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THE INCLUSION THEOREMS FOR GENERALIZED VARIABLE EXPONENT GRAND LEBESGUE SPACES

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Abstract. In this paper, we discuss and investigate the existence of the inclusion $L^{p(\cdot),\theta}(\mu) \subseteq L^{q(\cdot),\theta}(\nu)$, where μ and ν are two finite measures on (X, Σ). Moreover, we show that the generalized variable exponent grand Lebesgue space $L^{p(\cdot),\theta}(\Omega)$ has a potential-type approximate identity, where Ω is a bounded open subset of $\mathbb{R}^d$.

1. Introduction

Let (X, Σ, μ) and (X, Σ, ν) be two finite measure spaces. It is known that $l^p(X) \subseteq l^q(X)$ for $0 < p \le q \le \infty$. Subramanian [25] characterized all positive measures μ on (X, Σ) for which $L^p(\mu) \subseteq L^q(\mu)$ whenever $0 < p \le q \le \infty$. Also, Romero [23] investigated and developed several results of [25]. Moreover, Miamee [21] obtained the more general result as $L^p(\mu) \subseteq L^q(\nu)$ with respect to μ and ν. Aydin and Gurkanli [3] proved some inclusion results for which $L^{p(\cdot)}(\mu) \subseteq L^{q(\cdot)}(\nu)$. Moreover, these results was generalized by Gurkanli [14] and Kulak [20] to the classical and variable exponent Lorentz spaces.

In 1992, Iwaniec and Sbordone [17] introduced grand Lebesgue spaces $L^{p)}(\Omega)$, $1 < p < \infty$, on bounded sets $\Omega \subset \mathbb{R}^d$. Also, Greco et al. [16] obtained a generalized version $L^{p),\theta}(\Omega)$. Recently, these spaces have intensively studied for various applications, see [4], [12], [13], [18], [22]. The variable exponent Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^d)$ was considered by Kováčik and Rákosník [19]. They presented some basic properties of $L^{p(\cdot)}(\mathbb{R}^d)$ including reflexivity, Holder inequalities etc. These spaces have many applications such as elastic mechanics, electrorheological fluids, image restoration and nonlinear degenerated partial differential equations. For more information, we refer to [7], [10] and [11]. Gurkanli [15] studied the inclusion $L^{p),\theta}(\mu) \subseteq L^{q),\theta}(\nu)$ under some conditions for two different measures μ and ν on (X, Σ), and proved that $L^{p),\theta}(\mu)$ has no an approximate identities. The generalized variable exponent grand Lebesgue space $L^{p(\cdot),\theta}(\Omega)$ was introduced and studied by Kokilashvili and Meskhi [18]. The authors established the boundedness of maximal and Calderon operators in these spaces. It is note that, the space $L^{p(\cdot),\theta}(\Omega)$ is not reflexive, separable, rearrangement invariant and translation invariant.

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In this paper, we investigate the inclusion $L^{p(.),\theta}(\mu) \subseteq L^{q(.),\theta}(\nu)$ for two different finite measures μ and ν on (X,Σ). Also, we consider the problem of the convergence of approximate identities in the generalized variable exponent grand Lebesgue space $L^{p(.),\theta}(\mu)$. Moreover, we will show the existence of a potential-type approximate identity for the space $L^{p(.),\theta}(\mu)$. These problems were considered several authors such as Cruz-Uribe and Fiorenza [6], Diening [9], Gurkanli [15]. Finally, we obtain more general results than [6] and [15].

2. Notations and Preliminaries

In this section, we give some essential definitions, theorems and remarks in generalized variable exponent grand Lebesgue space $L^{p(.),\theta}(\mu)$.

Definition 2.1. (see [1]) Let $(X, \|.\|_X)$ and $(Y, \|.\|_Y)$ be two normed spaces. We say that the space X is continuously embedded in Y, briefly $X \hookrightarrow Y$, if $X \subset Y$ and there exists $c > 0$ such that $\|f\|_Y \leq c\|f\|_X$ for every $f \in X$.

Definition 2.2. Assume that (X,Σ,μ) is a finite measure space. Also, let $p(.): X \longrightarrow [1,\infty)$ be a measurable function (variable exponent) such that

$$1 < p^- = \underset{x\in X}{essinf}\ p(x) \leq p^+ = \underset{x\in X}{esssup}\ p(x) < \infty.$$

The variable exponent Lebesgue space $L^{p(.)}(\mu)$ is defined as the set of all measurable functions f on X such that $\varrho_{p(.)}(\lambda f) < \infty$ for some $\lambda > 0$ equipped with the Luxemburg norm

$$\|f\|_{p(.)} = \inf\left\{\lambda > 0 : \varrho_{p(.)}\left(\frac{f}{\lambda}\right) \leq 1\right\},$$

where $\varrho_{p(.)}(f) = \int\limits_X |f(x)|^{p(x)}\,d\mu(x)$. It is known that the space $L^{p(.)}(\mu)$ is a Banach space in sense to the norm $\|.\|_{p(.)}$. Moreover, the norm $\|.\|_{p(.)}$ coincides with the usual Lebesgue norm $\|.\|_p$ whenever $p(.) = p$ is a constant function. Also, it is known that $f \in L^{p(.)}(\mu)$ if and only if $\varrho_{p(.)}(f) < \infty$, see [7, 10, 11].

