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CYCLES OF CHARACTERISTIC MATRICES OF CELLULAR AUTOMATA WITH PERIODIC BOUNDARY CONDITION

Jae-Gyeom Kim

Abstract. In this note, we will investigate some relations among powers of characteristic matrices of uniform cellular automata configured with rule 102 and periodic boundary condition.

1. Introduction

Cellular automata have been demonstrated by many researchers to be a good computational model for physical systems simulation since the concept of cellular automata first introduced by John Von Neumann in the 1950s. Cellular automata configured with rule 102 and null boundary condition has been studied [1, 4-7, 9, 10]. And researches about cellular automata with periodic boundary condition mainly focused on reversibility [2, 3, 8].

In this note, we will investigate some relations among powers of characteristic matrices of uniform cellular automata configured with rule 102 and periodic boundary condition.

2. Preliminaries

A cellular automaton (CA) is an array of sites (cells) where each site is in any one of the permissible states. At each discrete time step (clock cycle) the evolution of a site value depends on some rule (the combinational logic) which is a function of the present state of its k

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neighbors for a k-neighborhood CA. For 2-state 3-neighborhood CA, the evolution of the (i)th cell can be represented as a function of the present states of $(i-1)$th, (i)th, and $(i+1)$th cells as: $x_i(t+1) = f(x_{i-1}(t), x_i(t), x_{i+1}(t))$, where f represents the combinational logic. For such 2-state site value calculation of CA, the modulo-2 logic is always applied.

For 2-state 3-neighborhood CA there are 2^3 distinct neighborhood configurations and 2^{2^3} distinct mappings from all these neighborhood configurations to the next state, each mapping representing a CA rule. The CA, characterized by a rule known as rule 102, specifies an evolution from neighborhood configuration to the next state as:

111	110	101	100	011	010	001	000	
0	1	1	0	0	1	1	0	decimal 102.

The corresponding combinational logic of rule 102 is

$$x_i(t+1) = x_i(t) \oplus x_{i+1}(t),$$

that is, the next state of (i)th cell depends on the present states of self and its right neighbors.

If in a CA the same rule applies to all cells, then the CA is called a uniform CA; otherwise the CA is called a hybrid CA. There can be various boundary conditions; namely, null (where extreme cells are connected to logic '0', periodic (extreme cells are adjacent), etc. In the sequel, we will deal with periodic boundary condition.

The characteristic matrix T of a CA is the transition matrix of the CA. The next state $f_{t+1}(x)$ of a CA is given by $f_{t+1}(x) = T \times f_t(x)$, where $f_t(x)$ is the current state, t is the time step. The length of a CA is the number of cells of the CA. Some powers of the characteristic matrix T, for examples, of the CA of length 6 configured with rule 102 and periodic boundary condition are as follows;

$$[T]=\begin{pmatrix}1&1&0&0&0&0\\0&1&1&0&0&0\\0&0&1&1&0&0\\0&0&0&1&1&0\\0&0&0&0&1&1\\1&0&0&0&0&1\end{pmatrix},\quad [T^2]=\begin{pmatrix}1&0&1&0&0&0\\0&1&0&1&0&0\\0&0&1&0&1&0\\0&0&0&1&0&1\\1&0&0&0&1&0\\0&1&0&0&0&1\end{pmatrix},$$

$$[T^3]=\begin{pmatrix}1&1&1&1&0&0\\0&1&1&1&1&0\\0&0&1&1&1&1\\1&0&0&1&1&1\\1&1&0&0&1&1\\1&1&1&0&0&1\end{pmatrix},\quad [T^4]=\begin{pmatrix}1&0&0&0&1&0\\0&1&0&0&0&1\\1&0&1&0&0&0\\0&1&0&1&0&0\\0&0&1&0&1&0\\0&0&0&0&0&1\end{pmatrix}.$$

Note that (i)th row of $[T^m]$ is an $(i-1)$-step shift of the first row of $[T^m]$ from left to right with modulo l, where a 1-step shift is one site (cell) shift. So the first row of $[T^m]$ completely characterizes $[T^m]$. $[T^0]$ denotes the identity matrix.

