

GENERALIZED SEQUENTIAL CONVOLUTION PRODUCT FOR THE GENERALIZED SEQUENTIAL FOURIER-FEYNMAN TRANSFORM

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ABSTRACT. This paper is a further development of the recent results by the authors on the generalized sequential Fourier-Feynman transform for functionals in a Banach algebra $\hat{\mathcal{S}}$ and some related functionals. We investigate various relationships between the generalized sequential Fourier-Feynman transform and the generalized sequential convolution product of functionals. Parseval's relation for the generalized sequential Fourier-Feynman transform is also given.

1. Introduction

In [16], the authors introduced the concept of generalized sequential Feynman integral to define the generalized sequential Fourier-Feynman transform for functionals in the Banach algebra $\hat{\mathcal{S}}$ which was introduced by Cameron and Storvick [1]. Their results are generalization of the previous results on the sequential Feynman integral and sequential Fourier-Feynman transform introduced and studied by Cameron and Storvick [1–3].

In the present paper we introduce the concept of generalized sequential convolution product for the generalized sequential Fourier-Feynman transform. Existence of the generalized sequential convolution product for functionals in the Banach algebra $\hat{\mathcal{S}}$ and some related functionals are established in Theorems 3.3, 3.4, 3.5 and 3.6.

In Section 4, we show that the generalized sequential Fourier-Feynman transform of the generalized sequential convolution product of functionals is a product of the generalized sequential Fourier-Feynman transforms of these functionals. We also obtain Parseval's relation for the generalized sequential Fourier-Feynman transform for some functionals.

Results in [6] on the sequential Fourier-Feynman transform and the sequential convolution product can be obtained as corollaries of our results here.

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Let $C_0[0, T]$ be the space of continuous functions $x(t)$ on $[0, T]$ such that $x(0) = 0$. Let a subdivision σ of $[0, T]$ be given:

$$\sigma : 0 = \tau_0 < \tau_1 < \dots < \tau_m = T,$$

and let $X = X(t)$ be a polygonal curve in $C_0[0, T]$ based on a subdivision σ and the real numbers $\vec{\xi} = \{\xi_k\}$, that is,

$$X(t) = X(t, \sigma, \vec{\xi})$$

where

$$X(t, \sigma, \vec{\xi}) = \frac{\xi_{k-1}(\tau_k - t) + \xi_k(t - \tau_{k-1})}{\tau_k - \tau_{k-1}}$$

when $\tau_{k-1} \leq t \leq \tau_k$, $k = 1, 2, \dots, m$ and $\xi_0 = 0$. If there is a sequence of subdivisions $\{\sigma_n\}$, then σ, m and τ_k will be replaced by σ_n, m_n and $\tau_{n,k}$.

Let Z_h be the Gaussian process

$$Z_h(x, t) = \int_0^t h(s) dx(s),$$

where $h(\neq 0 \text{ a.e.})$ is in $L_2[0, T]$ and the integral $\int_0^t h(s) dx(s)$ denotes the Paley-Wiener-Zygmund (PWZ) integral [5, 8, 9, 13].

Note that Z_h is a Gaussian process with mean zero and covariance function

$$\int_{C_0[0, T]} Z_h(x, s) Z_h(x, t) dm(x) = \int_0^{\min\{s, t\}} h^2(u) du,$$

where the integral on the left-hand side of the last expression denotes the Wiener integral. Of course if $h \equiv 1$ on $[0, T]$, then $Z_h(x, t) = x(t)$ is the standard Wiener process.

The standard Wiener process is stationary in time, while the Gaussian process Z_h is non-stationary in time, unless h is equal to the constant function 1.

Let $q \neq 0$ be a given real number and let $F(x)$ be a functional defined on a subset of $C_0[0, T]$ containing all the polygonal curves in $C_0[0, T]$. Let $\{\sigma_n\}$ be a sequence of subdivisions such that the norm $\|\sigma_n\| \rightarrow 0$ and let $\{\lambda_n\}$ be a sequence of complex numbers with $\text{Re } \lambda_n > 0$ such that $\lambda_n \rightarrow -iq$. Then if the integral in the right hand side of (1.1) exists for all n and if the following limit exists and is independent of the choice of the sequences $\{\sigma_n\}$ and $\{\lambda_n\}$, we say that the generalized sequential Feynman integral with parameter q exists and it is denoted by

$$(1.1) \quad \int^{\text{sf}_q} F(Z_h(x, \cdot)) dx = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^{m_n}} W_{\lambda_n}(\sigma_n, \vec{\xi}) F(Z_h(X(\cdot, \sigma_n, \vec{\xi}), \cdot)) d\vec{\xi},$$

where

$$(1.2) \quad \begin{aligned} W_\lambda(\sigma, \vec{\xi}) &= \gamma_{\sigma, \lambda} \exp\left\{-\frac{\lambda}{2} \int_0^T \left|\frac{dX}{dt}(t, \sigma_n, \vec{\xi})\right|^2 dt\right\} \\ &= \left(\frac{\lambda}{2\pi}\right)^{m/2} \left[\prod_{k=1}^m (\tau_k - \tau_{k-1})^{-1/2}\right] \exp\left\{-\frac{\lambda}{2} \sum_{k=1}^m \frac{(\xi_k - \xi_{k-1})^2}{\tau_k - \tau_{k-1}}\right\} \end{aligned}$$

and

$$(1.3) \quad \gamma_{\sigma, \lambda} = \left(\frac{\lambda}{2\pi}\right)^{m/2} \prod_{k=1}^m (\tau_k - \tau_{k-1})^{-1/2}.$$

When $h \equiv 1$ on $[0, T]$, the generalized sequential Feynman integral is reduced to the sequential Feynman integral $\int^{\text{sf}_q} F(x) dx$ defined and studied in [1–3, 6].

