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OSCILLATION CRITERIA OF DIFFERENTIAL EQUATIONS OF SECOND ORDER

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Abstract. We give sufficient conditions that the homogeneous differential equations : for $t \geq t_0 (> 0)$,

$$x''(t) + q(t)x'(t) + p(t)x(t) = 0,$$
$$x''(t) + q(t)x'(t) + F(t, x(\phi(t))) = 0$$

are oscillatory where $0 \leq \phi(t)$, $0 < \phi'(t)$, $\lim_{t\to\infty} \phi(t) = \infty$ and $F(t, u) \cdot \operatorname{sgn} u \geq p(t)|u|$. We obtain comparison theorems.

1. Introduction

In this paper, we are concerned with the differential equations of the types : for $t \in I = [t_0, \infty)$, $t_0 > 0$

$$x''(t) + q(t)x'(t) + p(t)x(t) = 0 \tag{1}$$

and

$$x''(t) + q(t)x'(t) + F(t, x(\phi(t))) = 0 \tag{2}$$

where $0 \leq \phi(t)$, $0 < \phi'(t)$ and $\lim_{t\to\infty} \phi(t) = \infty$. Throughout of this paper the coefficients $p(t)$ and $q(t)$ satisfy

(A) $p(t)$ and $q(t)$ are real valued and locally integrable over I.
(B) $p(t)$ is not identically zero in any neighborhood of ∞.

We assume that

$$\operatorname{sgn} F(t, u) = \operatorname{sgn} u \quad \text{and} \quad |F(t, u)| \geq p(t)|u|. \tag{H}$$

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By a solution to (1) we mean a real valued function u that satisfies (1) in I and that u and u' are locally absolutely continuous over I. We consider only nontrivial continuable solutions of (1). The usual existence theorems hold(see Naimark [6]). That is, given any real numbers c_1 and c_2 there is a unique solution u to (1) in I which satisfies $u(t_0) = c_1$ and $u'(t_0) = c_2$.

Definition. A solution $x(t)$ of (1) is said to be oscillatory if it has arbitrarily large zeros over I, otherwise it is said to be nonoscillatory.

It is well known (see Reid [7]) that either all the solutions of (1) are nonoscillatory, or all the solutions are oscillatory. In the former case, we call the differential equation (1) nonoscillatory and in the later case, (1) oscillatory.

The investigation of the oscillation for the equation

$$(E) \qquad (r(t)x'(t))' + q(t)x(t) = 0$$

may be done in the following many directions([1], [3]-[6], [10]) : among these, an often considered way is to determine "integral tests" involving functions r and q in order to obtain oscillatory criteria. An example is the following well-known Leighton's result(see [9]) : Every solution of (E) is oscillatory if

$$\int_0^\infty \frac{1}{r(\sigma)}\, d\sigma = \infty, \qquad \int_0^\infty q(\sigma)\, d\sigma = \infty.$$

2. Main results

We need the following lemma which is due to Agarwal[8].

Lemma 2.1. *Suppose that the following conditions are valid :*

(i) $u \in C^2[T, \infty)$ for some $T > 0$.

(ii) $u(t) > 0$, $u'(t) > 0$ *and* $u''(t) \le 0$ for $t \ge T > 0$.

Then,

(a) *for each* $k_1 \in (0, 1)$, *there exists a constant* $T_{k_1} \ge T$ *such that*

$$u(\phi(t)) \ge \frac{k_1 \phi(t)}{t} u(t), \quad \text{for} \quad t \ge T_{k_1}$$

(b) *for each* $k_2 \in (0, 1)$, *there exists a constant* $T_{k_2} \ge T$ *such that*

$$u(t) \ge k_2 t u'(t), \quad \text{for} \quad t \ge T_{k_2}.$$

Put $U(t) = \exp \int_{t_0}^{t} q(\sigma)\, d\sigma$.