For the characteristic matrix T of the CA of length l configured with rule 102 and periodic boundary condition, $[T_p]$ is the matrix of column size l and sufficiently large row size defined by $[T_p]_{i,j}=[T^{i-1}]_{1,j}$ where $i=1,\ 2,\ \cdots$ and $j=1,\ \cdots,\ l$. And $[T_p]$ of column size l characterizes completely the powers of the characteristic matrix T of the CA of length l configured with rule 102 and periodic boundary condition. The matrix $[T_p]$ of column size 9, for example, is as follows;

$$[T_p]=\begin{pmatrix}1&0&0&0&0&0&0&0&0\\1&1&0&0&0&0&0&0&0\\1&0&1&0&0&0&0&0&0\\1&1&1&1&0&0&0&0&0\\1&0&0&0&1&0&0&0&0\\1&1&0&0&1&1&0&0&0\\1&0&1&0&1&0&1&0&0\\1&1&1&1&1&1&1&1&0\\1&0&0&0&0&0&0&0&1\\&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\\&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\\&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\cdots&\end{pmatrix}.$$

Lemma 2.1. [5] *Let T be the characteristic matrix of the CA of length l configured with rule 102 and periodic boundary condition. And let $2 \le 2^{t+1} \le l$. Then three 2^t entries, which are the first 2^t entries of the first row of $[T_p]$ and the first 2^t entries and the second 2^t entries of (2^t+1)th row of $[T_p]$, have the same pattern of which only the first value is* 1.

Lemma 2.2. [5] *Let T be the characteristic matrix of the CA of length l configured with rule 102 and periodic boundary condition. And let $1 \le n \le l-1$ and $n = a_0 2^0 + a_1 2^1 + a_2 2^2 + \cdots$ with $a_i = 0$ or 1 for $i = 1, 2, \cdots$. Then we have entry values of $[T_p]$ as in Table* 1.

Table 1. Values of $[T_p]_{i,j}$

r	a_r	$[T_p]_{i,j}$ ($n+1 \le l$ and $2^m + k \le l$)	k
0	$a_0 = 0$	$[T_p]_{n+1,2^0+k} = [T^n]_{1,2^0+k} = 0$	1
	$a_0 = 1$	$[T_p]_{n+1,2^0+k} = [T^n]_{1,2^0+k} = [T^n]_{1,k} = [T_p]_{n+1,k}$	
1	$a_1 = 0$	$[T_p]_{n+1,2^1+k} = [T^n]_{1,2^1+k} = 0$	1,2
	$a_1 = 1$	$[T_p]_{n+1,2^1+k} = [Tn]_{1,2^1+k} = [T^n]_{1,k} = [T_p]_{n+1,k}$	
2	$a_2 = 0$	$[T_p]_{n+1,2^2+k} = [T^n]_{1,2^2+k} = 0$	1,2,3,4
	$a_2 = 1$	$[T_p]_{n+1,2^2+k} = [T^n]_{1,2^2+k} = [T^n]_{1,k} = [T_p]_{n+1,k}$	
$\cdots$			
m	$a_m = 0$	$[T_p]_{n+1,2^m+k} = [T^n]_{1,2^m+k} = 0$	$1, \cdots, 2^m$
	$a_m = 1$	$[T_p]_{n+1,2^m+k} = [T^n]_{1,2^m+k} = [T^n]_{1,k} = [T_p]_{n+1,k}$	
$\cdots$			

3. Cycles of characteristic matrices

We will investigate some relations among powers of the characteristic matrix T of the CA of length l configured with rule 102 and periodic boundary condition.

The following lemma is quite similar to Lemma 2.1.

