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ON f-KENMOTSU MANIFOLDS ADMITTING SCHOUTEN-VAN KAMPEN CONNECTION

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Abstract. In the present paper, we study three-dimensional f-Kenmotsu manifolds admitting the Schouten-Van Kampen connection. We study the concircular curvature tensor of a three-dimensional f-Kenmotsu manifold with respect to the Schouten-Van Kampen connection. Finally, we have cited an example of a three-dimensional f-Kenmotsu manifold admitting Schouten-Van Kampen connection which verify our results.

1. Introduction

In 1978, Solov'ev investigated hyperdistributions in Riemannian manifolds using the Schouten-Van Kampen connection [15]. In 2006, Bejancu studied Schouten-Van Kampen connection on Foliated manifolds [2]. In 2014, Olszak studied the Schouten-van Kampen connection to adapt it to an almost contact metric structure [13]. He characterized some classes of an almost contact metric manifolds with the Schouten-Van Kampen connection. Recently, G. Ghosh [4], Yildiz [19], Nagaraj [10] and D. L. Kiran Kumar [7] have studied the Schouten-Van Kampen connection in Sasakian manifolds, f-Kenmotsu manifolds and Kenmotsu manifolds respectively. Also Y. S. Perktas and A. Yildiz [14] have studied on f-Kenmotsu 3-manifolds with respect to the Schouten-van Kampen connection.

A transformation of an n-dimensional differential manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation [6], [16]. A concircular transformation is always a conformal transformation [6]. Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus the geometry of concircular transformations, i.e., the concircular geometry, is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism. An interesting inverient of a concircular transformation is the concircular curvature tensor $\mathbb{W}$ with respect to Levi-Civita connection. It is defined by [16], [17]

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$$\mathbb{W}(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{1}$$

where $X, Y, Z \in \chi(M)$, R and r are the curvature tensor and the scalar curvature with respect to the Levi-Civita connection.

The concircular curvature tensor $\tilde{\mathbb{W}}$ with respect to the Schouten-Van Kampen connection is defined by

$$\tilde{\mathbb{W}}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{\tilde{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y], \tag{2}$$

where $\tilde{R}$ and $\tilde{r}$ are the curvature tensor and the scalar curvature with respect to the Schouten-Van Kampen connection. Riemannian manifolds with vanishing concircular curvature tensor are of constant curvature. Thus the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature. In the present paper we have studied f-Kenmotsu manifolds admitting Schouten-Van Kampen connection.

The present paper is organized as follows:

After the introduction, we give some required preliminaries in Section 2. In Section 3, we study the curvature tensor, the Ricci tensor, scalar curvature of a three-dimensional f-Kenmotsu manifold with respect to the Schouten-Van Kampen connection. Section 4 is devoted to obtain ξ-concircularly flat f-Kenmotsu manifolds with respect to the Schouten-Van Kampen connection. In this section we also prove that a f-Kenmotsu manifold admitting the Schouten-Van Kampen connection is ξ-concircularly flat if and only if the scalar curvature of the manifold vanishes. Section 5, we study f-Kenmotsu manifold admitting Schouten-Van Kampen connection satisfying $\tilde{\mathbb{W}}.\tilde{S} = 0$, where $\tilde{S}$ denotes the Ricci tensor with respect to the Schouten-Van Kampen connection. In the next section we study locally ϕ-Ricci symmetric three-dimensional f-Kenmotsu manifolds with respect to Schouten-Van Kampen connection. In the last section, we have cited an example of a three-dimensional f-Kenmotsu manifold admitting the Schouten-Van Kampen connection to support the results obtained in Section 3 and Section 4.

2. Preliminaries

Let M be a connected almost contact metric manifold with an almost contact metric structure (ϕ, ξ, η, g), that is, ϕ is an $(1,1)$ tensor field, ξ is a vector field, η is an 1-form and g is compatible Riemannian metric such that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\phi = 0, \tag{3}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \tag{4}$$

$$g(X, \phi Y) = -g(\phi X, Y), \quad g(X,\xi) = \eta(X), \tag{5}$$

for all $X, Y \in T(M)$.