Remark 2.3. (see [11]) If $f \in L^{p(.)}(\mu)$, then we have
(i) $\|f\|_{p(.)}^{p^-} \leq \rho_{p(.)}(f) \leq \|f\|_{p(.)}^{p^+}$ for $\|f\|_{p(.)} \geq 1$.
(ii) $\|f\|_{p(.)}^{p^+} \leq \rho_{p(.)}(f) \leq \|f\|_{p(.)}^{p^-}$ for $\|f\|_{p(.)} \leq 1$.

Definition 2.4. Let $\theta > 0$. The generalized variable exponent grand Lebesgue space $L^{p(.),\theta}(\mu)$ is the class of all measurable functions such that

$$\|f\|_{p(.),\theta,\mu} = \sup_{0<\varepsilon<p^- -1} \varepsilon^{\frac{\theta}{p^- -\varepsilon}} \|f\|_{p(.)-\varepsilon,\mu} < \infty.$$

It is note that these spaces coincide with the grand Lebesgue spaces $L^{p),\theta}(\mu)$ whenever $p(.) = p$ is a constant function. Moreover, it is easy to see that the following continuous embeddings hold;

$$L^{p(.)} \hookrightarrow L^{p(.),\theta} \hookrightarrow L^{p(.)-\varepsilon} \hookrightarrow L^1,\ 0 < \varepsilon < p^- - 1 \tag{1}$$

due to $\mu(X) < \infty$, see [8, 18, 22].

The following proposition is called Nesting Property, see [8, 18].

PROPOSITION 2.5. *Assume that $\theta_1 < \theta_2$. Then we have*

$$L^{p(.)} \hookrightarrow L^{p(.),\theta_1} \hookrightarrow L^{p(.),\theta_2} \hookrightarrow L^{p(.)-\varepsilon} \hookrightarrow L^1,\ 0 < \varepsilon < p^- - 1$$

due to $\mu(X) < \infty$.

REMARK 2.6. There are several differences between $L^{p(.)}(\mu)$ and $L^{p(.),\theta}(\mu)$. For instance, the set of bounded functions is not dense in $L^{p(.),\theta}(\mu)$, and the closure of L^∞ in the norm of $L^{p(.),\theta}(\mu)$ can be characterized by the functions f such that

$$\lim_{\varepsilon \longrightarrow 0} \sup \varepsilon^{\frac{\theta}{p^- - \varepsilon}} \|f\|_{p(.)-\varepsilon,\mu} = 0,$$

see [2]. Moreover, the space $L^{p(.),\theta}(\mu)$ is not reflexive, separable and rearrangement invariant, see [8, 18].

Throughout this paper assume that $p^+, q^+ < \infty$.

3. Inclusions of The Space $L^{p(.),\theta}(\mu)$

Throughout this section, we assume that (X, Σ, μ) is a finite measure space. We say that μ is absolutely continuous with respect to ν (denoted by $\mu \ll \nu$) if $\mu(E) = 0$ for every $E \in \Sigma$ such that $\nu(E) = 0$. If two measures μ and ν are absolutely continuous with respect to each other, that is $\mu \ll \nu$ and $\nu \ll \mu$, then we denote it by the symbol $\mu \approx \nu$.

The notation $L^{p(.),\theta}(\mu) \subseteq L^{q(.),\theta}(\nu)$ means that every equivalence class of functions (i.e. the class of all μ-measurable functions on X equal to each other μ-almost everywhere) of $L^{p(.),\theta}(\mu)$ belongs to $L^{q(.),\theta}(\nu)$ as a equivalence class. There is, however, another possible interpretation for $L^{p(.),\theta}(\mu) \subseteq L^{q(.),\theta}(\nu)$, namely any individual function f with $\|f\|_{p(.),\theta,\mu} < \infty$ has the property $\|f\|_{q(.),\theta,\nu} < \infty$.