Lemma 3.1. *Let T be the characteristic matrix of the CA of length l configured with rule 102 and periodic boundary condition. If $2^t < l$ for some $t \ge 0$, then we have*

$$[T_p]_{2^t+1,j} = [T^{2^t}]_{1,j} = \begin{cases} 1, & \text{if } j = 1 \text{ or } 2^t + 1, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. If $2^{t+1} \le l$, then the lemma is equal to Lemma 2.1. Thus let $2^t < l < 2^{t+1}$. If we can extend the column size l of T to 2^{t+1} without changing the value of (2^t+1)th row of $[T_p]$, we have the conclusion by comparing with Lemma 2.1. In fact, we can do that by assigning value 0 to every extended entry because $[T_p]_{i,j} = 0$ if $i \le l$ and $i < j$. Hence we have the lemma. □

Proposition 3.2. *Let T be the characteristic matrix of the CA of length l configured with rule* 102 *and periodic boundary condition. And let $l = 2^t + 2^s$ with $0 \le s < t$. Then (2^t+1)th row of $[T_p]$ is a (2^s)-step shift of (2^s+1)th row of $[T_p]$ from right to left, in other words, $[T^{2^t}]$ is a (2^s)-step shift of $[T^{2^s}]$ from right to left.*

Proof. By Lemma 3.1, we have

$$[T_p]_{2^t+1,j} = \begin{cases} 1, & \text{if } j = 1 \text{ or } 2^t+1, \\ 0, & \text{otherwise.} \end{cases}$$

If $s = 0$, then $l = 2^t + 1$, and so both of the first and the last entries of (2^t+1)th row of $[T_p]$ are 1 and the remaining entries are all 0. Thus the (2^t+1)th row is a (2^0)-step shift of (2^0+1)th row of $[T_p]$ from right to left. And if $s \ge 1$, then both of the first and (2^t+1)th entries of the (2^t+1)th row are 1 and remaining entries including the last $2^s - 1$ entries are all 0, and so the (2^t+1)th row is a 2^s-step shift of (2^s+1)th row of $[T_p]$ from right to left. □

Now we can give a relation among some powers of the characteristic matrix T of the CA of length l configured with rule 102 and periodic boundary condition.

Theorem 3.3. *Let T be the characteristic matrix of the CA of length l configured with rule* 102 *and periodic boundary condition. And let $l = 2^t + 2^s$ with $0 \le s < t$. Then all of the rows of $[T_p]$ except the first 2^s rows form a cycle under evolution with period $2^{2t-s} - 2^s$. In other words, if $r = 2^{2t-s} - 2^s$, then $[T^{m+r}] = [T^m]$ for all $m \ge 2^s$.*

Proof. An l-step shift of any row of $[T_p]$ is the row itself since the length of the CA is l. So $l/(2^s)$ times 2^s-step shift of any row of $[T_p]$ is the row itself. But, by Proposition 3.2, (2^t+1)th row of $[T_p]$ is a 2^s-step shift of (2^s+1)th row of $[T_p]$ from right to left. And the (2^t+1)th row is the $(2^t - 2^s)$-step evolution of the (2^s+1)th row. Thus $l/(2^s)$ times $(2^t - 2^s)$-step evolution of the (2^s+1)th row becomes the row of

$l/(2^s)$ times 2^s-step shift of the (2^s+1)th row, which is consequently the (2^s+1)th row itself. Therefore $(n+1)$th row of $[T_p]$ coincides with the (2^s+1)th row when $(n-2^s) = k\cdot(2^t-2^s)\cdot l/(2^s)$ for some positive integer k. To get the least such n, let $k=1$, then we have

$$\begin{aligned} n &= (2^t-2^s)l/(2^s)+2^s \\ &= (2^t-2^s)(2^t+2^s)/(2^s)+2^s \\ &= (2^{2t}-2^{2s})/(2^s)+2^s \\ &= 2^{2t-s}. \end{aligned}$$

So all of the rows of $[T_p]$ except the first 2^s rows form a cycle under evolution with period $2^{2t-s}-2$. Hence we have the conclusion. $\square$