The fundamental 2-form Φ of the manifold is defined by

$$\Phi(X,Y) = g(X,\phi Y), \tag{6}$$

for $X, Y \in T(M)$.

An almost contact metric manifold is normal if $[\phi,\phi](X,Y) + 2d\eta(X,Y)\xi = 0$. An almost contact metric structure (ϕ,ξ,η,g) on a manifold M is called f-Kenmotsu manifold if this may be expressed by the condition [11]

$$(\nabla_X\phi)Y = f\{g(\phi X,Y)\xi - \eta(Y)\phi X\}, \tag{7}$$

where $f \in C^\infty(M)$ such that $df \wedge \eta = 0$ and ∇ is Levi-Civita connection on M. If $f = \alpha$=constant$\neq 0$, then the manifold is an α-Kenmotsu manifold [5]. 1-Kenmotsu manifold is a Kenmotsu [8]. If f=0, then the manifold is cosymplectic [5]. An f-Kenmotsu manifold is said to be to be regular if $f^2 + f' \neq 0$, where $f' = \xi(f)$.

For an f-Kenmotsu manifold it follows that

$$\nabla_X\xi = f\{X - \eta(X)\xi\}. \tag{8}$$

Then using (8), we have

$$(\nabla_X\eta)Y = f(g(X,Y) - \eta(X)\eta(Y)). \tag{9}$$

The condition $df \wedge \eta = 0$ holds if dim $M \geqslant 5$. This does not hold in general if dim M=3 [12]. In a 3-dimensional f-Kenmotsu manifold M, we have [12]

$$\begin{aligned} R(X,Y)Z &= (\frac{r}{2} + 2f^2 + 2f')\{g(Y,Z)X - g(X,Z)Y\} \\ &- (\frac{r}{2} + 3f^2 + 3f')\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}, \end{aligned} \tag{10}$$

$$S(X,Y) = (\frac{r}{2} + f^2 + f')g(X,Y) - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\eta(Y), \tag{11}$$

$$QX = (\frac{r}{2} + f^2 + f')X - (\frac{r}{2} + 3f^2 + 3f')\eta(X)\xi, \tag{12}$$

where R denotes the curvature tensor, S is the Ricci tensor of type $(0,2)$, Q is the Ricci operator and r is the scalar curvature of the manifold M.

From (10) and (11), we have

$$R(X,Y)\xi = -(f^2 + f')\{\eta(Y)X - \eta(X)Y\}, \tag{13}$$

$$R(\xi,X)Y = -(f^2 + f')\{g(X,Y)\xi - \eta(Y)X\}, \tag{14}$$

$$S(X,\xi) = -2(f^2 + f')\eta(X), \tag{15}$$

$$\eta(R(\xi,X)Y) = -(f^2 + f')\{g(X,Y) - \eta(Y)\eta(X)\}, \tag{16}$$

3. Curvature tensor of a three-dimensional f-Kenmotsu manifold with respect to the Schouten-Van Kampen connection

The Schouten-Van Kampen connections [9], [13] $\tilde{\nabla}$ and the Levi-Civita connection ∇ are related by

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi, \tag{17}$$

for all vector fields X, Y on M.

With the help of (8) and (9), the above equation takes the form

$$\tilde{\nabla}_X Y = \nabla_X Y + f\{g(X,Y)\xi - \eta(Y)X\}, \tag{18}$$

for a f-Kenmotsu manifold.