LEMMA 3.1. *Let (X, Σ, μ) and (X, Σ, ν) be two finite measure spaces. Then we have $L^{p(.),\theta}(\mu) \subseteq L^{q(.),\theta}(\nu)$ in the sense of equivalence classes if and only if $\mu \approx \nu$ and $L^{p(.),\theta}(\mu) \subseteq L^{q(.),\theta}(\nu)$ in the sense of individual functions.*

Proof. Suppose that $L^{p(.),\theta}(\mu) \subseteq L^{q(.),\theta}(\nu)$ holds in the sense of equivalence classes. Let $f \in L^{p(.),\theta}(\mu)$ be any individual function. This implies that $\|f\|_{p(.),\theta,\mu} < \infty$ and $f \in L^{p(.),\theta}(\mu)$ in the sense of equivalence classes. Hence, we have $f \in L^{q(.),\theta}(\nu)$ in the sense of equivalent classes by the assumption. This implies $\|f\|_{q(.),\theta,\nu} < \infty$ and $f \in L^{q(.),\theta}(\nu)$ in the sense of individual functions. Therefore, we get

$$L^{p(.),\theta}(\mu) \subseteq L^{q(.),\theta}(\nu)$$

in the sense of individual functions. Now, let $E \in \Sigma$ such that $\mu(E) = 0$. If χ_E is the characteristic function of E, then we have $\chi_E = 0$ μ-almost everywhere. Hence we have

$$\varrho_{p(.)-\varepsilon,\mu}(\chi_E) = \int_X |\chi_E(x)|^{p(x)-\varepsilon} d\mu = \mu(E) = 0.$$

Since $p^+ < \infty$, we get $\|\chi_E\|_{p(.)-\varepsilon,\mu} = 0$ and $\chi_E \in L^{p(.)-\epsilon}(\mu)$ for all $\varepsilon \in (0, p^- - 1)$. Therefore χ_E is in the equivalence class $0 \in L^{p(.)-\varepsilon}(\mu)$ for any $\varepsilon \in (0, p^- - 1)$. By definition of $\|.\|_{p(.),\theta,\mu}$, we obtain

$$\|\chi_E\|_{p(.),\theta,\mu} = \sup_{0<\varepsilon<p^- -1} \varepsilon^{\frac{\theta}{p^- -\varepsilon}} \|\chi_E\|_{p(.)-\varepsilon,\mu} = 0$$

and $0 \in L^{p(.),\theta}(\mu)$ in the sense of equivalence classes. Since the equivalence class of 0 (with respect to μ) is also an element of $L^{q(.),\theta}(\nu)$, then χ_E is in the equivalent classes of $0 \in L^{q(.),\theta}(\nu)$ with respect to ν. That means $\|\chi_E\|_{q(.),\theta,\nu} = 0$. Moreover, by (1), we have $L^{q(.),\theta} \hookrightarrow L^{q(.)-\varepsilon}$ for all $\varepsilon \in (0, q^- - 1)$. This yields $\|\chi_E\|_{q(.)-\varepsilon,\nu} = 0$ and then

$$\nu(E) = \varrho_{q(.)-\varepsilon,\nu}(\chi_E) = \int_X |\chi_E|^{q(x)-\varepsilon} d\nu = 0.$$

This yields $\nu \ll \mu$. In similar way, one can prove that $\mu \ll \nu$. The proof of sufficiency is easy to see. □

THEOREM 3.2. *$L^{p(.),\theta}(\mu) \subseteq L^{q(.),\theta}(\nu)$ in the sense of equivalence classes if and only if $\mu \approx \nu$ and there exists a $C > 0$ such that*

$$\|f\|_{q(.),\theta,\nu} \leq C\|f\|_{p(.),\theta,\mu} \tag{2}$$

for all $f \in L^{p(.),\theta}(\mu)$.

Proof. Let $L^{p(.),\theta}(\mu) \subseteq L^{q(.),\theta}(\nu)$ in the sense of equivalence classes. Now, we denote the sum norm on $L^{p(.),\theta}(\mu)$ by

$$\|.\| = \|.\|_{p(.),\theta,\mu} + \|.\|_{q(.),\theta,\nu}.$$

The space $L^{p(.),\theta}(\mu)$ is a Banach space with respect to $\|.\|$. To prove this, we assume that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^{p(.),\theta}(\mu)$. Then for all $\eta > 0$ there exists $N(\eta) > 0$ whenever $n, m > N(\eta)$ such that

$$\|f_n - f_m\|_{p(.),\theta,\mu} = \sup_{0<\varepsilon<p^- -1} \varepsilon^{\frac{\theta}{p^- -\varepsilon}} \|f_n - f_m\|_{p(.)-\varepsilon,\mu} < \eta$$

and

$$\|f_n - f_m\|_{q(.),\theta,\nu} = \sup_{0<\varepsilon<q^- -1} \varepsilon^{\frac{\theta}{p^- -\varepsilon}} \|f_n - f_m\|_{q(.)-\varepsilon,\nu} < \eta.$$