2. Generalized sequential Fourier-Feynman transform

Let $D[0, T]$ be the class of elements $x \in C_0[0, T]$ such that x is absolutely continuous on $[0, T]$ and its derivative $x' \in L_2[0, T]$. For $u, v \in L_2[0, T]$ and $x \in C_0[0, T]$, we let

$$\langle u, v \rangle = \int_0^T u(t)v(t) dt,$$

and for a subdivision σ of $[0, T]$, we let

$$\langle u, v \rangle_k = \int_{\tau_{k-1}}^{\tau_k} u(t)v(t) dt$$

for $k = 1, \dots, m$. If there is a sequence of subdivision $\{\sigma_n\}$, then $\langle u, v \rangle_k$ will be replaced by $\langle u, v \rangle_{n,k}$.

Let $\mathcal{M} = \mathcal{M}(L_2[0, T])$ be the class of complex measures of finite variation defined on $\mathcal{B}(L_2[0, T])$, the Borel measurable subsets of $L_2[0, T]$. A functional F defined on a subset of $C_0[0, T]$ that contains $D[0, T]$ is said to be an element of $\hat{\mathcal{S}} = \hat{\mathcal{S}}(L_2[0, T])$ if there exists a measure $f \in \mathcal{M}$ such that for $x \in D[0, T]$,

$$(2.1) \quad F(x) = \int_{L_2[0, T]} \exp\{i\langle v, x' \rangle\} df(v).$$

Note that $\hat{\mathcal{S}}$ with the norm $\|F\| = \|f\|$ is a Banach algebra [1].

Next we consider one more class of functionals which is different from but are closely related with the expression (2.1).

Let \mathcal{T} be the set of functions Ψ defined on \mathbb{R} by

$$(2.2) \quad \Psi(r) = \int_{\mathbb{R}} \exp\{irs\} d\rho(s)$$

where ρ is a complex Borel measure of bounded variation on \mathbb{R} .

For $s \in \mathbb{R}$, let $\phi(s)$ be the function $v \in L_2[0, T]$ such that $v(t) = s$ for $0 \leq t \leq T$; thus $\phi : \mathbb{R} \rightarrow L_2[0, T]$ is continuous. For $E \in \mathcal{B}(L_2[0, T])$, let

$$(2.3) \quad \psi(E) = \rho(\phi^{-1}(E)).$$

Thus $\psi \in \mathcal{M}$. Transforming the right hand member of (2.2), we have for $x \in D[0, T]$,

$$(2.4) \quad \Psi(x(T)) = \int_{L_2[0, T]} \exp\{i\langle u, x' \rangle\} d\psi(u),$$

and $\Psi(x(T))$, considered as a functional of x , is an element of $\hat{\mathcal{S}}$. These functionals were studied in [3, 4, 6, 16].

In the application of the Feynman integral to quantum theory, the function Ψ in (2.2) corresponds to the initial condition associated with Schrödinger equation.

Cameron and Storvick defined the sequential Fourier-Feynman transform for the first time in [2]. And in [6] the authors and coworkers modified the definition so as to be useful to study the sequential Fourier-Feynman transform in conjunction with the convolution product. For discussions on the differences between the two definitions of sequential Fourier-Feynman transform, see Remarks 1.4 and 3.5 in [6] and Remark

3.2 in [16]. We introduce the definition of the generalized sequential Fourier-Feynman transform from [16].

DEFINITION 2.1. Let q be a nonzero real number. For $y \in D[0, T]$, we define a generalized sequential Fourier-Feynman transform $\Gamma_{q,h}(F)$ of F by the formula

$$(2.5) \quad \Gamma_{q,h}(F)(y) = \int^{sf_q} F(Z_h(x, \cdot) + y) dx$$

if it exists.

For the convenience of the readers, we introduce some results from [16] on the existences and explicit expressions for the generalized sequential Fourier-Feynman transforms of functionals that we work with in this paper.