LEMMA 3.4. *Let T be the characteristic matrix of the CA of length l configured with rule 102 and periodic boundary condition. If $2^t \le l$ for some $t \ge 1$, then we have*

$$[T_p]_{2^t,j} = [T^{2^t-1}]_{1,j} = \begin{cases} 1, & \text{if } j \le 2^t, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We will use induction on $t \ge 1$. If $t=1$ then it is clear. Let $t>1$, then

$$[T_p]_{2^{t-1},j} = [T^{2^{t-1}-1}]_{1,j} = \begin{cases} 1, & \text{if } j \le 2^{t-1}, \\ 0, & \text{otherwise,} \end{cases}$$

by induction hypothesis, and so we have

$$[T_p]_{2^{t-1}+1,j} = [T^{2^{t-1}}]_{1,j} = \begin{cases} 1, & j = 1 \text{ or } 2^{t-1}+1, \\ 0, & \text{otherwise.} \end{cases}$$

In fact, the first 2^{t-1} entries of $(2^{t-1}+1)$th row of $[T_p]$ has the same pattern with the first 2^{t-1} entries of the first row of $[T_p]$ of which the $(2^{t-1}-1)$-step evolution makes 2^{t-1} entries which are all 1 by induction hypothesis again. Thus the $(2^{t-1}-1)$-step evolution of the first 2^{t-1} entries of the $(2^{t-1}+1)$th row makes 2^{t-1} entries which are all 1. But the first 2^{t-1} entries and the second 2^{t-1} entries of the $(2^{t-1}+1)$th row have the same pattern by the equation above. And the $(2^{t-1}-1)$-step evolution of the second 2^{t-1} entries is independent of other entries, because the last $2^{t-1}-1$ entries of the second 2^{t-1} entries are all 0 and because the $2^{t-1}-1$ entries preceded before the second 2^{t-1} entries are all 0. Therefore the $(2^{t-1}-1)$-step evolution of the second 2^{t-1} entries also makes 2^{t-1} entries which are all 1. This says that

$$[T_p]_{2^t,j} = [T^{2^t-1}]_{1,j} = \begin{cases} 1, & \text{if } j \le 2^t, \\ 0, & \text{otherwise.} \end{cases}$$

This completes the induction. □

By Lemma 3.4, we have the following theorem obviously.

THEOREM 3.5. *Let T be the characteristic matrix of the CA of length l configured with rule* 102 *and periodic boundary condition. Let $l = 2^t$ for some $t \ge 1$. Then $[T_p]_{2^t+1,j} = [T^{2^t}]_{1,j} = 0$ for all j and so (r)th row of $[T_p]$ is vanished if $r > l$. In other words, if $r > l$ then $[T^r] = 0$.*

Finally, we will give another relation among some powers of the characteristic matrix T of the CA of length l configured with rule 102 and periodic boundary condition. For the purpose, we will start with two lemmas. For explicit patterns of some rows in the proofs, refer to Table 2.

LEMMA 3.6. *Let T be the characteristic matrix of the CA of length l configured with rule* 102 *and periodic boundary condition. And let $3 \cdot 2^s \le l$ with $s \ge 1$. Assume that there are consecutive r iterations of 2^{s+1} entries of which the first half and second half are all* 1 *and all* 0*, respectively, in (n)th row of $[T_p]$ and the 2^s entries preceded before the iterations are all* 0*, where $r \ge 1$. Then the (2^s)-step evolution of the iterations makes r iterations of 2^{s+1} entries, which are all* 1*, in $(n+s)$th row of $[T_p]$.*