We define the curvature tensor of a three-dimensional f-Kenmotsu manifold with respect to the Schouten-Van Kampen connection $\tilde{\nabla}$ by

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z. \tag{19}$$

In view of (18) we obtain

$$\begin{aligned}\tilde{R}(X,Y)Z &= R(X,Y)Z + f^2\{g(Y,Z)X - g(X,Z)Y\} \\ &\quad + f'\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &\quad + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}.\end{aligned} \tag{20}$$

Taking inner product of (19) with W we have

$$\begin{aligned}\tilde{R}(X,Y,Z,W) &= R(X,Y,Z,W) + f^2\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ &\quad + f'\{g(Y,Z)\eta(X)\eta(W) - g(X,Z)\eta(Y)\eta(W) \\ &\quad + g(X,W)\eta(Y)\eta(Z) - g(Y,W)\eta(X)\eta(Z)\}.\end{aligned} \tag{21}$$

where $\tilde{R}(X,Y,Z,W) = g(\tilde{R}(X,Y)Z,W)$.

From (20) we have

$$\tilde{R}(X,Y)Z + \tilde{R}(Y,X)Z = 0, \tag{22}$$

and

$$\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0. \tag{23}$$

Putting $Z = \xi$ in (19) and using (13) we get

$$\tilde{R}(X,Y)\xi = 0. \tag{24}$$

Again putting $W = \xi$ in (21) we have

$$\eta(\tilde{R}(X,Y)Z) = R(X,Y,Z,\xi) + (f^2 + f')\{g(Y,Z)\eta(X) - g(X,Z)\eta(Y)\}, \tag{25}$$

Putting $X = W = e_i, \{i = 1, 2, 3\}$, in (21), we get

$$\tilde{S}(Y,Z) = S(Y,Z) + (2f^2 + f')g(Y,Z) + f'\eta(Y)\eta(Z), \tag{26}$$

From (26), implies that

$$\tilde{S}(Y,Z) = \tilde{S}(Z,Y), \tag{27}$$

$$\tilde{Q}X = QX + (2f^2 + f')X + f'\eta(X)\xi. \tag{28}$$

$$\tilde{r} = r + 6f^2 + 4f', \tag{29}$$

where $\tilde{r}$ and r are the scalar curvatures of the connection $\tilde{\nabla}$ and ∇ respectively.

Putting $Z = \xi$ in (26) and using (15) we get

$$\tilde{S}(Y,\xi) = -2f'\eta(Y), \tag{30}$$

Hence we can state the following :

PROPOSITION 3.1. *For a three-dimensional f-Kenmotsu manifold M with respect to the Schouten-Van Kampen connection* $\tilde{\nabla}$

(a) *the curvature tensor $\tilde{R}$ is given by (20),*
(b) *the Ricci tensor $\tilde{S}$ is given by (26),*
(c) $\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0$,
(d) $\tilde{R}(X,Y)Z + \tilde{R}(Y,X)Z = 0$,
(e) *the scalar curvature $\tilde{r}$ is given by $\tilde{r} = r + 6f^2 + 4f'$,*
(f) *the Ricci tensor $\tilde{S}$ is symmetric.*

4. ξ-Concircularly flat and ϕ-Concircularly flat f-Kenmotsu manifolds with respect to the Schouten-Van Kampen connection

Definition 4.1. A f-Kenmotsu manifold M with respect to the Schouten-Van Kampen connection is said to be ξ-concircularly flat if

$$\tilde{\mathbb{W}}(X,Y)\xi = 0, \tag{31}$$

for all vector fields $X, Y \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M.

Theorem 4.1. *A three-dimensional f-Kenmotsu manifold with respect to the Schouten-Van Kampen connection is ξ-concircularly flat if and only if the manifold M with respect to the Levi-Civita connection is also ξ-concircular flat provided f is a constant.*

Proof. Combining (1), (2), (20) and (29), we get

$$\begin{aligned}\tilde{\mathbb{W}}(X,Y)Z = \mathbb{W}(X,Y)Z &- \frac{2}{3}f'\{g(Y,Z)X - g(X,Z)Y\}\\ &+ f'\{g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi\\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}.\end{aligned} \tag{32}$$