THEOREM 2.2 (Theorem 3.4 in [16]). *Let $F \in \hat{\mathcal{S}}$ be given by (2.1) and q be a nonzero real number. Then the generalized sequential Fourier-Feynman transform $\Gamma_{q,h}(F)(y)$ exists and is given by the formula*

$$(2.6) \quad \Gamma_{q,h}(F)(y) = \int_{L_2[0,T]} \exp\left\{i\langle v, y' \rangle - \frac{i}{2q} \|vh\|_2^2\right\} df(v)$$

for $y \in D[0, T]$. Furthermore, as a function of y , $\Gamma_{q,h}(F)$ is an element of $\hat{\mathcal{S}}$. In fact,

$$(2.7) \quad \Gamma_{q,h}(F)(y) = \int_{L_2[0,T]} \exp\{i\langle v, y' \rangle\} df_{q,h}^t(v)$$

for $y \in D[0, T]$, where $f_{q,h}^t$ is the measure in \mathcal{M} defined by

$$(2.8) \quad f_{q,h}^t(E) = \int_E \exp\left\{-\frac{i}{2q} \|vh\|_2^2\right\} df(v)$$

for $E \in \mathcal{B}(L_2[0, T])$.

THEOREM 2.3 (Theorem 3.7 in [16]). *For $x \in D[0, T]$, let $F(x) = G(x)\Psi(x(T))$ where $G \in \hat{\mathcal{S}}$ and $\Psi \in \mathcal{T}$ are given by (2.1) with corresponding measure g in \mathcal{M} and (2.2), respectively. Then the generalized sequential Fourier-Feynman transform $\Gamma_{q,h}(F)(y)$ exists and is given by the formula*

$$(2.9) \quad \Gamma_{q,h}(F)(y) = \int_{L_2[0,T]} \int_{\mathbb{R}} \exp\left\{i\langle v + s, y' \rangle - \frac{i}{2q} \|(v + s)h\|_2^2\right\} d\rho(s) dg(v)$$

for $y \in D[0, T]$. Furthermore, as a function of y , $\Gamma_{q,h}(F)$ belongs to $\hat{\mathcal{S}}$. In fact,

$$(2.10) \quad \Gamma_{q,h}(F)(y) = \int_{L_2[0,T]} \exp\{i\langle v, y' \rangle\} d\hat{f}_{q,h}^t(v)$$

for $y \in D[0, T]$, where $\hat{f}_{q,h}^t$ is the measure given by (2.8) with corresponding measure \hat{f} defined by $\hat{f}(E) = \int_{L_2[0,T]} g(E - u) d\psi(u)$ for $E \in \mathcal{B}(L_2[0, T])$.

THEOREM 2.4 (Theorem 3.8 in [16]). *Let Φ be a bounded measurable functional defined on $L_2[0, T]$, and let*

$$(2.11) \quad F(x) = \int_{L_2[0,T]} \exp\{i\langle v, x' \rangle\} \Phi(v) df(v).$$

Then the generalized sequential Fourier-Feynman transform $\Gamma_{q,h}(F)(y)$ exists and is given by the formula

$$(2.12) \quad \Gamma_{q,h}(F)(y) = \int_{L_2[0,T]} \exp\left\{i\langle v, y' \rangle - \frac{i}{2q} \|vh\|_2^2\right\} \Phi(v) df(v)$$

for $y \in D[0, T]$. Furthermore, as a function of y , $\Gamma_{q,h}(F)$ belongs to $\hat{\mathcal{S}}$. In fact,

$$(2.13) \quad \Gamma_{q,h}(F)(y) = \int_{L_2[0,T]} \exp\{i\langle v, y' \rangle\} d\tilde{f}_{q,h}^t(v)$$

for $y \in D[0, T]$, where $\tilde{f}_{q,h}^t$ is the measure given by (2.8) with corresponding measure \tilde{f} defined by $\tilde{f}(E) = \int_E \Phi(v) df(v)$ for $E \in \mathcal{B}(L_2[0, T])$.

3. Generalized sequential convolution product

In this section we define and study the generalized sequential convolution product for the generalized sequential Fourier-Feynman transform. We investigate the existence of the generalized sequential convolution product for functionals in Section 2. Also we show that the generalized sequential Fourier-Feynman transform of the generalized sequential convolution product is a product of the generalized sequential Fourier-Feynman transforms for these functionals.

DEFINITION 3.1. Let q be a nonzero real number. For $y \in D[0, T]$, we define the generalized sequential convolution $(F * G)_q$ of F and G by the formula

$$(3.1) \quad (F * G)_{q,h}(y) = \int^{\text{sf}_q} F\left(\frac{y + Z_h(x, \cdot)}{\sqrt{2}}\right) G\left(\frac{y - Z_h(x, \cdot)}{\sqrt{2}}\right) dx$$

if it exists.

- REMARK 3.2.
1. When $h \equiv 1$ on $[0, T]$, the generalized sequential convolution product $(F * G)_{q,h}$ is reduced to the convolution product $(F * G)_q$ for the sequential Fourier-Feynman transform which was defined and studied on [6].
 2. Hence some of the results in [6] can be obtained as corollaries of the results in this section. For example, Theorems 3.1, 3.2, 3.3 and 3.8 in [6] follow from Theorems 3.3, 3.9, 3.10 and 3.4 below, respectively.
 3. Convolution product for the analytic Fourier-Feynman transform was introduced and studied in many literatures including [5, 7, 10–12, 14, 15].
 4. For $\vec{\eta} = -\vec{\xi}$, we know that $|\frac{\partial(\vec{\xi})}{\partial(\vec{\eta})}| = 1$, $W_{\lambda_n}(\sigma_n, \vec{\xi}) = W_{\lambda_n}(\sigma_n, \vec{\eta})$ and $Z_h(X(\cdot, \sigma_n, \vec{\xi}), \cdot) = -Z_h(X(\cdot, \sigma_n, \vec{\eta}), \cdot)$. So we conclude that the generalized (sequential) convolution product, in common with the convolution product for the analytic Fourier-Feynman transform [7, 10], is commutative.