This yields that $(f_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence in $L^{p(.),\theta}(\mu)$ and $L^{q(.),\theta}(\nu)$, and $(f_n)_{n\in\mathbb{N}}$ converges to functions $f \in L^{p(.),\theta}(\mu)$ and $g \in L^{q(.),\theta}(\nu)$, respectively. If we use the embedding $L^{p(.),\theta}(\mu) \hookrightarrow L^{p(.)-\varepsilon}(\mu)$ for $\varepsilon \in (0, p^- - 1)$, then we obtain that there is a subsequence $(f_{n_i})_{i\in\mathbb{N}}$ of $(f_n)_{n\in\mathbb{N}}$ such that $f_{n_i} \longrightarrow f$ (μ-almost everywhere). Also since $(f_n)_{n\in\mathbb{N}}$ converges to g in $L^{q(.),\theta}(\nu)$, then it is easy to prove that $(f_{n_i})_{n\in\mathbb{N}}$ converges to g in $L^{q(.),\theta}(\nu)$ and $f_{n_i} \longrightarrow g$ (ν-almost everywhere) due to $L^{q(.),\theta}(\nu) \hookrightarrow L^{q(.)-\varepsilon}(\nu)$ for $\varepsilon \in (0, q^- - 1)$. Therefore, one can find a subsequence $\left(f_{n_{i_k}}\right)$ of (f_{n_i}) such that $f_{n_{i_k}} \longrightarrow g$ (ν-almost everywhere). If we consider the space $L^{p(.),\theta}(\mu)$ is a subspace of $L^{q(.),\theta}(\nu)$ in the sense of equivalence classes, then we have $\mu \approx \nu$ by Lemma 3.1. This follows the inequality

$$|f(x) - g(x)| \leq \left|f(x) - f_{n_{i_k}}(x)\right| + \left|f_{n_{i_k}}(x) - g(x)\right|,$$

that we have $f = g$ (μ-almost everywhere). Since $\mu \approx \nu$, we obtain $f = g$ (ν-almost everywhere), and $f_n \longrightarrow f$ in $L^{p(.),\theta}(\mu)$ with respect to the norm $\|.\|$. Now, we define the identity operator I from $\left(L^{p(.),\theta}(\mu), \|.\|\right)$ into $\left(L^{p(.),\theta}(\mu), \|.\|_{p(.),\theta,\mu}\right)$. Since

$$\|I(f)\|_{p(.),\theta,\mu} = \|f\|_{p(.),\theta,\mu} \leq \|f\|,$$

then I is continuous. If we consider the Banach's theorem, then I is a homeomorphism, see [5]. This yields the norms $\|.\|$ and $\|.\|_{p(.),\theta,\mu}$ are equivalent. Thus there exists a $C > 0$ such that

$$\|f\| \leq C \|f\|_{p(.),\theta,\mu}$$

for all $f \in L^{p(.),\theta}(\mu)$. Finally, we have

$$\|f\|_{q(.),\theta,\nu} \leq \|f\| \leq C \|f\|_{p(.),\theta,\mu}.$$

This completes the necessity part of the proof. Now, we suppose that $\mu \approx \nu$ and the inequality (2) holds for $L^{p(.),\theta}(\mu)$. Then, we have $L^{p(.),\theta}(\mu) \subseteq L^{q(.),\theta}(\nu)$ in the sense of individual functions. By Lemma 3.1, the space $L^{p(.),\theta}(\mu)$ is a subspace of $L^{q(.),\theta}(\nu)$ in the sense of equivalence classes. That is the desired result. □

Proposition 3.3. *Assume that the space $L^1(\mu)$ is continuously embedded in $L^1(\nu)$. Then we have $L^{p(.)}(\mu) \subseteq L^{p(.),\theta}(\nu)$.*

Proof. By the assumption, there exists a $C_1 > 0$ such that

$$\|h\|_{1,\nu} \leq C_1 \|h\|_{1,\mu} \tag{3}$$

for all $h \in L^1(\mu)$. Now, let $f \in L^{p(.)}(\mu)$ be given. Since the space $L^{p(.)}(\mu)$ is continuously embedded in $L^{p(.)-\varepsilon}(\mu)$ for all $\varepsilon \in (0, p^- - 1)$ and $p^+ < \infty$, we have

$$\varrho_{p(.)-\varepsilon,\mu}(f) = \int_X |f|^{p(x)-\varepsilon} d\mu < \infty,$$

that is $|f|^{p(.)-\varepsilon} \in L^1(\mu)$ for any $\varepsilon \in (0, p^- - 1)$. By (3), we get $|f|^{p(.)-\varepsilon} \in L^1(\nu)$ and

$$\varrho_{p(.)-\varepsilon,\nu}(f) \leq C_1 \int_X |f|^{p(x)-\varepsilon} d\mu = C_1 \varrho_{p(.)-\varepsilon,\mu}(f).$$