THEOREM 2.2. *The equation (1) is oscillatory if for $t \geq t_0$, $p(t) > 0$ and*

$$\int_{t_0}^{\infty} 1/U(\sigma)\, d\sigma = \infty, \tag{3}$$

$$\int_{t_0}^{\infty} \left(p(\sigma) - \frac{q^2(\sigma)}{4} \right) d\sigma = \infty. \tag{4}$$

Proof. Assume that (1) is nonoscillatory. Then there exists a nonoscillatory solution $x(t)$ of (1). So we may assume that $x(t) > 0$ on $[t_1, \infty)$ for some $t_1 \geq t_0$. In the case of $x(t) < 0$, we put $y(t) = -x(t)$. Since

$$(U(t)x'(t))' = -U(t)p(t)x(t) \leq 0. \tag{5}$$

$U(t)x'(t)$ is decreasing for $t \geq t_1$. Assume that $U(t_1)x'(t_1) < 0$ for some $t_1 \geq t_0$. Put $C := U(t_1)x'(t_1)$. Then for $t \geq t_1$, we have

$$U(t)x'(t) \leq C. \tag{6}$$

Dividing both sides by $U(t)$ and integrating from t_1 to $t\,(\geq t_1)$ we obtain for $t \geq t_1$,

$$x(t) \leq x(t_1) + C \int_{t_1}^{t} 1/U(\sigma)\, d\sigma. \tag{7}$$

Thus it follows that $x(t) < 0$ for sufficiently large t and that $x'(t) > 0$ for $t \geq t_1$. Considering Ricatti transform

$$W(t) = \frac{x'(t)}{x(t)} \quad \text{for } t \geq t_1, \tag{8}$$

then we have

$$\begin{aligned} W'(t) &= -q(t)W(t) - p(t) - W^2(t) \\ &= -\left(W(t) + \frac{q(t)}{2} \right)^2 - \left(p(t) - \frac{q(t)^2}{4} \right). \end{aligned} \tag{9}$$

Integrating (9) from t_1 to $t(\geq t_1)$ we have

$$W(t) - W(t_1) + \int_{t_1}^{t} \left(p(\sigma) - \frac{q^2(\sigma)}{4} \right) d\sigma = - \int_{t_1}^{t} \left(W(\sigma) + \frac{q(\sigma)}{2} \right)^2 d\sigma. \tag{10}$$

By means of (4) there exists a $t_2 \geq t_1$ such that for $t \geq t_2$,

$$W(t) \leq -\int_{t_1}^{t} \left(W(\sigma) + \frac{q(\sigma)}{2}\right)^2 d\sigma, \tag{11}$$

which is impossible because $W(t) > 0$ for $t \geq t_1$. □

We note (see [9])that the equation $x''(t) + p(t)x(t) = 0$ is oscillatory if

$$\int_{t_0}^{\infty} p(\sigma)\, d\sigma = \infty. \tag{12}$$

Hence we can conclude that the differential equations (1) and $x''(t) + p(t)x(t) = 0$ are oscillatory if the estimates (3), (12) and $q(t) \in L^2[t_0, \infty)$ are valid.

Theorem 2.3. *Assume that for $t \geq t_0$, $p(t) \geq 0$ and that the differential equation (1) has a solution $x(t)$ satisfying $x(t)x'(t) < 0$ for $t \geq t_1 (> t_0)$. If*

$$\int_{t_0}^{\infty} \int_{t_0}^{\tau} \left(p(\sigma) - \frac{q^2(\sigma)}{4}\right) d\sigma\, d\tau = \infty \tag{13}$$

then $\lim_{t\to\infty} x(t) = 0$.

Proof. Let $x(t)$ be a solution of (1) such that $x(t) \cdot x'(t) < 0$ for $t \geq t_1$. Let $x(t) > 0$ and $x'(t) < 0$ for $t \geq t_1$. Put $W(t) = x'(t)/x(t)$ for $t \geq t_1$. By the method similar to the proof of theorem 2.2, we have

$$W(t) \leq W(t_1) - \int_{t_1}^{t} \left(p(\sigma) - \frac{q^2(\sigma)}{4}\right) d\sigma.$$