THEOREM 3.3. Let $F_j \in \hat{\mathcal{S}}$ be given by (2.1) with corresponding measures f_j in \mathcal{M} for $j = 1, 2$. Then for each nonzero real number q , the generalized convolution product $(F_1 * F_2)_{q,h}$ exists, belongs to $\hat{\mathcal{S}}$ and is given by

$$(3.2) \quad (F_1 * F_2)_{q,h}(y) = \int_{L_2^2[0,T]} \exp\left\{\frac{i}{\sqrt{2}}\langle v_1 + v_2, y' \rangle - \frac{i}{4q} \|(v_1 - v_2)h\|_2^2\right\} \times df_1(v_1) df_2(v_2)$$

for $y \in D[0, T]$. Furthermore, as a function of $y \in D[0, T]$, $(F_1 * F_2)_{q,h}(y)$ is an element of $\hat{\mathcal{S}}$. In fact

$$(3.3) \quad (F_1 * F_2)_{q,h}(y) = \int_{L_2[0,T]} \exp\{i\langle w, y' \rangle\} df_{q,h}^c(w),$$

where $f_{q,h}^c = f_{q,h} \circ \phi^{-1}$ and

$$(3.4) \quad f_{q,h}(E) = \int_E \exp\left\{-\frac{i}{4q} \|(v_1 - v_2)h\|_2^2\right\} df_1(v_1) df_2(v_2)$$

for $E \in \mathcal{B}(L_2^2[0, T])$ and $\phi : L_2^2[0, T] \rightarrow L_2[0, T]$ is a function defined by $\phi(v_1, v_2) = \frac{v_1 + v_2}{\sqrt{2}}$.

Proof. Let $\sigma : 0 = \tau_0 < \tau_1 < \dots < \tau_m = T$ be a subdivision of $[0, T]$. Then

$$\begin{aligned} & F_1\left(\frac{y + Z_h(X(\cdot, \sigma, \vec{\xi}), \cdot)}{\sqrt{2}}\right) F_2\left(\frac{y - Z_h(X(\cdot, \sigma, \vec{\xi}), \cdot)}{\sqrt{2}}\right) \\ &= \int_{L_2^2[0,T]} \exp\left\{\frac{i}{\sqrt{2}}\langle v_1 + v_2, y' \rangle + \frac{i}{\sqrt{2}} \sum_{k=1}^m \frac{\xi_k - \xi_{k-1}}{\tau_k - \tau_{k-1}} \langle v_1 - v_2, h \rangle_k\right\} \\ & \quad \times df_1(v_1) df_2(v_2). \end{aligned}$$

Let λ be a complex number with $\operatorname{Re} \lambda > 0$, and let

$$\begin{aligned} I_{\sigma,\lambda}(F_1, F_2) &= \int_{\mathbb{R}^m} W_\lambda(\sigma, \vec{\xi}) F_1\left(\frac{y + Z_h(X(\cdot, \sigma, \vec{\xi}), \cdot)}{\sqrt{2}}\right) \\ & \quad \times F_2\left(\frac{y - Z_h(X(\cdot, \sigma, \vec{\xi}), \cdot)}{\sqrt{2}}\right) d\vec{\xi}. \end{aligned}$$

By the Fubini theorem, we have for $y \in D[0, T]$

$$\begin{aligned} I_{\sigma,\lambda}(F_1, F_2) &= \int_{L_2^2[0,T]} J_{\sigma,\lambda}\left(\frac{v_1 - v_2}{\sqrt{2}}\right) \exp\left\{\frac{i}{\sqrt{2}}\langle v_1 + v_2, y' \rangle\right\} \\ & \quad \times df_1(v_1) df_2(v_2), \end{aligned}$$

where

$$\begin{aligned} J_{\sigma,\lambda}\left(\frac{v_1 - v_2}{\sqrt{2}}\right) &= \gamma_{\sigma,\lambda} \int_{\mathbb{R}^m} \exp\left\{-\frac{\lambda}{2} \sum_{k=1}^m \frac{(\xi_k - \xi_{k-1})^2}{\tau_k - \tau_{k-1}}\right. \\ & \quad \left. + \frac{i}{\sqrt{2}} \sum_{k=1}^m \frac{\xi_k - \xi_{k-1}}{\tau_k - \tau_{k-1}} \langle v_1 - v_2, h \rangle_k\right\} d\vec{\xi}. \end{aligned}$$

