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ON SEQUENTIALLY g-CONNECTED COMPONENTS AND SEQUENTIALLY LOCALLY g-CONNECTEDNESS

Palanichamy Vijayashanthi

Abstract. In this paper, we introduce the definition of sequentially g-connected components and sequentially locally g-connected by using sequentially g-closed sets. Moreover, we investigate some characterization of sequentially g-connected components and sequentially locally g-connected.

1. Introductions

Generalized closed sets is a vital role in General Topology. The concept of generalized closed set (briefly, g-closed set) of a topological space and a class of topological spaces called $T_{1/2}$-spaces was introduced by Levine [7]. Also, these sets were considered first by Dunham and Levine [4] and then by Dunham [3].

The purpose of this paper is to introduce and study the concepts of a sequentially g-connected components and sequentially locally g-connected space by using sequentially g-closed sets. Throughout this paper, we consider a topological space (X, τ) and investigate some results in this generalized setting.

2. Preliminaries

We recall the following definitions.

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is called *g-closed* [7] if $cl(A) \subset G$ holds whenever $A \subset G$ and G is open in X.

A is called *g-open* of X if its complement A^c is g-closed in X. Every open set is g-open [7].

Lemma 2.2. *A topological space X is said to be $T_{1/2}$ [1] if every g-closed set in X is closed in X.*

Definition 2.3. Let (X, τ) be a topological space. A subset A of X is called *sequentially closed* [5] if for every sequence (x_n) in A with $(x_n) \to x$, then $x \in A$.

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Definition 2.4. Let (X, τ) be a topological space. A sequence (x_n) in a space X *g-converges* to a point $x \in X$ [2] if (x_n) is eventually in every g-open set containing x and is denoted by $(x_n) \xrightarrow{g} x$ and x is called the g-limit of the sequence (x_n), denoted by $glim\ x_n$.

Definition 2.5. Let (X, τ) be a topological space. A subset A of X is called *sequentially g-closed* [2] if every sequence in A g-converges to a point in A. A *sequentially g-open* subset U (which is the complement of a sequentially g-closed set) is one in which every sequence in X which g-converges to a point in U is eventually in U.

Definition 2.6. Let (X, τ) and (Y, σ) be any two topological spaces. Then a map $f : (X, \tau) \to (Y, \sigma)$ is said to be *sequentially g-continuous at* $x \in X$ [2] if the sequence $(f(x_n)) \xrightarrow{g} f(x)$ whenever the sequence $(x_n) \xrightarrow{g} x$. If f is sequentially g-continuous at each $x \in X$, then it is said to be a sequentially g-continuous function.

3. Sequentially g-connected

In this section, we discuss characterization of Sequentially g-connected in topological spaces.

Definition 3.1. A subset A of a topological space (X, τ) is called a *g-neighborhood* of a point $x \in X$ if there exists a g-open set U with $x \in U \subset A$.

Definition 3.2. Let (X, τ) be a topological space, $A \subset X$ and let $S[A]$ be the set of all sequences in A. Then the sequential g-closure of A, denoted by $[A]_{g_{seq}}$, is defined as

$$[A]_{g_{seq}} = \{x \in X \mid x = glim\ x_n \text{ and } (x_n) \in S[A] \cap c_g(X)\}$$

$c_g(X)$ denote the set of all g-convergent sequences in X.

Lemma 3.3. *Let (X, τ) be a topological space. Then the following hold.*

(a) *Every g-convergence sequence is convergence sequence.*

(b) *In $T_{1/2}$ space, convergence coincides with g-convergence.*

Proof. (a) Suppose that (x_n) be a sequence in X such that $(x_n) \xrightarrow{g} x$. Let U be a neighborhood of x. Since every open set is g-open, U is a g-open neighborhood of x. Therefore, there exists $N \in \mathbb{N}$ such that $(x_n) \in U$ for all $n \geq N$. Thus, $(x_n) \to x$.

(b) Let (x_n) be a sequence in X. Suppose $(x_n) \xrightarrow{g} x$, then by (a), every g-convergence sequence is a convergence sequence. Suppose that $(x_n) \to x$. Let U be a g-open neighborhood of x. Since X is a $T_{1/2}$ space, U is a open neighborhood of x. Since $(x_n) \to x$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq n_0$. Therefore, $(x_n) \xrightarrow{g} x$. Hence the convergence g-convergence of sequences coincide in a $T_{1/2}$ space.