Proof. By the assumption, the (2^s)-step evolution of each of the iterations is independent of other entries. And each of the iterations and the first 2^{s+1} entries of (2^s)th row of $[T_p]$ have the same pattern by Lemma 3.4. So the (2^s)-step evolution of each of the iterations and the (2^s)-step evolution of the first 2^{s+1} entries of the (2^s)th row have the same pattern. But the (2^s)-step evolution of the (2^s)th row is (2^{s+1})th row of $[T_p]$ and the first 2^{s+1} entries of the (2^{s+1})th row are all 1 by Lemma 3.4 again. Thus the (2^s)-step evolution of each of the iterations makes 2^{s+1} entries which are all 1. Hence we have the conclusion. □

LEMMA 3.7. *Let T be the characteristic matrix of the CA of length l configured with rule* 102 *and periodic boundary condition. And let $2^{s+2} \le l$ with $s \ge 1$. Assume that there is consecutive $2^{s+1} + 2^s$ entries* *of which the first* 2^{s+1} *entries are all* 1 *and the remaining* 2^s *entries are all* 0 *in* (n)*th row of* $[T_p]$ *and the* 2^s *entries preceded before the iterations are all* 0*. Then the* (2^s)*-step evolution of the consecutive* $2^{s+1}+2^s$ *entries makes* $2^{s+1}+2^s$ *entries, of which the first and the last* 2^s *entries are all* 1 *and the middle* 2^s *entries are all* 0*, in* $(n+s)$*th row of* $[T_p]$*.*

Proof. By the assumption, the (2^s)-step evolution of the consecutive $2^{s+1}+2^s$ entries is independent of other entries. And the consecutive $2^{s+1}+2^s$ entries and the first $2^{s+1}+2^s$ entries of the (2^{s+1})th row have the same pattern. So the (2^s)-step evolution of the consecutive $2^{s+1}+2^s$ entries and the (2^s)-step evolution of the first $2^{s+1}+2^s$ entries of the (2^{s+1})th row have the same pattern. But the (2^s)-step evolution of the (2^{s+1})th row is $(2^{s+1}+2^s)$th row of $[T_p]$ and the first $2^{s+1}+2^s$ entries of the $(2^{s+1}+2^s)$th row are as in Table 2 by Lemma 2.2. Hence we have the conclusion. □

Table 2. Patterns of some rows

row	values *
(l)th row	$1\cdots1\ 0\cdots0\ 1\cdots1\ 0\cdots0\ \cdots\ 1\cdots1\ 0\cdots0\ 1\cdots1$
(2^s)th row	$1\cdots1\ 0\cdots0\ 0\cdots0\ 0\cdots0\ \cdots\ 0\cdots0\ 0\cdots0\ 0\cdots0$
(2^{s+1})th row	$1\cdots1\ 1\cdots1\ 0\cdots0\ 0\cdots0\ \cdots\ 0\cdots0\ 0\cdots0\ 0\cdots0$
$(2^{s+1}+2^s)$th row	$1\cdots1\ 0\cdots0\ 1\cdots1\ 0\cdots0\ \cdots\ 0\cdots0\ 0\cdots0\ 0\cdots0$
(2^s)-step shift of (2^{s+1})th row from right to left	$1\cdots1\ 0\cdots0\ 0\cdots0\ 0\cdots0\ \cdots\ 0\cdots0\ 0\cdots0\ 1\cdots1$
(2^s)-step shift of $(2^{s+1}+2^s)$th row from right to left	$0\cdots0\ 1\cdots1\ 0\cdots0\ 0\cdots0\ \cdots\ 0\cdots0\ 0\cdots0\ 1\cdots1$
$(l+2^s)$th row	$0\cdots0\ 1\cdots1\ 1\cdots1\ 1\cdots1\ \cdots\ 1\cdots1\ 1\cdots1\ 1\cdots1$
$(l+2^s+1)$th row	$1\cdots0\ 1\cdots0\ 0\cdots0\ 0\cdots0\ \cdots\ 0\cdots0\ 0\cdots0\ 0\cdots0$

$*$: $1\cdots1$ means 2^s consecutive 1, $1\cdots0$ means one 1 and 2^s-1 consecutive 0, and $0\cdots0$ means 2^s consecutive 0.