Letting $\eta_k = \xi_k - \xi_{k-1}$ for $k = 1, 2, \dots, m$ and using the integration formula $\int_{\mathbb{R}} e^{-a\eta^2 + ib\eta} d\eta = (\frac{\pi}{a})^{1/2} \exp\{-\frac{b^2}{4a}\}$ for $\operatorname{Re} a > 0$ and $b \in \mathbb{R}$, we have

$$J_{\sigma,\lambda}\left(\frac{v_1 - v_2}{\sqrt{2}}\right) = \exp\left\{-\frac{1}{4\lambda} \sum_{k=1}^m \frac{\langle v_1 - v_2, h \rangle_k^2}{\tau_k - \tau_{k-1}}\right\}.$$

Let $\{\sigma_n\}$ be a sequence of subdivisions of $[0, T]$ such that $\|\sigma_n\| \rightarrow 0$, and let $\{\lambda_n\}$ be a sequence of complex numbers such that $\text{Re } \lambda_n > 0$ and $\lambda_n \rightarrow -iq$ as $n \rightarrow \infty$. Let

$$v_{\sigma_n, h}(t) = \begin{cases} \frac{\langle \frac{v_1 - v_2}{\sqrt{2}}, h \rangle_{n, k}}{\tau_{n, k} - \tau_{n, k-1}}, & \text{if } \tau_{n, k-1} \leq t < \tau_{n, k}, \quad k = 1, \dots, m \\ 0, & \text{if } t = T. \end{cases}$$

Then since

$$\|v_{\sigma_n, h}\|_2^2 = \sum_{k=1}^{m_n} \int_{\tau_{n, k-1}}^{\tau_{n, k}} (v_{\sigma_n, h}(t))^2 dt = \frac{1}{2} \sum_{k=1}^{m_n} \frac{\langle v_1 - v_2, h \rangle_{n, k}^2}{\tau_{n, k} - \tau_{n, k-1}},$$

we have by Lemma 2.1 of [16],

$$\lim_{n \rightarrow \infty} J_{\sigma_n, \lambda_n} \left(\frac{v_1 - v_2}{\sqrt{2}} \right) = \exp \left\{ -\frac{i}{4q} \|v_1 - v_2\|_2^2 \right\}.$$

Now applying the bounded convergence theorem, we obtain

$$\begin{aligned} (F_1 * F_2)_{q, h}(y) &= \lim_{n \rightarrow \infty} I_{\sigma_n, \lambda_n}(F_1, F_2) \\ &= \int_{L_2^2[0, T]} \exp \left\{ \frac{i}{\sqrt{2}} \langle v_1 + v_2, y' \rangle - \frac{i}{4q} \|v_1 - v_2\|_2^2 \right\} \\ &\quad \times df_1(v_1) df_2(v_2) \end{aligned}$$

as we wished. Furthermore since $f_{q, h}$ in (3.4) belongs to \mathcal{M} and ϕ is a Borel measurable function, we see that $f_{q, h}^c \in \mathcal{M}$ and $(F_1 * F_2)_{q, h}(y)$ can be expressed as (3.3) which completes the proof. \square

THEOREM 3.4. For $x \in D[0, T]$, let $F_j(x) = G_j(x)\Psi_j(x(T))$ where $G_j \in \hat{\mathcal{S}}$ and $\Psi_j \in \mathcal{T}$ are given by (2.1) with corresponding measures g_j in \mathcal{M} and (2.2), respectively for $j = 1, 2$. Then for each nonzero real number q , the generalized convolution product $(F_1 * F_2)_{q, h}$ exists, belongs to $\hat{\mathcal{S}}$ and is given by

$$\begin{aligned} (F_1 * F_2)_{q, h}(y) &= \int_{L_2^2[0, T]} \int_{\mathbb{R}^2} \exp \left\{ \frac{i}{\sqrt{2}} \langle v_1 + v_2 + s_1 + s_2, y' \rangle \right. \\ (3.5) \quad &\quad \left. - \frac{i}{4q} \|(v_1 - v_2 + s_1 - s_2)h\|_2^2 \right\} \\ &\quad \times d\rho_1(s_1) d\rho_2(s_2) dg_1(v_1) dg_2(v_2) \end{aligned}$$

for $y \in D[0, T]$. Furthermore, as a function of $y \in D[0, T]$, $(F_1 * F_2)_{q, h}(y)$ is an element of $\hat{\mathcal{S}}$. In fact

$$(3.6) \quad (F_1 * F_2)_{q, h}(y) = \int_{L_2[0, T]} \exp\{i\langle w, y' \rangle\} d\hat{f}_{q, h}^c(w),$$

for $y \in D[0, T]$, where $\hat{f}_{q, h}^c = \hat{f}_{q, h} \circ \phi^{-1}$ is the measure in \mathcal{M} with $\hat{f}_{q, h}$ defined by (3.4) replaced f_1 and f_2 with \hat{f}_1 and \hat{f}_2 , respectively, and \hat{f}_1 and \hat{f}_2 are the measures defined in the proof below.