Integrating from t_1 to $t (> t_1)$ we obtain

$$\log \frac{x(t)}{x(t_1)} \leq W(t_1)(t - t_1) - \int_{t_1}^{t} \int_{t_1}^{\tau} \left(p(\sigma) - \frac{q^2(\sigma)}{4}\right) d\sigma\, d\tau.$$

By means of (13) we have our theorem. If $x(t) < 0$ and $x'(t) > 0$ for $t \geq t_1$, a similar argument holds. □

Corollary 2.4. *Let $F(t, u)$ satisfy the condition (H). We assume that for $t \geq t_0$, $p(t) > 0$, (3) and*

$$\int_{t_0}^{\infty} p(\sigma) U(\sigma)\, d\sigma = \infty \tag{14}$$

are satisfied. Then the functional differential equation

$$x''(t) + q(t)x'(t) + F(t, x(t)) = 0 \tag{15}$$

is oscillatory.

Proof. Multiplying (15) by the integrating factor $U(t)$ we obtain

$$(U(t)x'(t))' = -U(t)F(t, x(t)).$$

Assume that (15) is nonoscillatory. Then we may assume that there exists a nonoscillatory solution $x(t) > 0$ on $[t_1, \infty)$ for some $t_1 \geq t_0$. Put

$$W(t) = \frac{U(t)x'(t)}{x(t)} \tag{16}$$

for $t \geq t_1$. It is not difficult to show that $x'(t) > 0$ for $t \geq t_1$. Thus $W(t) > 0$ for $t \geq t_1$. After differentiating $W(t)$, integrating this term from t_1 to $t(> t_1)$, we have

$$W(t) \leq W(t_1) - \int_{t_1}^{t} p(\sigma)U(\sigma)\, d\sigma - \int_{t_1}^{t} \frac{W^2(\sigma)}{U(\sigma)}\, d\sigma.$$

In view of (14) there exists a $t_2 \geq t_1$ such that for $t \geq t_2$,

$$W(t) \leq -\int_{t_1}^{t} \frac{W^2(\sigma)}{U(\sigma)}\, d\sigma,$$

which is impossible. □

THEOREM 2.5. *Assume that (4) is valid. Then equation (1) is oscillatory if*

$$q(t) \leq 0 \quad \text{and} \quad q'(t) \leq 0 \text{ for } \quad t \geq t_0. \tag{17}$$

Proof. Suppose that this is not the case. Then the solution $x(t)$ of (1) eventually nonzero exists. Without loss of generality, we may assume that $x(t) > 0$ on $[t_1, \infty)$ for some $t_1 \geq t_0$. The process of proof is similar to that of theorem 2.3. Putting $W(t) = x'(t)/x(t)$ we have the equation (9). In view of (4), it follows that there exists a $t_3 \geq t_1$ such that (11) is valid for $t \geq t_3$. Put

$$V(t) = -\int_{t_1}^{t} \left(W(\sigma) + \frac{q(\sigma)}{2}\right)^2 d\sigma. \tag{18}$$

Immediately we have

$$V'(t) = -\left(W(t) + \frac{q(t)}{2}\right)^2.$$

In view of (17) we obtain

$$V'(t) + \frac{q'(t)}{2} \le V'(t) \le -\left(V(t) + \frac{q(t)}{2}\right)^2. \tag{19}$$

Multiplying both sides by $-1/\left(V(t)+q(t)/2\right)^2$ and integrating this term from t_3 to $t\,(\ge t_3)$ we have

$$\frac{1}{V(t)+q(t)/2} - \frac{1}{V(t_3)+q(t_3)/2} \ge t - t_3. \tag{20}$$

But this is impossible because

$$-\frac{1}{V(t_3)+q(t_3)/2} \ge \frac{1}{V(t)+q(t)/2} - \frac{1}{V(t_3)+q(t_3)/2} \tag{21}$$

and $\lim_{t\to\infty}(t-t_3) = +\infty$. □

Corollary 2.6. *Let $F(t,u)$ satisfy the condition (H). We assume that for $t \ge t_0$, (3) and (14) are satisfied. Then the equation (15) is oscillatory.*

