{k_n} - L) < \frac{1}{r} \right\}$ for $r = 1, 2, ...$ Then $\delta_{p,q}(K_r) = 0$,

$$M_1 \supset M_2 \supset ... \supset M_i \supset M_{i+1} \supset ..., \tag{2.1}$$

$$\delta_{p,q}(M_r) = 1, \quad r = 1, 2, ... \tag{2.2}$$

We have to show that, for $n \in M_r$, (x_{k_n}) is g-convergent to L. On contrary suppose that (x_{k_n}) is not g-convergent to L. Therefore, there is $\varepsilon > 0$ such that $g(x_{k_n} - L) \ge \varepsilon$ for infinitely many terms. Let $M_\varepsilon = \{ n \in \mathbb{N} : g(x_{k_n} - L) < \varepsilon \}$ and $\varepsilon > 1/r, r \in \mathbb{N}$. Then $\delta_{p,q}(M_\varepsilon) = 0$ and by (2.1), $M_r \subset M_\varepsilon$. Hence, $\delta_{p,q}(M_r) = 0$. This contradicts (2.2) and hence (x_{k_n}) is g-convergent to L. □

Now, consider the set $K = \{k_1 < k_2 < k_3 < ...\} \subset \mathbb{N}$ with $\delta_{p,q}(K) = 1$ and $g(x_{k_n} - L) \to 0\ \ (n \to \infty)$. So we can find a positive integer n_0 such that $g(x_k - L) < \varepsilon$ for $n \ge n_0$. $K_\varepsilon = \{k : g(x_k - L) \ge \varepsilon\} \subseteq \mathbb{N} - \{k_{n_0+1}, k_{n_0+2}, k_{n_0+3}, ...\}$ and therefore $\delta_{p,q}(K_\varepsilon) = 0$. This shows that (x_k) is deferred statistically convergent to L in (X, g).

Theorem 2.9 *A sequence* (x_k) *in a complete paranormed space* (X, g) *is deferred statistically Cauchy if and only if it is deferred statistically convergent.*

Proof. Assume that (x_k) is $g(DS[p,q])$-Cauchy but not $g(DS[p,q])$-convergent. Then, we have $m \in \mathbb{N}$ such that $\delta_{p,q}(G(\varepsilon)) = 0$, where $G(\varepsilon) = \{n \in \mathbb{N} : g(x_n - x_m) \geq \varepsilon\}$ and $\delta_{p,q}(D(\varepsilon)) = 0$, where $D(\varepsilon) = \{n \in \mathbb{N} : g(x_n - L) < \varepsilon/2\}$, i.e., $\delta_{p,q}\left(D^C(\varepsilon)\right) = 1$. If $g(x_n - L) < \frac{\varepsilon}{2}$, then $g(x_n - x_m) \leq 2g(x_n - L) < \varepsilon$. Moreover, $\delta_{p,q}\left(G^C(\varepsilon)\right) = 0$, i.e., $\delta(G(\varepsilon)) = 1$, which leads to a contradiction, since (x_k) is $g(DS[p,q])$-Cauchy. Hence (x_k) must be $g(DS[p,q])$-convergent.

Conversely, let us assume that $g(DS[p,q]) - \lim\limits_{k\to\infty} x_k = L$. Then, we have $\delta(K(\varepsilon)) = 0$ where

$$K(\varepsilon) = \left\{n \in \mathbb{N} : g(x_n - L) \geq \frac{\varepsilon}{2}\right\}.$$

This implies that

$$\delta(\mathbb{N}\backslash K(\varepsilon)) = \delta\left(\left\{n \in \mathbb{N} : g(x_n - L) < \frac{\varepsilon}{2}\right\}\right) = 1.$$

Let $m, n \notin K(\varepsilon)$, then $g(x_m - x_n) < \varepsilon$. Let

$$M(\varepsilon) = \{n \in \mathbb{N} : g(x_m - x_n) < \varepsilon\}$$

for a fix $m \notin K(\varepsilon)$. Then $\mathbb{N}\backslash K(\varepsilon) \subset M(\varepsilon)$. Hence

$$1 = \delta(\mathbb{N}\backslash K(\varepsilon)) \leq \delta(M(\varepsilon)) \leq 1.$$

This implies $\delta(\mathbb{N}\backslash M(\varepsilon)) = 0$, where $\mathbb{N}\backslash M(\varepsilon) = \{n \in \mathbb{N} : g(x_m - x_n) \geq \varepsilon\}$. This implies that (x_n) is deferred statistically Cauchy in (X, g). □