Proof. We know from the proof of Theorem 2.3 of [16], that F_j belongs to $\hat{\mathcal{S}}$ and can be expressed as

$$F_j(x) = \int_{L_2[0, T]} \exp\{i\langle w_j, y' \rangle\} df_j(v_j)$$

for $y \in D[0, T]$, where $\hat{f}_j \in \mathcal{M}$ is given by $\hat{f}_j(E) = \int_{L_2[0, T]} g_j(E - u_j) d\psi_j(u_j)$ for $E \in \mathcal{B}(L_2[0, T])$ and $j = 1, 2$. Now we apply Theorem 3.3 to obtain

$$(F_1 * F_2)_{q,h}(y) = \int_{L_2^2[0, T]} \exp\left\{\frac{i}{\sqrt{2}}\langle w_1 + w_2, y' \rangle - \frac{i}{4q}\|(w_1 - w_2)h\|_2^2\right\} \\ \times d\hat{f}_1(w_1) d\hat{f}_2(w_2)$$

for $y \in D[0, T]$. By the unsymmetric Fubini theorem and the transformation $v_j = w_j - u_j$, $j = 1, 2$, we have

$$(F_1 * F_2)_{q,h}(y) = \int_{L_2^4[0, T]} \exp\left\{\frac{i}{\sqrt{2}}\langle v_1 + v_2 + u_1 + u_2, y' \rangle - \frac{i}{4q}\|(v_1 - v_2 + u_1 - u_2)h\|_2^2\right\} \\ \times dg_1(v_1) d\psi_1(u_1) dg_2(v_2) d\psi_2(u_2)$$

for $y \in D[0, T]$. By the definition of Ψ_j and ψ_j , $j = 1, 2$, and the Fubini theorem, we obtain (3.5). Moreover it is obvious to see that $(F_1 * F_2)_{q,h}(y)$ can be expressed as (3.6) and that $(F_1 * F_2)_{q,h}$ belongs to $\hat{\mathcal{S}}$. □

THEOREM 3.5. *Let $F_j(x) = \int_{L_2[0, T]} \exp\{i\langle v_j, x' \rangle\} \Phi_j(v_j) df_j(v_j)$, $x \in D[0, T]$, where Φ_j is a bounded measurable functional defined on $L_2[0, T]$ for $j = 1, 2$. Then for each nonzero real number q , the generalized convolution product $(F_1 * F_2)_{q,h}$ exists, belongs to $\hat{\mathcal{S}}$ and is given by*

$$(3.7) \quad (F_1 * F_2)_{q,h}(y) = \int_{L_2^2[0, T]} \exp\left\{\frac{i}{\sqrt{2}}\langle v_1 + v_2, y' \rangle - \frac{i}{4q}\|(v_1 - v_2)h\|_2^2\right\} \\ \times \Phi_1(v_1)\Phi_2(v_2) df_1(v_1) df_2(v_2)$$

for $y \in D[0, T]$. Furthermore, as a function of $y \in D[0, T]$, $(F_1 * F_2)_{q,h}(y)$ is an element of $\hat{\mathcal{S}}$. In fact

$$(3.8) \quad (F_1 * F_2)_{q,h}(y) = \int_{L_2[0, T]} \exp\{i\langle w, y' \rangle\} d\tilde{f}_{q,h}^c(w),$$

for $y \in D[0, T]$, where $\tilde{f}_{q,h}^c = \tilde{f}_{q,h} \circ \phi^{-1}$ is the measure in \mathcal{M} with $\tilde{f}_{q,h}$ defined by (3.4) replaced f_1 and f_2 with \tilde{f}_1 and \tilde{f}_2 , respectively, and \tilde{f}_1 and \tilde{f}_2 are the measures defined in the proof below.

Proof. We know from the proof of Theorem 2.4 of [16], that F_j belongs to $\hat{\mathcal{S}}$ and can be expressed as

$$F_j(x) = \int_{L_2[0, T]} \exp\{i\langle v_j, x' \rangle\} d\tilde{f}_j(v_j)$$

for $y \in D[0, T]$, where $\tilde{f}_j \in \mathcal{M}$ is given by $\tilde{f}_j(E) = \int_E \Phi_j(v_j) df_j(v_j)$ for $E \in \mathcal{B}(L_2[0, T])$. Now we apply Theorem 3.3 to obtain

$$(F_1 * F_2)_{q,h}(y) = \int_{L_2^2[0, T]} \exp\left\{\frac{i}{\sqrt{2}}\langle v_1 + v_2, y' \rangle - \frac{i}{4q}\|(v_1 - v_2)h\|_2^2\right\} \\ \times d\tilde{f}_1(v_1) d\tilde{f}_2(v_2)$$

for $y \in D[0, T]$. Replacing $df_j(v_j)$ by $\Phi(v_j)df_j(v_j)$, we completes the proof. Moreover it is obvious to see that $(F_1 * F_2)_{q,h}(y)$ can be expressed as (3.8) and that $(F_1 * F_2)_{q,h}$ belongs to $\hat{\mathcal{S}}$. \square

In Theorems 3.3, 3.4 and 3.5, we considered the generalized convolution product $(F_1 * F_2)_{q,h}$ of the same type of functionals F_1 and F_2 . That is, for example, in Theorem 3.3 F_1 and F_2 were given by (2.1), while in Theorem 3.4, F_j was given by $G(x)\Psi_j(x(T))$ for $j = 1, 2$.