Proof. Multiplying (15) by the integrating factor $U(t)$ we obtain

$$(U(t)x'(t))' = -U(t)F(t,x(t)).$$

Assume that (15) is nonoscillatory. Then we may assume that there exist a nonoscillatory solution $x(t)$ and $t_1(> t_0)$ such that $x(t) > 0$ on $[t_1,\infty)$. Put

$$W(t) = \frac{U(t)x'(t)}{x(t)}$$

for $t \ge t_1$. After differentiating $W(t)$, integrating this term from t_1 to $t(> t_1)$, we have

$$W(t) \le W(t_1) - \int_{t_1}^{t} p(s)U(s)\,ds - \int_{t_1}^{t} \frac{W^2(\sigma)}{U(\sigma)}\,d\sigma.$$

In view of (14) there exists a $t_2 \ge t_1$ such that for $t \ge t_2$,

$$W(t) \le -\int_{t_1}^{t} \frac{W^2(\sigma)}{U(\sigma)}\,d\sigma.$$

Put

$$X(t) = -\int_{t_1}^{t} \frac{W^2(\sigma)}{U(\sigma)}\,d\sigma. \tag{22}$$

Then $W(t) \leq X(t) < 0$ for $t \geq t_2$. Since $X'(t) \leq -\dfrac{X^2(t)}{U(t)}$, we get

$$\frac{1}{X(t)} - \frac{1}{X(t_2)} \geq \int_{t_2}^{t} \frac{1}{U(\sigma)}\, d\sigma. \tag{23}$$

But from the fact that

$$-\frac{1}{X(t_2)} \geq \frac{1}{X(t)} - \frac{1}{X(t_2)}$$

and (3), (23) is impossible. □

Let $\phi(t) \leq t$ and $g(t) = \sup\{s \geq t_0 \,|\, \phi(s) \leq t\}$. It is obvious that $t \leq g(t)$, and $\phi(s) = t$ if $g(t) \leq s$.

THEOREM 2.7. *Let $F(t,u)$ satisfy the condition (H). Assume that for $t \geq t_0$, $p(t) \geq 0$, $q(t) \geq 0$ and (3) are satisfied. Then the equation (2) is oscillatory if*

$$\int_{t_0}^{\infty} \left(p(\sigma)\frac{\phi(\sigma)}{\sigma} - \frac{q^2(\sigma)}{4} \right) d\sigma = \infty \tag{24}$$

is valid.

Proof. Assume the contrary that (2) is nonoscillatory. Let $x(t)$ be a nonoscillatory solution of (2). We may assume that there exists a $t_1(\geq t_0)$ such that $x(t)$ and $x(\phi(t))$ are positive for $t \geq t_1$. It follows that $x(t) > 0$, $x'(t) > 0$ and that $x''(t) \leq 0$ for $t \geq t_1$. By Lemma 2.1, for each $k_1 \in (0,1)$, there exists a constant $T_{k_1} \geq t_1$ such that

$$x(\phi(t)) \geq \frac{k_1\phi(t)}{t} x(t), \quad \text{for} \quad t \geq T_{k_1}.$$

Putting $W(t) = x'(t)/x(t)$, for $t \geq T_{k_1}$ we have

$$W'(t) \leq -\left(k_1 p(t)\frac{\phi(t)}{t} - \frac{q^2(t)}{4} \right).$$

Integrating from T_{k_1} to $t(> T_{k_1})$ we obtain

$$W(t) \leq W(T_{k_1}) - \int_{T_{k_1}}^{t} \left(k_1 p(\sigma)\frac{\phi(\sigma)}{\sigma} - \frac{q^2(\sigma)}{4} \right) d\sigma, \tag{25}$$

which leads us to a contradiction. □

THEOREM 2.8. *Assume that for $t \geq t_0$, $p(t) \geq 0$, $q(t) \geq 0$ and (3) are satisfied. Then the equation (2) is oscillatory if either*

$$\limsup_{t\to\infty} \frac{t}{U(t)} \int_t^\infty p(\sigma)\frac{\phi(\sigma)}{\sigma} U(\sigma)\, d\sigma > 1 \tag{26}$$

or

$$\limsup_{t\to\infty} \frac{t}{U(t)} \int_{g(t)}^\infty p(\sigma) U(\sigma)\, d\sigma > 1 \tag{27}$$

is valid.