Theorem 3.4. *Let (X, τ) be a topological space and A be a subset of X. Then A is sequentially g-open if and only if each $b \in A$ has a sequentially g-neighborhood U_b such that $U_b \subseteq A$.*

Proof. It is enough to prove that the union of any collection of sequentially g-open subsets of X is sequentially g-open.

Let $\{A_\alpha \mid \alpha \in \Delta\}$ be a collection of sequentially g-open subsets of X where α is an arbitrary index set. If $\cup A_\alpha = \emptyset$, then there is nothing to prove. Suppose $\cup A_\alpha \neq \emptyset$.

Then $x \in [\cup A_\alpha]_{g_{seq}}$ and so there exists a sequence (x_n) of points in $\bigcup_{\alpha\in\Delta} A_\alpha$ such that $(x_n) \xrightarrow{g} x$. Therefore, for each $\alpha \in \Delta$, (x_n) are points in A_α such that $(x_n) \xrightarrow{g} x$. Therefore, $x \in [A_\alpha]_{g_{seq}}$ for each $\alpha \in \Delta$. As each A_α is sequentially g-closed, $x \in A_\alpha$ for each $\alpha \in \Delta$. Thus, $x \in \bigcup_{\alpha\in\Delta} A_\alpha$ and so $[\bigcup_{\alpha\in\Delta} A_\alpha]_{g_{seq}} \subset \bigcup_{\alpha\in\Delta} A_\alpha$. By the definition of sequentially g-open, $\bigcup_{\alpha\in\Delta} A_\alpha$ is sequentially g-open in X.

Definition 3.5. A nonempty subset A of a topological space (X, τ) is called *sequentially g-connected* if there are no nonempty and disjoint sequentially g-closed subsets U and V such that $A \subseteq U \cup V$, and $A \cap U$ and $A \cap V$ are nonempty. In particular, X is called sequentially g-connected if there are no nonempty, disjoint sequentially g-closed subsets of X whose union is X.

Lemma 3.6. *Let A be a sequentially g-connected subset of X. If U and V are nonempty disjoint sequentially g-closed subsets of X such that $A \subseteq U \cup V$, then either $A \subseteq U$ or $A \subseteq V$.*

Proof. Suppose that $A \nsubseteq U$ and $A \nsubseteq V$. Now, $A \nsubseteq U$ implies that there exists an $x \in A$ such that $x \notin U$. Since $A \subseteq U \cup V$, $x \in V$ and so $A \cap V$ is not empty. Similarly, $A \nsubseteq V$ implies that $A \cap U$ is not empty. This contradiction completes the proof.

Theorem 3.7. *Let (X, τ) be a topological space and A is a sequentially g-connected subset of X, then $[A]_{g_{seq}}$ is also sequentially g-connected.*

Proof. If $A \subseteq [A]_{g_{seq}}$, then $A \subseteq [A]_{g_{seq}} \cap A = [A]_{g|_{A_{seq}}}$. On the other hand, $[A]_{g|_{A_{seq}}} \subseteq A$. Therefore, $[A]_{g|_{A_{seq}}} = A$, where $[A]_{g|_{A_{seq}}}$ is the sequential g-closure of A in A. Conversely, suppose that $[A]_{g_{seq}}$ is not sequentially g-connected. So there are nonempty and disjoint sequentially g-closed subsets U and V of X such that $[A]_{g_{seq}} \subseteq U \cup V$ and $[A]_{g_{seq}} \cap U$ and $[A]_{g_{seq}} \cap V$ are nonempty. Since A is sequentially g-connected and by Lemma 3.6, either $A \subseteq U$ or $A \subseteq V$. If $A \subset U$, then $[A]_{g_{seq}} \subseteq [U]_{g_{seq}}$, and so $[A]_{g|_{A_{seq}}} = [U]_{g_{seq}} \cap A$. Since U is sequentially g-closed in X, $[U]_{g_{seq}} = U$. So we have that $A = [A]_{g|_{A_{seq}}} = U \cap A$, which implies that $A = A \cap U$. Similarly, if $A \subseteq V$, then $A = A \cap V$. This contradiction completes the proof.