Putting $Z = \xi$ in (32) we get

$$\tilde{\mathbb{W}}(X,Y)\xi = \mathbb{W}(X,Y)\xi + \frac{f'}{3}\{\eta(Y)X - \eta(X)Y\}. \tag{33}$$

Hence the proof of theorem is completed. □

Theorem 4.2. *A three-dimensional f-Kenmotsu manifold is ξ-concircularly flat with respect to the Schouten-Van Kampen connection if and only if the scalar curvature with respect to the Schouten-Van Kampen connection vanishes.*

Proof. Putting $Z = \xi$ in (2) and using (4) and (24), we have

$$\tilde{\mathbb{W}}(X,Y)\xi = -\frac{\tilde{r}}{6}\{\eta(Y)X - \eta(X)Y\}. \tag{34}$$

Thus the theorem is proved. □

Theorem 4.3. *A f-Kenmotsu manifold admitting Schouten-Van Kampen connection is ϕ-concircular flat if and only if the manifold is an η-Einstein manifold with respect to the Levi-Civita connection.*

Proof. From (2) it follows that

$$g(\tilde{\mathbb{W}}(\phi X,\phi Y)\phi Z,\phi U)=g(\tilde{R}(\phi X,\phi Y)\phi Z,\phi U) -\frac{\tilde{r}}{6}\{g(\phi Y,\phi Z)g(\phi X,\phi U) - g(\phi X,\phi Z)g(\phi Y,\phi W)\}. \tag{35}$$

Suppose

$$g(\tilde{\mathbb{W}}(\phi X,\phi Y)\phi Z,\phi U)=0. \tag{36}$$

Then from (35) we get

$$0=g(\tilde{R}(\phi X,\phi Y)\phi Z,\phi U) -\frac{\tilde{r}}{6}\{g(\phi Y,\phi Z)g(\phi X,\phi U) - g(\phi X,\phi Z)g(\phi Y,\phi W)\}. \tag{37}$$

Let $\{e_1,e_2,\xi\}$ be a local orthonormal basis of the vector fields in M and using the fact that $\{\phi e_1,\phi e_2,\xi\}$ is also a local orthonormal basis, putting $X=U=e_i$ in (37) and summing up with respect to i, we have

$$0=\sum_{i=1}^{2} g(\tilde{R}(\phi X,\phi Y)\phi Z,\phi U) -\frac{\tilde{r}}{6}\sum_{i=1}^{2}\{g(\phi Y,\phi Z)g(\phi X,\phi U) - g(\phi X,\phi Z)g(\phi Y,\phi W)\}. \tag{38}$$

From the above equation it follows that

$$\tilde{S}(\phi Y,\phi Z)=\frac{\tilde{r}}{6}g(\phi Y,\phi Z). \tag{39}$$

Putting $X=\phi X$, $Y=\phi Y$ in (39) and using (3), (15) and (26) we get

$$S(Y,Z)=[\frac{r-6f^2-2f'}{6}]g(Y,Z)-[\frac{r+6f^2+10f'}{6}]\eta(Y)\eta(Z). \tag{40}$$

Conversely, let S be of the form (40), then obviously

$$g(\tilde{\mathbb{W}}(\phi X,\phi Y)\phi Z,\phi U)=0.$$

Thus the theorem is proved. □

5. Almost η-Ricci soliton on f-Kenmotsu manifold admitting Schouten-Van Kampen connection $\tilde{\nabla}$ satisfying $\tilde{\mathbb{W}}.\tilde{S}$=0

In this section we consider f-Kenmotsu manifold admitting Schouten-Van Kampen connection $\tilde{\nabla}$ satisfying $\tilde{\mathbb{W}}.\tilde{S} = 0$.

THEOREM 5.1. *A three-dimensional f-Kenmotsu manifold with respect to the Schouten-Van Kampen connection satisfies $\tilde{\mathbb{W}}.\tilde{S}$=0, then the manifold M is an Einstein manifold with respect to the Schouten-Van Kampen connection provided f is not a constant and $\tilde{r} \neq 0$.*

Proof. We suppose that the manifold under consideration is the Schouten-Van Kampen connection M, that is

$$(\tilde{\mathbb{W}}(X,Y).\tilde{S})(U,V) = 0,$$

where $X, Y, U, V \in \chi(M)$, $\chi(M)$ is the set of all differentiable vector fields on M.