This follows by Remark 2.3 that

$$\begin{aligned}
&\|f\|_{p(.),\theta,\nu} \\
\leq &\sup_{0<\varepsilon<p^--1} \varepsilon^{\frac{\theta}{p^- - \varepsilon}} \left[\max\left\{\left(\varrho_{p(.)-\varepsilon,\nu}(f)\right)^{\frac{1}{p^- - \varepsilon}}, \left(\varrho_{p(.)-\varepsilon,\nu}(f)\right)^{\frac{1}{p^+ - \varepsilon}}\right\}\right] \\
\leq &\ C_1 \sup_{0<\varepsilon<p^--1} \varepsilon^{\frac{\theta}{p^- - \varepsilon}} \left[\max\left\{\|f\|_{p(.)-\varepsilon,\mu}^{\frac{p^+ - \varepsilon}{p^- - \varepsilon}}, 1\right\}\right] \\
\leq &\ C_1 \sup_{0<\varepsilon<p^--1} \varepsilon^{\frac{\theta}{p^- - \varepsilon}} \left[\max\left\{\|f\|_{p(.)-\varepsilon,\mu}^{p^+}, 1\right\}\right]
\end{aligned}$$

Again, by $L^{p(.)}(\mu) \hookrightarrow L^{p(.)-\varepsilon}(\mu)$ for all $\varepsilon \in (0, p^- - 1)$, we get

$$\begin{aligned}
\|f\|_{p(.),\theta,\nu} &\leq (\mu(X) + 1) C_1 \sup_{0<\varepsilon<p^--1} \varepsilon^{\frac{\theta}{p^- - \varepsilon}} \left[\max\left\{\|f\|_{p(.),\mu}^{p^+}, 1\right\}\right] \\
&= (\mu(X) + 1) C_1 \left(p^- - 1\right)^{\theta} \max\left\{\|f\|_{p(.),\mu}^{p^+}, 1\right\} < \infty.
\end{aligned}$$

This yields $L^{p(.)}(\mu) \subseteq L^{p(.),\theta}(\nu)$. $\square$

Proposition 3.4. *Assume that $L^{p(.),\theta}(\mu) \subseteq L^{p(.),\theta}(\nu)$. Then $\mu \approx \nu$ and there exists a $C > 0$ such that*

$$\nu(E) \leq C(\mu(E)+1)$$

for all $E \in \Sigma$.

Proof. Let $E \in \Sigma$. By Theorem 3.2, we have $\mu \approx \nu$ and there exists a $C > 0$ such that

$$\|f\|_{p(.),\theta,\nu} \leq C\|f\|_{p(.),\theta,\mu}$$

for all $f \in L^{p(.),\theta}(\mu)$. By [7, Lemma 2.39], we get that $\chi_E \in L^{p(.)-\varepsilon,\mu}$, $\chi_E \in L^{p(.)-\varepsilon,\nu}$, $\|\chi_E\|_{p(.)-\varepsilon,\mu} \leq \mu(E)+1$, $\|\chi_E\|_{p(.)-\varepsilon,\nu} \leq \nu(E)+1$ and $\chi_E \in L^{p(.),\theta}(\mu) \subseteq L^{p(.),\theta}(\nu)$ for all $\varepsilon \in (0, p^- - 1)$. If we consider the fact that $L^{p(.),\theta}(\nu) \hookrightarrow L^1(\nu)$, then we obtain

$$\begin{aligned} \nu(E) &\leq C\|\chi_E\|_{p(.),\theta,\nu} \leq C^*\|\chi_E\|_{p(.),\theta,\mu} \\ &\leq C^*\left(p^- - 1\right)^\theta (\mu(E)+1). \end{aligned}$$

This completes the proof. $\square$

Proposition 3.5. *Let $\theta_1 < \theta_2$ and $1 < q(.) < p(.)$. Then we have*

$$L^{p(.),\theta_1}(\mu) \hookrightarrow L^{q(.),\theta_2}(\mu),$$

or equivalently there exists a $C > 0$ such that

$$\|f\|_{q(.),\theta_2,\mu} \leq C(p,q)\|f\|_{p(.),\theta_1,\mu}$$

for all $f \in L^{p(.),\theta_1}(\mu)$.

Proof. Let $f \in L^{p(.),\theta_1}(\mu)$ be given. If we consider the Proposition 2.5, then we have $f \in L^{p(.),\theta_2}(\mu)$. Since $\mu(X) < \infty$ and $q(.)-\varepsilon < p(.)-\varepsilon$, we get $L^{p(.)-\varepsilon}(\mu) \hookrightarrow L^{q(.)-\varepsilon}(\mu)$, i.e. there exists a $C(\varepsilon) > 0$ such that

$$\|f\|_{q(.)-\varepsilon,\mu} \leq C(\varepsilon)\|f\|_{p(.)-\varepsilon,\mu}$$

for $f \in L^{p(.)-\varepsilon}(\mu)$ and $\varepsilon \in (0, p^- - 1)$. It is note that identity operator does not exceed $\mu(X)+1$, see [19]. Thus, for all $\varepsilon \in (0, p^- - 1)$ we have $C(\varepsilon) \leq \mu(X)+1$. This yields