Proof. Assume that (2) is nonoscillatory. Let $x(t)$ be a nonoscillatory solution of (2). We may assume that $x(t)$ and $x(\phi(t))$ are positive for $t \geq t_1$ for some $t_1 \geq t_0$. It is clear that there exists a $t_2(\geq t_1)$ such that $x'(t) > 0$ for $t \geq t_2$. Then it follows that $x''(t) \leq 0$ for $t \geq t_2$. Thus (a) and (b) of lemma 2.1 hold. For each $k_1 \in (0,1)$, there exists a constant $T_{k_1} \geq t_0$ such that $x(\phi(t)) \geq \dfrac{k_1\phi(t)}{t} x(t)$ for $t \geq T_{k_1}$ and for each $k_2 \in (0,1)$, there exists a constant $T_{k_2} \geq t_0$ such that $x(t) \geq k_2 t x'(t)$ for $t \geq T_{k_2}$. Since $(U(t)x'(t))' = -U(t)F(t, x(\phi(t)))$, we have, for $t \geq \max\{t_2, T_{k_1}, T_{k_2}\}$,

$$\begin{aligned} U(t)x'(t) &\geq \int_t^\infty U(\sigma)F(\sigma, \phi(\sigma))\, d\sigma \\ &\geq \int_t^\infty p(\sigma)x(\phi(\sigma))U(\sigma)\, d\sigma \\ &\geq \int_t^\infty p(\sigma)\frac{k_1\phi(s)}{s} U(\sigma)\, d\sigma \cdot x(t). \end{aligned} \tag{28}$$

Moreover, since

$$x'(t) \geq \frac{1}{U(t)} \int_t^\infty p(\sigma)\frac{k_1\phi(\sigma)}{\sigma} U(\sigma)\, d\sigma \cdot x(t). \tag{29}$$

and $x(t) \geq k_2 t x'(t)$, we obtain

$$1 \geq \frac{k_1 k_2 t}{U(t)} \int_t^\infty p(\sigma)\frac{\phi(\sigma)}{\sigma} U(\sigma)\, d\sigma. \tag{30}$$

Thus it follows that there exists a constant $c > 0$ such that

$$c = \limsup_{t\to\infty} \frac{t}{U(t)} \int_t^\infty p(\sigma)\frac{\phi(\sigma)}{\sigma} U(\sigma)\, d\sigma \tag{31}$$

holds. Assume that $c > 1$. There exists a sequence $\{t_n\}$ such that $\lim_{n\to\infty} t_n = \infty$ and

$$c = \lim_{n\to\infty} \frac{t_n}{U(t_n)} \int_{t_n}^{\infty} p(\sigma)\frac{\phi(\sigma)}{\sigma}U(\sigma)\,d\sigma. \tag{32}$$

Choose $\epsilon = \dfrac{c-1}{2} > 0$. Then for large n, we have

$$\frac{c+1}{2} = c - \epsilon < \frac{t_n}{U(t_n)} \int_{t_n}^{\infty} p(\sigma)\frac{\phi(\sigma)}{\sigma}U(\sigma)\,d\sigma. \tag{33}$$

If we take $0 < \dfrac{2}{c+1} = M < 1$. Then from (31) and (33) we have

$$1 \geq \frac{Mt_n}{U(t_n)} \int_{t_n}^{\infty} p(\sigma)\frac{\phi(\sigma)}{\sigma}U(\sigma)\,d\sigma > M \cdot \frac{c+1}{2} = 1,$$

which is a contradiction. Since $\phi(\sigma) = t$ if $g(t) \leq \sigma$, $x'(t) > 0$ and $t \geq \max\{t_2, T_{k_1}, T_{k_2}\}$, we find

$$\begin{aligned} x(t) &\geq k_2 t x'(t) \\ &\geq \frac{k_2 t}{U(t)} \int_t^{\infty} p(\sigma)x(\phi(\sigma))U(\sigma)\,d\sigma \\ &\geq \frac{k_2 t}{U(t)} \int_{g(t)}^{\infty} p(\sigma)x(\phi(\sigma))U(\sigma)\,d\sigma \\ &\geq \frac{k_2 t}{U(t)} \int_{g(t)}^{\infty} p(\sigma)U(\sigma)\,d\sigma \cdot x(t). \end{aligned}$$