Theorem 3.8. *Let $\{A_j \mid j \in I\}$ be a class of sequentially g-connected subsets of X. If $\bigcap_{j\in I} A_j \neq \emptyset$, then $\bigcup_{j\in I} A_j$ is sequentially g-connected.*

Proof. Suppose that A is not sequentially g-connected, then there exist nonempty disjoint sequentially g-closed subsets U and V of X such that $A \subseteq U \cup V$. For each A_j is sequentially g-connected, by Lemma 3.6, either $A_j \subseteq U$ or $A_j \subseteq V$. If $A_j \subseteq U$ and $A_k \subseteq V$ for $j \neq k$, then $A_j \cap A_k = \emptyset$. Because $\bigcup_{j\in I} A_j$ is nonempty, for all $j \in I$, either $A_j \subseteq U$ or $A_j \subseteq V$. Therefore, either $A \subseteq U$ or $A \subseteq V$. If $A \subseteq U$, then $A = A \cap U$. If $A \subseteq V$, then $A = A \cap V$, which is a contradiction. Thus, A is sequentially g-connected.

Lemma 3.9. *Let $f : (X, \tau) \to (Y, \sigma)$ be a sequentially g-continuous. If A is sequentially g-closed, then $f^{-1}(A)$ is sequentially g-closed.*

Proof. Let $B = f^{-1}(A)$ and suppose that $x \in [B]_{g_{seq}}$. Then there is a sequence (x_n) such that $(x_n) \xrightarrow{g} x$. Since f is sequentially g-continuous, $(f(x_n)) \xrightarrow{g} f(x)$ and A is sequentially g-closed implies that $f(x) \in A$. But $x \in B$. Thus, $[B]_{g_{seq}} \subset B$ and so $x \in f^{-1}(A)$. Therefore, $f^{-1}(A)$ is sequentially g-closed.

Corollary 3.10. *Let $f : (X, \tau) \to (Y, \sigma)$ be a sequentially g-continuous. If A is sequentially g-open, then $f^{-1}(A)$ is sequentially g-open.*

Proof. It follows from by Lemma 3.9.

Theorem 3.11. *A sequentially g-continuous image of any sequentially g-connected subset of X is sequentially g-connected.*

Proof. Suppose that $f(A)$ is not sequentially g-connected and let U and V be two disjoint sequentially g-closed subsets of X. Then $f(A)$ can be covered as a union $U \cup V$ of nonempty, both meeting $f(A)$. Since f is sequentially g-continuous, inverse image of a sequentially g-closed subset of X is sequentially g-closed, by Lemma 3.9 and so $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty, disjoint sequentially g-closed subsets of X and cover A. It follows that A is not sequentially g-connected, which is a contradiction to our assumptions.

4. Sequentially g-connected components

In this section, we introduce a concept of sequentially g-connected component of a point x in X.

Definition 4.1. The largest sequentially g-connected subset containing a point x in X is called sequentially g-connected component of x and denoted by C_{g_x}.

We note that C_{g_x} coincides with the ordinary sequentially connected component of x when $glimx_n = limx_n$. We denote $\zeta(X_g)$ is the set of sequentially g-connected components of all points in X and similarly we denote $\zeta(A_g)$ is the set of all sequentially g-connected components of all points in a subset A.

Lemma 4.2. *Let (X, τ) be a topological space and let $x, y \in X$. If x and y are in a sequentially g-connected subset A of X, then x and y are in the same sequentially g-component of X.*

Proof. Let x and y are in a sequentially g-connected subset A of X. Then $x, y \in A \subseteq C_{g_x}$ and $x, y \in A \subseteq C_{g_y}$. So $C_{g_x} \subseteq C_{g_y}$ and $C_{g_y} \subseteq C_{g_x}$. Therefore, $C_{g_x} = C_{g_y}$.

Lemma 4.3. *The sequentially g-connected components of X form a partition of X.*

Proof. It is clear that sequentially g-connected components form a cover of X. It is enough to prove that for $x, y \in X$ if the components C_{g_x} and C_{g_y} intersect, then $C_{g_x} = C_{g_y}$. Let $w \in C_{g_x} \cap C_{g_y}$. Since C_{g_w} is the largest sequentially g-connected subset including w, $w \in C_{g_x} \subseteq C_{g_w}$ and $w \in C_{g_y} \subseteq C_{g_z}$ On the other hand $C_{g_w} \subseteq C_{g_x}$ and $C_{g_w} \subseteq C_{g_y}$ since $x \in C_{g_x} \subseteq C_{g_w}$ and $y \in C_{g_y} \subseteq C_{g_w}$. Therefore $C_{g_x} = C_{g_y} = C_{g_w}$. Hence the proof is completed.