$$\begin{aligned} \|f\|_{q(.),\theta_2,\mu} &\leq (\mu(X)+1) \sup_{0<\varepsilon<q^- -1} \varepsilon^{\frac{\theta_2}{q^- -\varepsilon}} \|f\|_{p(.)-\varepsilon,\mu} \\ &= (\mu(X)+1) \sup_{0<\varepsilon<q^- -1} \varepsilon^{\frac{\theta_2}{q^- -\varepsilon}} \varepsilon^{\frac{\theta_2}{p^- -\varepsilon}} \varepsilon^{\frac{-\theta_2}{p^- -\varepsilon}} \|f\|_{p(.)-\varepsilon,\mu} \\ &\leq (\mu(X)+1) C^* \sup_{0<\varepsilon<p^- -1} \varepsilon^{\frac{\theta_2}{p^- -\varepsilon}} \|f\|_{p(.)-\varepsilon,\mu} \\ &= (\mu(X)+1) C^* \|f\|_{p(.),\theta_2,\mu} < \infty \end{aligned}$$

where $C^* = \sup_{0<\varepsilon<q^- -1} \varepsilon^{\frac{\theta_2\left(p^- - q^-\right)}{(q^- -\varepsilon)(p^- -\varepsilon)}}$. This completes the proof. $\square$

Proposition 3.6. *Let $1 < p(.) \leq p^+ < q^- \leq q(.)$. If $L^{p(.),\theta}(\mu) \subseteq L^{q(.),\theta}(\mu)$, then there exists a constant $m > 0$ such that $\mu(E) \geq m$ for every μ-non null set $E \in \Sigma$.*

Proof. By Theorem 3.2, there is a $C>0$ such that

$$\|f\|_{q(.),\theta,\mu} \leq C \|f\|_{p(.),\theta,\mu} \tag{4}$$

for all $f \in L^{p(.),\theta}(\mu)$. Let $E \in \Sigma$ be a μ-non null set and $\mu(E) < \infty$. Therefore, we get

$$\begin{aligned} \|\chi_E\|_{p(.),\theta,\mu} &= \sup_{0<\varepsilon<p^- -1} \varepsilon^{\frac{\theta}{p^- -\varepsilon}} \|\chi_E\|_{p(.)-\varepsilon,\mu} \\ &\leq (\mu(E)+1)\left(p^- -1\right)^{\theta} < \infty \end{aligned}$$

and

$$\|\chi_E\|_{q(.),\theta,\mu} \leq C(\mu(E)+1)\left(p^- -1\right)^{\theta} < \infty.$$

This implies $\chi_E \in L^{q(.),\theta}(\mu)$. If we assume that $\mu(E) \geq 1$, then there is nothing to prove. Now, let $\mu(E) \leq 1$. Since $\frac{1}{\mu(E)} \geq 1$ and $\frac{p(.)-\varepsilon}{p^+} \leq 1$, we get

$$\begin{aligned} \varrho_{p(.)-\varepsilon,\mu}\left(\frac{\chi_E}{\mu(E)^{\frac{1}{p^+}}}\right) &= \int_X \frac{|\chi_E(x)|^{p(x)-\varepsilon}}{\mu(E)^{\frac{p(x)-\varepsilon}{p^+}}} d\mu \\ &\leq \frac{1}{\mu(E)} \int_X |\chi_E(x)|^{p(x)-\varepsilon} d\mu = 1. \end{aligned}$$

Thus we obtain

$$\|\chi_E\|_{p(.)-\varepsilon,\mu} \leq \mu(E)^{\frac{1}{p^+}} \tag{5}$$

by definition of $\|.\|_{p(.)-\varepsilon}$ for all $\varepsilon \in (0, p^- -1)$. By Remark 2.3, we have

$$\mu(E)^{\frac{1}{q^- -\varepsilon}} \leq \|\chi_E\|_{q(.)-\varepsilon,\mu}$$

for any $\varepsilon \in (0, q^- -1)$. This yields

$$\sup_{0<\varepsilon<q^- -1} \varepsilon^{\frac{\theta}{q^- -\varepsilon}} \mu(E)^{\frac{1}{q^- -\varepsilon}} \leq \sup_{0<\varepsilon<q^- -1} \varepsilon^{\frac{\theta}{q^- -\varepsilon}} \|\chi_E\|_{q(.)-\varepsilon,\mu}.$$

Thus, we have

$$\left(q^- -1\right)^{\theta} \mu(E)^{\frac{1}{q^-}} \leq \|\chi_E\|_{q(.),\theta,\mu}.$$

By (4), there exist a $C>0$ such that

$$\left(q^- -1\right)^{\theta} \mu(E)^{\frac{1}{q^-}} \leq C \|\chi_E\|_{p(.),\theta,\mu}. \tag{6}$$