Then we have

$$\tilde{S}(\tilde{\mathbb{W}}(X,Y)U,V) + \tilde{S}(U,\tilde{\mathbb{W}}(X,Y)V) = 0. \tag{41}$$

Putting $U = \xi$ in (41) and using (2) we have

$$\begin{aligned} 0 = {} & \frac{\tilde{r}}{6}[\eta(Y)\tilde{S}(X,V) - \eta(X)\tilde{S}(Y,V)] \\ & + 2f'[g(\tilde{R}(X,Y)V,\xi) - \frac{\tilde{r}}{6}\{g(Y,V)\eta(X) - g(X,V)\eta(Y)\}]. \end{aligned} \tag{42}$$

Again putting $X = \xi$ in (42) and using (25), (26), (29) and (30) we have

$$\frac{\tilde{r}}{6}\{\tilde{S}(Y,V) - 2f'g(Y,V)\} = 0. \tag{43}$$

Then above equation implies that

$$\tilde{S}(Y,V) = -2f'g(Y,V), \tag{44}$$

provided $\tilde{r} \neq 0$.
Hence the theorem is proved. □

6. Locally ϕ-Ricci symmetry on f-Kenmotsu Manifold with respect to the Schouten-Van Kampen connection

DEFINITION 6.1. A f-Kenmotsu manifold is said to be ϕ-Ricci symmetric if the Ricci operator satisfies

$$\phi^2((\nabla_X Q)Y) = 0, \tag{45}$$

for all vector fields X, Y in M and $S(X,Y) = g(QX,Y)$. If X, Y are orthogonal to ξ, then the manifold is manifold is said to be locally ϕ-Ricci symmetric. The notion of ϕ-symmetry was introduced by E. Boeckx, P. Buecken and L. Vanhecke [1]. In [3], De and Sarkar studied ϕ-Ricci symmetric sasakian manifolds.

THEOREM 6.1. *A three-dimensional f-Kenmotsu manifold locally ϕ-Ricci symmetry with respect to the Schouten-Van Kampen connection and the Levi-Civita connection are equivalent.*

Proof. We have

$$(\tilde{\nabla}_X \tilde{Q})Y = \tilde{\nabla}\tilde{Q}Y - \tilde{Q}(\tilde{\nabla}_X Y). \tag{46}$$

Using (18) and (28) in (46)

$$(\tilde{\nabla}_X \tilde{Q})Y = \tilde{\nabla}\tilde{Q}Y + f'(\tilde{\nabla}_X \eta)(Y)\xi + f'\eta(Y)\tilde{\nabla}_X \xi - Q(\tilde{\nabla}_X Y) - f'\eta(\tilde{\nabla}_X Y)\xi. \tag{47}$$

Again using (12), (18) and (28) in (47) we have

$$\begin{aligned}(\tilde{\nabla}_X \tilde{Q})Y = (\nabla_X Q)Y + f\{g(X,QY)\xi - \eta(QY)X\} + f'(\tilde{\nabla}_X \eta)(Y)\xi \\ - f\{g(X,Y)Q\xi - \eta(Y)QX\} - f'\eta(\tilde{\nabla}_X Y)\xi.\end{aligned} \tag{48}$$