But F_1 and F_2 are not necessarily of the same type of functionals. In the following theorem we express explicitly $(F_1 * F_2)_{q,h}$ when F_1 and F_2 are different type of functionals. We state these results without proofs because they can be proved by similar methods as in Theorems 3.3, 3.4 and 3.5.

THEOREM 3.6. *Let F_1, F_2 and F_3 are given as in Theorems 3.3, 3.4 and 3.5, respectively. Then for each nonzero real number q , the following three generalized convolution products exist, belong to $\hat{\mathcal{S}}$ and are given by*

$$(3.9) \quad \begin{aligned} (F_1 * F_2)_{q,h}(y) = & \int_{L^2_2[0,T]} \int_{\mathbb{R}} \exp\left\{ \frac{i}{\sqrt{2}} \langle v_1 + v_2 + s_2, y' \rangle \right. \\ & \left. - \frac{i}{4q} \|(v_1 - v_2 - s_2)h\|_2^2 \right\} d\rho_2(s_2) df_1(v_1) dg_2(v_2), \end{aligned}$$

$$(3.10) \quad \begin{aligned} (F_1 * F_3)_{q,h}(y) = & \int_{L^2_2[0,T]} \exp\left\{ \frac{i}{\sqrt{2}} \langle v_1 + v_3, y' \rangle \right. \\ & \left. - \frac{i}{4q} \|(v_1 - v_3)h\|_2^2 \right\} \Phi_3(v_3) df_1(v_1) df_3(v_3) \end{aligned}$$

and

$$(3.11) \quad \begin{aligned} (F_2 * F_3)_{q,h}(y) = & \int_{L^2_2[0,T]} \int_{\mathbb{R}} \exp\left\{ \frac{i}{\sqrt{2}} \langle v_2 + v_3 + s_2, y' \rangle \right. \\ & \left. - \frac{i}{4q} \|(v_2 - v_3 + s_2)h\|_2^2 \right\} \Phi_3(v_3) d\rho_2(s_2) dg_2(v_2) df_3(v_3) \end{aligned}$$

for $y \in D[0, T]$.

4. Transform and convolution product

In this section, we examine the generalized sequential Fourier-Feynman transform of the generalized convolution product of functionals in $\hat{\mathcal{S}}$. It turns out that the relationship (4.1) below is the same as Theorem 3.3 in [12] for the generalized analytic Fourier-Feynman transform.

THEOREM 4.1. *Let $F_j, j = 1, 2$, be given as in Theorem 3.3. Then for all nonzero real number q , $\Gamma_{q,h}((F_1 * F_2)_{q,h})(y)$ exists and*

$$(4.1) \quad \Gamma_{q,h}((F_1 * F_2)_{q,h})(y) = \Gamma_{q,h}(F_1)\left(\frac{y}{\sqrt{2}}\right)\Gamma_{q,h}(F_2)\left(\frac{y}{\sqrt{2}}\right),$$

for $y \in D[0, T]$.

Proof. From the expression (3.3) for $(F_1 * F_2)_{q,h}(y)$ and Theorem 2.2, we have

$$\Gamma_{q,h}((F_1 * F_2)_{q,h})(y) = \int_{L_2[0,T]} \exp\left\{i\langle w, y' \rangle - \frac{i}{2q} \|wh\|_2^2\right\} df_{q,h}^c(w)$$

for $y \in D[0, T]$. Using the definition of $f_{q,h}^c$ in Theorem 3.3, we rewrite $\Gamma_{q,h}(F_1 * F_2)_{q,h}(y)$ as

$$\begin{aligned} \Gamma_{q,h}((F_1 * F_2)_{q,h})(y) &= \int_{L_2^2[0,T]} \exp\left\{\frac{i}{\sqrt{2}} \langle v_1 + v_2, y' \rangle \right. \\ &\quad \left. - \frac{i}{2q} (\|v_1 h\|_2^2 + \|v_2 h\|_2^2)\right\} df_1(v_1) df_2(v_2), \end{aligned}$$

and, by Theorem 2.2, it is easy to see that the last expression is equal to the right hand side of (4.1) and this completes the proof. \square

Next we show that the Parseval's relation for the generalized sequential Fourier-Feynman transform holds for functionals in $\hat{\mathcal{S}}$, which is a generalization of the Parseval's relation for the sequential Fourier-Feynman transform established in Theorem 3.3 of [6].