Moreover, by (5) and (6), we have

$$\left(q^- -1\right)^{\theta} \mu(E)^{\frac{1}{q^-}} \leq C\left(p^- -1\right)^{\theta} \mu(E)^{\frac{1}{p^+}}$$

or equivalently

$$\frac{1}{C}\left(\frac{q^- -1}{p^- -1}\right)^{\theta} \leq \mu(E)^{\frac{1}{p^+}-\frac{1}{q^-}}.$$

Since $p^+ < q^-$, we get $\frac{1}{p^+} - \frac{1}{q^-} > 0$. Therefore, we obtain

$$\mu(E) \geq m$$

where $m = \left(\frac{1}{C}\left(\frac{q^- -1}{p^- -1}\right)^{\theta}\right)^{\frac{p^+ q^-}{q^- -p^+}}$. That is the desired result. □

4. Approximate Identities in $L^{p(.),\theta}(\Omega)$

Let $\Omega \subset \mathbb{R}^d$ be bounded and open set. It is well known that the classical Lebesgue space $L^p(\Omega)$ has a bounded approximate identity in $L^1(\Omega)$. Gurkanli considered $L^{p),\theta}(\Omega)$ does not admit a bounded approximate identity in $L^1(\Omega)$ in [15, Theorem 4], and also $[L^p(\Omega)]_{p),\theta}$, the closure of $C_0^\infty(\Omega)$ in $L^{p),\theta}(\Omega)$, admits a bounded approximate identity in $L^1(\Omega)$ in [15, Theorem 6]. Moreover, Cruz-Uribe and Fiorenza proved the convergence of a potential-type approximate identities, both pointwise and in norm, in variable exponent Lebesgue space $L^{p(.)}(\Omega)$ where $\Omega \subset \mathbb{R}^d$ is unbounded and open set (see Theorem 2.2 and Theorem 2.3 in [6]). Also, a weaker version of Theorem 2.2 in [6] was considered by Diening [9]. In this section, we will discuss that the convergence of potential-type approximate identity is valid for $L^{p(.),\theta}(\Omega)$.

Definition 4.1. Let $P_{loc}^{\log}(\Omega)$ be the class of exponents $p(.)$ satisfying the local logarithmic condition that there is a positive constant c_0 such that for all $x, y \in \Omega$ with $d(x,y) < \frac{1}{2}$,

$$|p(x) - p(y)| \leq \frac{c_0}{-\ln(d(x,y))}.$$

Moreover, let $\widetilde{P}_{loc}^{\log}(\Omega)$ be the class of exponents satisfying the condition, i.e. there exists positive constants a and b such that if $d(x,y) < b$, then

$$|p(x) - p(y)| \leq \frac{a}{-\ln(\mu(B(x,y)))}$$

where $B(x,y)$ is an open ball with center $x \in \Omega$ and radius $y > 0$. Also, if μ is a finite measure, then it is obvious that $P_{loc}^{\log}(\Omega) \subset \widetilde{P}_{loc}^{\log}(\Omega)$, see [18].

For $f \in L_{loc}^1(\Omega)$, we denote the (centered) Hardy-Littlewood maximal operator Mf of f by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy,$$

where the supremum is taken over all balls $B(x,r)$. It is well known that the Hardy-Littlewood maximal operator is bounded in $L^{p(.),\theta}(\Omega)$ if $p(.) \in \widetilde{P}_{loc}^{\log}(\Omega)$ and $\theta > 0$, see [18, Theorem 3.1].

Definition 4.2. Assume that φ is an integrable function defined on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} \varphi(x)dx = 1$. For each $t > 0$, define the function $\varphi_t(x) = t^{-d}\varphi\left(\frac{x}{t}\right)$. The sequence $\{\varphi_t\}$ is referred to as an approximate identity. It is known that for $1 < p < \infty$, the sequence $\{\varphi_t * f\}$ converges to f in $L^p(\Omega)$, i.e.

$$\lim_{t \longrightarrow \infty} \|\varphi_t * f - f\|_{p,\Omega} = 0,$$

see [24]. If we impose additional conditions on φ, then the entire sequence converges almost everywhere to f. Define the radial majorant of φ to be the function

$$\widetilde{\varphi}(x) = \sup_{|y| \geq |x|} |\varphi(y)|.$$

If the function $\widetilde{\varphi}$ is integrable, then $\{\varphi_t\}$ is called a potential-type approximate identity, see [6].

THEOREM 4.3. *(see [24, Theorem 2]) Let $\{\varphi_t\}$ be a potential-type approximate identity. Then*

(i) $\sup\limits_{t>0} |\varphi_t * f(x)| \leq AMf(x)$ *for* $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$ *where* $A = \int\limits_{\mathbb{R}^d} \widetilde{\varphi}(x)dx$.