Theorem 3.8. *Let* T *be the characteristic matrix of the CA of length* l *configured with rule* 102 *and periodic boundary condition. And let* $l=2^t-2^s$ *with* $t\geq 2$ *and* $0\leq s\leq t-2$*. Then all of the rows of* $[T_p]$ *except the first* 2^s *rows form a cycle under evolution with period* l*. In other words,* $[T^{m+l}]=[T^m]$ *for all* $m\geq 2^s$*.*

Proof. We have

$$\begin{aligned} l-1 &= (2^t-2^s)-1 \\ &= (2^t-1)-2^s \\ &= (1\cdot 2^0+1\cdot 2^1+\cdots+1\cdot 2^{t-1})-2^s \\ &= 1\cdot 2^0+\cdots+1\cdot 2^{s-1}+0\cdot 2^s+1\cdot 2^{s+1}+\cdots+1\cdot 2^{t-1} \end{aligned}$$

and

$$(l-2^s)/2^{s+1} = (2^t-2^s-2^s)/2^{s+1} = 2^{t-s-1}-1.$$

Thus, by Lemma 2.2, the entries of (l)th row of $[T_p]$ consist of d iterations of 2^{s+1} entries of which the first half and second half are all 1 and all 0, respectively, where $d = 2^{t-s-1}-1$ and the last 2^s entries of which values are all 1 because the number of the entries l is a odd multiple of 2^s. Then, by Lemma 3.6, 2^t-2^{s+2} entries from $(2^{s+1}+1)$th entry to $(l-2^s)$th entry of $(l+2^s)$th row of $[T_p]$ are all 1, because the 2^t-2^{s+2} entries are composed of $d-1$ iterations of 2^{s+1} entries which satisfies the assumption of Lemma 3.6. For the remaining $2^{s+1}+2^s$ entries of the (l)th row, since the boundary condition of the CA is periodic, the $2^{s+1}+2^s$entries can be considered as a consecutive entries of which the leading part is the last 2^s entries of the (l)th row. Thus the remaining $2^{s+1}+2^s$entries satisfies the assumption of Lemma 3.7 because 2^s entries preceded before the last 2^s entries are all 0, in fact, the remaining $2^{s+1}+2^s$ entries coincides with the corresponding entries of (2^s)-step shift of $(2^{s+1}+1)$th row of $[T_p]$ from right to left. Therefore, by Lemma 3.7, the last 2^s entries and the second 2^s entries of the $(l+2^s)$th row are all 1 and the first 2^s entries of the $(l+2^s)$th row are all 0. Consequently, we have that the first 2^s entries of the $(l+2^s)$th row are all 0 and the remaining entries of the $(l+2^s)$th row are all 1. Thus we easily have that both of the first and (2^s+1)th entries of $(l+2^s+1)$th row of $[T_p]$ are 1 and the remaining $l-2$ entries of the $(l+2^s+1)$th row are all 0. Therefore the $(l+2^s+1)$-th row coincides with (2^s+1)th row of $[T_p]$. And we can easily see that l is the least positive integer such that $(l+2^s+1)$th row of $[T_p]$ coincides with (2^s+1)th row of $[T_p]$ by Lemma 2.1 and 2.2. This says that all of the rows of $[T_p]$ except the first 2^s rows form a cycle under evolution with period l. Hence we have the conclusion. □

In Theorem 3.8, the case of $l = 2^t-2^s$ and $s = t-1$ was excluded. In fact, if $l = 2^t-2^s$ and $s = t-1$, then $l = 2^s$, and so the case is included in Theorem 3.5. And, in the proof of Theorem 3.8, the (2^s)-step evolution of the first 2^{s+1} entries of the (l)th row is not independent of the last 2^s entries of the (l)th row consists of 1's, that is why the first 2^{s+1} entries of the (l)th row is not included in the process in the first part of the proof.

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Department of Mathematics
Kyungsung University
Busan 608-736, Korea
E-mail: `jgkim@ks.ac.kr`