Considering X, Y orthogonal to ξ and using (3), (12) from (48) it follows that

$$\phi^2(\tilde{\nabla}_X \tilde{Q})Y = \phi^2(\nabla_X Q)Y. \tag{49}$$

Thus the theorem is proved. □

7. Example

We consider an example of a three-dimensional manifold $M = \{(x, y, z) \in R^3, z \neq 0\}$, where (x, y, z) are the standard coordinates in R^3 [14]. The vector fields

$$e_1 = z^2 \frac{\partial}{\partial x}, \quad e_2 = z^2 \frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z},$$

are linearly independent at each point of M. Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any $Z \in \chi(M)$. Let ϕ be the (1,1) tensor field defined by $\phi(e_1) = -e_2$, $\phi(e_2) = e_1$, $\phi(e_3) = 0$. Then using the linearity of ϕ and g we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(M)$. Thus for $e_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on M. Now, by direct computations we obtain

$$[e_1, e_2] = 0, \quad [e_2, e_3] = -\frac{2}{z}e_2, \quad [e_1, e_3] = -\frac{2}{z}e_1.$$

By Koszul formula we have

$$\begin{array}{lll} \nabla_{e_1} e_3 = -\frac{2}{z}e_1, & \nabla_{e_1} e_2 = 0, & \nabla_{e_1} e_1 = \frac{2}{z}e_3, \\ \nabla_{e_2} e_3 = -\frac{2}{z}e_2, & \nabla_{e_2} e_2 = \frac{2}{z}e_3, & \nabla_{e_2} e_1 = 0, \\ \nabla_{e_3} e_3 = 0, & \nabla_{e_3} e_2 = 0, & \nabla_{e_3} e_1 = 0. \end{array}$$

From above we see that the manifold satisfies $\nabla_X \xi = f(X - \eta(X)\xi)$ for $\xi = e_3$, where $f = -\frac{2}{z}$. Hence the manifold is a f-Kenmotsu manifold. Also $f^2 + f' \neq 0$. Hence M is a regular f-Kenmotsu manifold [18].

In [18] the authors obtained the expression of the curvature tensor as follows:

$$\begin{array}{lll} R(e_1, e_2)e_3 = 0, & R(e_2, e_3)e_3 = -\frac{6}{z^2}e_2, & R(e_1, e_3)e_3 = -\frac{6}{z^2}e_1, \\ R(e_1, e_2)e_2 = -\frac{4}{z^2}e_1, & R(e_2, e_3)e_2 = \frac{6}{z^2}e_3, & R(e_1, e_3)e_2 = 0, \\ R(e_1, e_2)e_1 = \frac{4}{z^2}e_2, & R(e_2, e_3)e_1 = 0, & R(e_1, e_3)e_1 = \frac{6}{z^2}e_3. \end{array}$$

Now using above relations we get from ([19]) as follows:

$$\begin{array}{lll} \tilde{\nabla}_{e_1} e_3 = (-\frac{2}{z} - f)e_1, & \tilde{\nabla}_{e_1} e_2 = 0, & \tilde{\nabla}_{e_1} e_1 = \frac{2}{z}(e_3 - \xi), \\ \tilde{\nabla}_{e_2} e_3 = (-\frac{2}{z} - f)e_2, & \tilde{\nabla}_{e_2} e_2 = \frac{2}{z}(e_3 - \xi), & \tilde{\nabla}_{e_2} e_1 = 0, \\ \tilde{\nabla}_{e_3} e_3 = -f(e_3 - \xi), & \tilde{\nabla}_{e_3} e_2 = 0, & \tilde{\nabla}_{e_3} e_1 = 0. \end{array}$$

From above we see that $\tilde{\nabla}_{e_i} e_j = 0, (0 \leq i, j \leq 3)$ for $\xi = e_3$ and $f = -\frac{2}{z}$. Hence the manifold is f-Kenmotsu manifold with respect to Schouten-Van Kampen connection.

From the above expressions of the curvature tensor we obtain the Ricci tensor as follows:

$$S(e_1, e_1) = \sum_{i=1}^{3} g(R(e_i, e_1)e_1, e_i) = -\frac{10}{z^2}.$$

Similarly, we have

$$S(e_2, e_2) = -\frac{10}{z^2} \quad \text{and} \quad S(e_3, e_3) = -\frac{12}{z^2}.$$

Therefore, the scalar curvature tensors $r = \sum_{i=1}^{3} S(e_i, e_i) = -\frac{32}{z^2}$ and $\tilde{r} = \sum_{i=1}^{3} \tilde{S}(e_i, e_i) = 0$ with respect to Levi-Civita connection and Schouten-Van Kampen connection respectively. Hence for $f = -\frac{2}{z}$, PROPOSITION 3.1. is verified. Also the THEOREM 4.2. is verified.