(ii) $\lim\limits_{t \longrightarrow 0} (\varphi_t * f)(x) = f(x)$ *almost everywhere.*

(iii) If $p < \infty$, *then we get* $\|\varphi_t * f - f\|_p \longrightarrow \infty$ *as* $t \longrightarrow 0^+$ *for* $f \in L^p(\mathbb{R}^d)$.

The following theorem is proved by Diening for $f \in L^{p(.)}(\Omega)$ where Ω is a bounded and open subset of $\mathbb{R}^d$, see [9, Corollary 3.6].

THEOREM 4.4. *Let $\{\varphi_t\}$ be a potential-type approximate identity. Then*

(i) $\sup\limits_{t>0} |\varphi_t * f(x)| \leq 2AMf(x)$ *for* $f \in L^{p(.)}(\Omega)$.

(ii) $\lim\limits_{t \longrightarrow 0} (\varphi_t * f)(x) = f(x)$ *almost everywhere.*

(iii) If $p^+ < \infty$, *then we have* $\|\varphi_t * f - f\|_{p(.)} \longrightarrow \infty$ *as* $t \longrightarrow 0^+$ *for* $f \in L^{p(.)}(\Omega)$. *Furthermore, we obtain*

$$\|\varphi_t * f\|_{p(.)} \leq C(A,p)\|Mf\|_{p(.)} \leq C(A,p)\|f\|_{p(.)}.$$

Now, we are ready to present the main theorem of this section for the space $L^{p(.),\theta}(\Omega)$.

THEOREM 4.5. *Let $\{\varphi_t\}$ be a potential-type approximate identity. Then*

(i) $\sup\limits_{t>0} |\varphi_t * f(x)| \leq 2AMf(x)$ *for* $f \in L^{p(.),\theta}(\Omega)$.

(ii) $\lim\limits_{t \longrightarrow 0} (\varphi_t * f)(x) = f(x)$ *almost everywhere.*

(iii) If $p^+ < \infty$, *then we have* $\|\varphi_t * f - f\|_{p(.),\theta,\mu} \longrightarrow \infty$ *as* $t \longrightarrow 0^+$ *for* $f \in L^{p(.),\theta}(\Omega)$. *Moreover, we get*

$$\|\varphi_t * f\|_{p(.),\theta,\mu} \leq C(A,p)\|Mf\|_{p(.),\theta,\mu} \leq C(A,p)\|f\|_{p(.),\theta,\mu}.$$

Proof. By (1), we have

$$L^{p(.),\theta} \hookrightarrow L^{p(.)-\varepsilon} \hookrightarrow L^1,\ 0 < \varepsilon < p^- - 1$$

due to $\mu(\Omega) < \infty$. This yields *(i)* and *(ii)* by Theorem 4.3. To prove *(iii)*, let $f \in L^{p(.),\theta}(\Omega)$ be given. If we consider [18, Theorem 3.1], then we have

$$\|\varphi_t * f\|_{p(.),\theta,\mu} \leq 2A\|Mf\|_{p(.),\theta,\mu} \leq 2AC\|f\|_{p(.),\theta,\mu} < \infty.$$

This yields $\varphi_t * f \in L^{p(.),\theta}(\Omega)$ and $\varphi_t * f \in L^{p(.)-\varepsilon}(\Omega)$ for all $t > 0$, $\varepsilon \in (0, p^- - 1)$. Since *(i)* holds, we obtain

$$\begin{aligned} |\varphi_t * f(x) - f(x)|^{p(x)-\varepsilon} &\leq (|\varphi_t * f(x)| + |f(x)|)^{p(x)-\varepsilon} \\ &\leq C(p)(|Mf(x)| + |f(x)|)^{p(x)-\varepsilon} \in L^1(\Omega) \end{aligned}$$

due to $f \in L^{p(.)-\varepsilon}(\Omega)$ and the boundedness of maximal operator in $L^{p(.)-\varepsilon}(\Omega)$ for all $\varepsilon \in (0, p^- - 1)$. Since $p^+ < \infty$, we get

$$\varrho_{p(.)-\varepsilon,\mu}(\varphi_t * f - f) \longrightarrow 0$$

if and only if

$$\|\varphi_t * f - f\|_{p(.)-\varepsilon,\mu} \longrightarrow 0$$

as $t \longrightarrow 0^+$ for any $\varepsilon \in (0, p^- - 1)$ by the Lebesgue dominated convergence theorem. Therefore, for every $\eta > 0$ there exists an $h > 0$ such that

$$\|\varphi_t * f - f\|_{p(.)-\varepsilon,\mu} < \eta$$

for all t satisfying $t < h$ and

$$\|\varphi_t * f - f\|_{p(.),\theta,\mu} < \eta \sup_{0<\varepsilon<p^- -1} \varepsilon^{\frac{\theta}{p^- -\varepsilon}} = \left(p^- - 1\right)^\theta \eta.$$

This completes the proof. $\square$

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