Since

$$1 \geq \frac{k_2 t}{U(t)} \int_{g(t)}^{\infty} p(\sigma)U(\sigma)\,d\sigma,$$

the limit

$$\limsup_{t\to\infty} \frac{t}{U(t)} \int_{g(t)}^{\infty} p(\sigma)U(\sigma)\,d\sigma = d \tag{34}$$

exists. Assume that $d > 1$. There exists a sequence $\{T_n\}$ such that $\lim_{n\to\infty} T_n = \infty$ and

$$d = \lim_{n\to\infty} \frac{T_n}{U(T_n)} \int_{g(T_n)}^{\infty} p(\sigma)U(\sigma)\,d\sigma.$$

Choose $\epsilon = \dfrac{d-1}{2} > 0$. Then there exists a N such that $n \geq N$ implies

$$\frac{d+1}{2} = d - \epsilon < \frac{T_n}{U(T_n)} \int_{g(T_n)}^{\infty} p(\sigma)U(\sigma)\, d\sigma. \tag{35}$$

If we take $0 < \dfrac{2}{d+1} = M' < 1$. Then from (34) and (35) we have

$$1 \geq \frac{M'T_n}{U(T_n)} \int_{g(T_n)}^{\infty} p(\sigma)U(\sigma)\, d\sigma > M' \cdot \frac{d+1}{2} = 1,$$

which is impossible. □

EXAMPLE 2.9. Let $\phi(t) = t/2$ and $t_0 = 1$. Consider the following functional differential equation:

$$x''(t) + x'(t) + \frac{3}{t^2} F(t, x(t/2)) = 0. \tag{E_1}$$

Since

$$\begin{aligned} \frac{t}{e^t} \int_t^{\infty} \frac{3}{\sigma^2} \frac{\sigma/2}{\sigma} e^{\sigma}\, d\sigma &\geq \frac{1}{2} \cdot \frac{t}{e^t} \int_t^{\infty} \frac{3}{\sigma^2}\, d\sigma \cdot e^t \\ &\geq \frac{t}{2} \cdot \frac{3}{t} \\ &= \frac{3}{2} > 1, \end{aligned}$$

the inequality (26) holds. It follows that (E_1) is oscillatory.

Now we obtain comparison theorems.

THEOREM 2.10. *Let $p_1(t)$ be real valued and locally integrable over I. Assume that (3) and (4) are satisfied. If $0 < p(t) \leq p_1(t)$ on I then*

$$x''(t) + q(t)x'(t) + p_1(t)x(t) = 0 \tag{36}$$

is oscillatory.

THEOREM 2.11. *Let $p_1(t)$ be real valued and locally integrable over I. Assume that $q(t) < 0$ on I and that the equation (36) is nonoscillatory. If $p(t) \leq p_1(t)U(t)$ on I, then*

$$x''(t) + p(t)x(t) = 0 \tag{37}$$

is also nonoscillatory.

Proof. We note that $0 < U(t) \le 1$ and $p(t) \le p_1(t)U(t)$. The equation (36) becomes

$$(U(t)x'(t))' + p_1(t)U(t)x(t) = 0$$

which is a Sturm majorant for (37)(See [2]). □

THEOREM 2.12. *Let $p_1(t)$, $q_1(t)$ be real valued and locally integrable over I. Assume that $q(t) \ge q_1(t)$ and $p(t)U(t) \le p_1(t) \exp \int_{t_0}^{t} q_1(\sigma)\, d\sigma$ on I.*

$$x''(t) + q_1(t)x'(t) + p_1(t)x(t) = 0$$

is also oscillatory if the differential equation (1) is oscillatory.

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