THEOREM 2.10 *A sequence (x_k) in (X, g) is strongly $D_{p,q}$-convergent to L, then (x_k) is deferred statistically convergent to L.*

Proof. Assume that (x_k) is strongly $D_{p,q}$-convergent to L. For an arbitrary $\varepsilon > 0$, following equality

$$\begin{aligned}
\frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} g(x_k - L) &= \frac{1}{q(n) - p(n)} \left(\sum_{\substack{k=p(n)+1 \\ g(x_k - L) \geq \varepsilon}}^{q(n)} + \sum_{\substack{k=p(n)+1 \\ g(x_k - L) < \varepsilon}}^{q(n)} \right) g(x_k - L) \\
&\geq \frac{1}{q(n) - p(n)} \sum_{\substack{k=p(n)+1 \\ g(x_k - L) \geq \varepsilon}}^{q(n)} g(x_k - L) \\
&\geq \varepsilon \frac{1}{q(n) - p(n)} \left| p(n) < k \leq q(n), g(x_k - L) \geq \varepsilon \right|
\end{aligned}$$

holds. As $n \to \infty$

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} \left| p(n) < k \leq q(n), g(x_k - L) \geq \varepsilon \right| = 0.$$

Therefore, desired result is obtained. □

THEOREM 2.11 *If $(x_k) \in (X, g)$ is deferred statistically convergent to L and $(x_k) \in \ell_\infty$, then (x_k) is strongly $D_{p,q}$-convergent to L.*

Proof. Suppose that $(x_k) \in \ell_\infty$ and (x_k) is deferred statistically convergent to L in (X,g). For arbitrary $\varepsilon > 0$, we have $\delta_{p,q}(K_\varepsilon) = 0$. Since $x \in \ell_\infty$, there exists $M > 0$ such that $g(x_k - L) \leq M$ $(k = 1, 2, 3, ...)$. We have

$$\begin{aligned} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} g(x_k - L) &= \frac{1}{q(n)-p(n)} \sum_{\substack{k=p(n)+1 \\ k \notin K_\varepsilon}}^{q(n)} g(x_k - L) \\ &\quad + \frac{1}{q(n)-p(n)} \sum_{\substack{k=p(n)+1 \\ k \in K_\varepsilon}}^{q(n)} g(x_k - L) \\ &= s_1(n) + s_2(n) \end{aligned}$$

where

$$s_1(n) = \frac{1}{q(n)-p(n)} \sum_{\substack{k=p(n)+1 \\ k \notin K_\varepsilon}}^{q(n)} g(x_k - L) \text{ and } s_2(n) = \frac{1}{q(n)-p(n)} \sum_{\substack{k=p(n)+1 \\ k \in K_\varepsilon}}^{q(n)} g(x_k - L).$$

Now if $k \notin K_\varepsilon$, then $s_1(n) < \varepsilon$. For $k \in K_\varepsilon$, since $\delta_{p,q}(K_\varepsilon) = 0$ we have

$$s_2(n) \leq (\sup g(x_k - L)) \left(\frac{|K_\varepsilon|}{q(n)-p(n)} \right) \leq M \frac{|K_\varepsilon|}{q(n)-p(n)} \to 0,$$

as $n \to \infty$. This inequality completes the proof. □

Theorem 2.12 *If (x_k) in (X,g) is statistically convergent to L and $\left(\frac{p(n)}{q(n)-p(n)} \right) \in \ell_\infty$, then it is deferred statistically convergent to L.*

Proof. Assume that (x_k) is statistically convergent to L in (X,g). Hence we have

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : g(x_k - L) \geq \varepsilon\}| = 0$$

for every $\varepsilon > 0$. From (1.1), we obtain

$$\lim_{n\to\infty} \frac{|\{k \leq q(n) : g(x_k - L) \geq \varepsilon\}|}{q(n)} = 0. \tag{2.3}$$

Moreover,we have

$$\{p(n) < k \leq q(n) : g(x_k - L) \geq \varepsilon\} \subset \{k \leq q(n) : g(x_k - L) \geq \varepsilon\}$$

and

$$|\{p(n) < k \leq q(n) : g(x_k - L) \geq \varepsilon\}| \leq |\{k \leq q(n) : g(x_k - L) \geq \varepsilon\}|.$$