THEOREM 4.2. *Let F_j , $j = 1, 2$, be given as in Theorem 3.3. Then for all nonzero real number q , the Parseval's relation*

$$\begin{aligned} (4.2) \quad & \int^{\text{sf}-q} \Gamma_{q,h}(F_1)\left(\frac{Z_h(y, \cdot)}{\sqrt{2}}\right) \Gamma_{q,h}(F_2)\left(\frac{Z_h(y, \cdot)}{\sqrt{2}}\right) dy \\ &= \int^{\text{sf}q} F_1\left(\frac{Z_h(y, \cdot)}{\sqrt{2}}\right) F_2\left(-\frac{Z_h(y, \cdot)}{\sqrt{2}}\right) dy \end{aligned}$$

for the generalized sequential Fourier-Feynman transform holds.

Proof. Since we know from Theorems 2.2 and 3.3 that both sides of (4.1) belong to the Banach algebra $\hat{\mathcal{S}}$ as a function of y , by Theorem 2.2 of [16], they are generalized sequential Feynman integrable and

$$\begin{aligned} & \int^{\text{sf}-q} \Gamma_{q,h}(F_1)\left(\frac{Z_h(y, \cdot)}{\sqrt{2}}\right) \Gamma_{q,h}(F_2)\left(\frac{Z_h(y, \cdot)}{\sqrt{2}}\right) dy \\ &= \int^{\text{sf}-q} \Gamma_{q,h}((F_1 * F_2)_{q,h})(Z_h(y, \cdot)) dy. \end{aligned}$$

To evaluate the generalized sequential Feynman integral on the right hand side of the above expression, we apply Theorem 2.2 to the expression (3.3) for $(F_1 * F_2)_{q,h}(y)$ to obtain

$$\Gamma_{q,h}((F_1 * F_2)_{q,h})(y) = \int_{L_2[0,T]} \exp\{i\langle v, y' \rangle\} d(f_{q,h}^c)^t_{q,h}(v),$$

where $(f_{q,h}^c)^t_{q,h}$ is the measure in \mathcal{M} defined by (2.8) replaced f with $f_{q,h}^c$, and $f_{q,h}^c$ is the measure defined in Theorem 3.3. Now applying the evaluation formula for the

generalized sequential Feynman integral in Theorem 2.2 of [16], we obtain

$$\begin{aligned} & \int^{\text{sf}_{-q}} \Gamma_{q,h}((F_1 * F_2)_{q,h}(Z_h(y, \cdot))) dy \\ &= \int_{L_2[0,T]} \exp\left\{\frac{i}{2q} \|vh\|_2^2\right\} d(f_{q,h}^c)_{q,h}^t(v) \\ &= \int_{L_2^2[0,T]} \exp\left\{-\frac{i}{4q} \|(v_1 - v_2)h\|_2^2\right\} df_1(v_1) df_2(v_2), \end{aligned}$$

where the second equality is obtained by the definition of the measure $(f_{q,h}^c)_{q,h}^t$. On the other hand, since $\hat{\mathcal{S}}$ is a Banach space,

$$F_1\left(\frac{y}{\sqrt{2}}\right)F_2\left(-\frac{y}{\sqrt{2}}\right) = \int_{L_2^2[0,T]} \exp\left\{\frac{i}{\sqrt{2}} \langle v_1 - v_2, y' \rangle\right\} df_1(v_1) df_2(v_2)$$

belongs to $\hat{\mathcal{S}}$, and we apply the evaluation formula in Theorem 2.2 of [16] once more to obtain

$$\begin{aligned} & \int^{\text{sf}_q} F_1\left(\frac{Z_h(y, \cdot)}{\sqrt{2}}\right)F_2\left(-\frac{Z_h(y, \cdot)}{\sqrt{2}}\right) dy \\ &= \int_{L_2^2[0,T]} \exp\left\{-\frac{i}{4q} \|(v_1 - v_2)h\|_2^2\right\} df_1(v_1) df_2(v_2), \end{aligned}$$

which completes the proof. □

REMARK 4.3. From a careful look at the relationship (4.2), we can modify the proof of Theorem 4.2 to obtain the following alternative form of the Parseval's relation for the generalized sequential Fourier-Feynman transform:

$$\begin{aligned} (4.3) \quad & \int^{\text{sf}_{-q}} \Gamma_{q/2,h}(F_1)(Z_h(y, \cdot))\Gamma_{q/2,h}(F_2)(Z_h(y, \cdot)) dy \\ &= \int^{\text{sf}_q} F_1(Z_h(y, \cdot))F_2(-Z_h(y, \cdot)) dy, \end{aligned}$$

for $F_j, j = 1, 2$, as in Theorem 3.3 and for every nonzero real number q .

Since all the functionals we worked with in Theorems 3.4 through 3.6 belong to the Banach algebra $\hat{\mathcal{S}}$, by Theorems 4.1 and 4.2, we have the following results.

THEOREM 4.4. *Let F_1 and F_2 be given as in Theorems 3.4, 3.5 or 3.6. Then for all nonzero real number q , $\Gamma_{q,h}((F_1 * F_2)_{q,h})(y)$ exists for $y \in D[0, T]$ and relationship (4.1) holds.*

THEOREM 4.5. *Let F_1 and F_2 be given as in Theorems 3.4, 3.5 or 3.6. Then for all nonzero real number q , the Parseval's relations (4.2) and (4.3) for the generalized sequential Fourier-Feynman transform holds.*

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