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References

[1] E. Boeckx, P. Buecken and L. Vanhecke, *ϕ-symmetric contact metric spaces,* Glasgow Math. J. **52** (2005), 97–112.

[2] A. Bejancu, *Schouten-van Kampen and Vranceanu connections on Foliated manifolds,* Anale Ştintifice Ale Universitati."AL. I. CUZA' IASI, Tomul LII, Mathematica, 2006, 37–60.

[3] U. C. De and A. Sarkar, *ϕ-Ricci symmetric Sasakian manifolds*, Proceedings of the Janjeon Mathematical Society **11** (2008), 47–52.

[4] G. Ghosh, *On Schouten-van Kampen connection in Sasakian manifolds*, Boletim da Sociedade Paranaense de Mathematica **36** (2018), 171–182.

[5] D. Janssens, and L. Vanheck, *Almost contact structures and curvature tensors*, Kodai Math. J. **4** (1981), 1–27.

[6] W. Kühnel, *Conformal transformatons between Einstein spaces*, Conformal geometry (Bonn, 1985/1986), 105-146, Aspects Math. E12, Friedr. Vieweg, Braunschweing, 1988.

[7] D. L. Kiran Kumar, H. G. Nagaraja and S. H. Naveenkumar, *Some curvature properties of Kenmotsu manifolds with Schouten-van Kampen connection*, Bull. of the Transilvania Univ. of Brasov., Series III: Math. Informatics, Phys. **2** (2019), 351–364.

[8] K. Kenmotsu, *A class of almost contact Riemannian manifolds*, Tohoku math. J. **24** (1972), 93–103.

[9] A. Kazan and H. B. Karadağ, *Trans-Sasakian manifolds with Schouten-Van Kampen connection*, Ilirias J. of Math. **7** (2018), 1–12.

[10] H. G. Nagaraja and D. L. Kiran Kumar, *Kenmotsu manifolds admitting Schouten-van Kampen connection*, Facta Univ. Series: Math. and Informations **34** (2019), 23–34.

[11] Z. Olszak, *Locally conformal almost cosymplectic manifolds*, Colloq. Math. **57** (1989), 73–87.

[12] Z. Olszak and R. Rosca, *Normal locally conformal almost cosymplectice manifolds*, Publ. Math. **39** (1991), 315–323.

[13] Z. Olszak, *The Schouten-Van Kampen affine connection adapted to an almost(para) contact metric structure*, Publications Delinstitut Mathematique **94** (2013), 31–42.

[14] Y. S. Perktas and A. Yildiz, *On f-Kenmotsu 3-manifolds with respect to the Schouten-van Kampen connection*, Turkis J. of Math. **45** (2021), 387–409.

[15] A. F. Solov'ev, *On the curvature of the connection induced on a hyperdistribution in a Riemannian space*, Geom. Sb. **19** (1978), 12–23 (in Russian).

[16] K. Yano, *Concircular geometry I. concircular transformations*, Proc. Inst. Acad. Tokyo **16** (1940), 195–200.

[17] K. Yano and S. Bochner, *Curvature and Betti numbers*, Annals of Math. Studies 32, Princeton university press, 1953.

[18] A. Yildiz, U. C. De and M. Turan, *On 3-dimensional f-Kenmotsu manifolds and Ricci soliton*, Ukrainian J. Math. **65** (2013), 620–628.

[19] A. Yildiz, *f-Kenmotsu manifolds with the Schouten-Van Kampen connection*, Pub. De L'institut Math. **102** (116) (2017), 93–105.

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