Definition 4.4. Let (X, τ) and (Y, σ) be two topological spaces. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *sequentially g-homeomorphic* if it satisfies the following conditions.
(a) f is bijection
(b) f is sequentially g-continuous.

Theorem 4.5. *Let (X, τ) be a topological space and let A and B be subsets of X. If A and B are sequentially g-homeomorphic, then $\zeta(A_g)$ and $\zeta(B_g)$ have the same cardinality, that is, there exists a bijection between them.*

Proof. Let $f : A \to B$ be a sequentially g-homeomorphism. Define a map $\zeta(f) : \zeta(A_g) \to \zeta(B_g)$, $f(C_{g_x}) = C_{g_{f(x)}}$ induced by the function $f : A \to B$. Since f is sequentially g-continuous, the image of a sequentially g-connected subset is sequentially g-connected, by Theorem 3.11. Suppose that $y \in C_{g_x}$, then $f(x)$ and $f(y)$ are in the same sequentially g-connected component, by Lemma 4.2. So, $C_{g_{f(x)}} = C_{g_{f(y)}}$. Therefore, the map $\zeta(f)$ is well defined. Since f^{-1} is sequentially g-continuous and $C_{g_{f(x)}} = C_{g_{f(y)}}$, then sequentially g-connected components of x and y are same, that is $C_{g_x} = C_{g_y}$. Therefore, $\zeta(f)$ is injective. Further since $f(C_{g_x}) = C_{g_y}$ with $y = f(x)$, the map $\zeta(f)$ is onto.

THEOREM 4.6. *Let (X, τ) be a topological space. Every sequentially g-connected component of a point x in X is sequentially g-closed.*

Proof. Since the sequentially g-connected component C_{g_x} is sequentially g-connected and by Theorem 3.7, $[C_x]_{g_{seq}}$ is sequentially g-connected. Therefore, $C_{g_x} \subseteq [C_x]_{g_{seq}}$. But the largest sequentially g-connected subset containing x is C_{g_x}. Therefore, $[C_x]_{g_{seq}} \subseteq C_{g_x}$ and so C_{g_x} is sequentially g-closed.

DEFINITION 4.7. A topological space (X, τ) is said to be *sequentially locally g-connected* if for any g-neighborhood U of x, there is a sequentially g-connected neighborhood V of x such that $x \in V \subseteq U$.

The following Theorem 4.8 shows that if X is sequentially locally g-connected, then each sequentially g-connected component of X is sequentially g-open.

THEOREM 4.8. *X is sequentially locally g-connected if and only if sequentially g-connected components of any sequentially g-open subset are sequentially g-open .*

Proof. Suppose that X be sequentially locally g-connected. Let A be a sequentially g-open subset of X, A is a sequentially connected component and $x \in C_{g_x}$. Since X is sequentially locally g-connected, there is a sequentially g-connected g-neighborhood U_x of x such that $U_x \subseteq A$. Since C_{g_x} is the largest sequentially g-connected subset of A containing x, we have that $x \in U_x \subseteq C_{g_x}$. Therefore, C_{g_x} is sequentially g-open, by Theorem 3.4. On the other hand, suppose that sequentially g-connected components of any sequentially g-open subset is sequentially g-open, then X becomes sequentially locally g-connected.

THEOREM 4.9. *Let (X, τ) be a topological space and A, B be subsets of X. Let $f : A \to B$ be an onto, sequentially g-continuous and sequentially g-open function. If A is sequentially locally g-connected, then B is also a sequentially locally g-connected.*

Proof. Suppose that $f : A \to B$ be an onto function which is sequentially g-continuous and sequentially g-open. Let $a \in A$ and $b \in B$ such that $f(a) = b$, and let U be a sequentially g-neighborhood of b in B. Since f is sequentially g-continuous, by Corollary 3.10, $f^{-1}(U)$ is a sequentially g-neighborhood of a. Since A is sequentially g-local connected, there is a sequentially g-connected neighborhood of a such that $V \subseteq f^{-1}(U)$. This implies that $f(V) \subseteq U$. Since f is sequentially g-open, $f(V)$ is sequentially g-open and since f is sequentially g-continuous, $f(V)$ is sequentially g-connected. Therefore, B is also sequentially locally g-connected.

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Palanichamy Vijayashanthi
Centre for Research and Post Graduate Studies in Mathematics,
Ayya Nadar Janaki Ammal College, Sivakasi 626124, Tamilnadu, India.
E-mail: vijayashanthi26892@gmail.com