Consequently

$$\begin{aligned} &\frac{1}{q(n)-p(n)} |p(n) < k \leq q(n) : g(x_k - L) \geq \varepsilon| \\ &5 \leq \left(1 + \frac{p(n)}{q(n)-p(n)} \right) . \frac{1}{q(n)} |k \leq q(n) : g(x_k - L) \geq \varepsilon|. \end{aligned}$$

Thus using (2.3), we conclude that (x_k) is deferred statistically convergent to L in (X,g). □

Corollary 2.13 *Let $q(n)$ be an arbitrary sequence with $q(n) < n$ for all $n \in \mathbb{N}$ and $\left(\frac{n}{q(n)-p(n)}\right) \in \ell_\infty$. Then, statistical convergence implies deferred statistical convergence in (X, g).*

Theorem 2.14 *Let $\acute{p} = (\acute{p}(n))$ and $\acute{q} = (\acute{q}(n))$ be sequences of positive natural numbers satisfiying*

$$p(n) \leq \acute{p}(n) < \acute{q}(n) \leq q(n) \tag{2.4}$$

and $\{k : p(n) < k \leq \acute{p}(n)\}$, $\{k : \acute{q}(n) < k \leq q(n)\}$ are finite sets for all $n \in \mathbb{N}$. Then, $g(DS[\acute{p}, \acute{q}])$-convergence implies $g(DS[p, q])$-convergence in (X, g).

Proof. Let us consider (x_k) is $g(DS[\acute{p}, \acute{q}])$-convergent to L. For an arbitrary $\varepsilon > 0$, the inequality

$$\begin{aligned}\{k : p(n) < k \leq q(n), g(x_k - L) \geq \varepsilon\} &= \{k : p(n) < k \leq \acute{p}(n), g(x_k - L) \geq \varepsilon\} \\ &\quad \cup \{k : \acute{p}(n) < k \leq \acute{q}(n), g(x_k - L) \geq \varepsilon\} \\ &\quad \cup \{k : \acute{q}(n) < k \leq q(n), g(x_k - L) \geq \varepsilon\}\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{q(n) - p(n)} |\{k : p(n) < k \leq q(n), g(x_k - L) \geq \varepsilon\}| \\ &\leq \frac{1}{\acute{q}(n) - \acute{p}(n)} |\{k : p(n) < k \leq \acute{p}(n), g(x_k - L) \geq \varepsilon\}| \\ &\quad + \frac{1}{\acute{q}(n) - \acute{p}(n)} |\{k : \acute{p}(n) < k \leq \acute{q}(n), g(x_k - L) \geq \varepsilon\}| \\ &\quad + \frac{1}{\acute{q}(n) - \acute{p}(n)} |\{k : \acute{q}(n) < k \leq q(n), g(x_k - L) \geq \varepsilon\}|\end{aligned}$$

hold. As $n \to \infty$, we have

$$\lim_{n\to\infty} \frac{1}{q(n) - p(n)} |\{k : p(n) < k \leq q(n), g(x_k - L) \geq \varepsilon\}| = 0.$$

This completes the proof. □

Theorem 2.15 *Let $\acute{p} = (\acute{p}(n))$ and $\acute{q} = (\acute{q}(n))$ be sequences of positive natural numbers satisfiying (2.4) and*

$$\lim_{n\to\infty} \frac{q(n) - p(n)}{\acute{q}(n) - \acute{p}(n)} > 0.$$

Then, $g(DS[p, q])$-statistical convergence implies $g(DS[\acute{p}, \acute{q}])$-statistical convergence in (X, g).

Proof. The inclusion

$$\{k : \acute{p}(n) < k \leq \acute{q}(n) : g(x_k - L) \geq \varepsilon\} \subset \{k : p(n) + 1 \leq k \leq q(n) : g(x_k - L) \geq \varepsilon\}$$

holds. Then we have the inequality

$$|\{k : \acute{p}(n) < k \leq \acute{q}(n) : g(x_k - L) \geq \varepsilon\}| \leq |\{k : p(n) + 1 \leq k \leq q(n) : g(x_k - L) \geq \varepsilon\}|.$$

Hence , we have

$$\frac{1}{q\'(n) - p\'(n)} \left|\left\{k : p\'(n) < k \le q\'(n) : g(x_k - L) \ge \varepsilon\right\}\right|$$
$$\le \frac{q(n) - p(n)}{q\'(n) - p\'(n)} . \frac{1}{q(n) - p(n)} \left|\left\{k : p(n) + 1 < k \le q(n) : g(x_k - L) \ge \varepsilon\right\}\right| .$$

As $n \to \infty$, the desired result is obtained. □

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