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SIMPSON'S AND NEWTON'S TYPE QUANTUM INTEGRAL INEQUALITIES FOR PREINVEX FUNCTIONS

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Abstract. In this research, we offer two new quantum integral equalities for recently defined q^{ε_2}-integral and derivative, the derived equalities then used to prove quantum integral inequalities of Simpson's and Newton's type for preinvex functions. We also considered the special cases of established results and offer several new and existing results inside the literature of Simpson's and Newton's type inequalities.

1. Introduction

A lot of research work has been carried out in the field of q-analysis, initially initiated by Euler. It provides a suitable framework to study models in quantum computing q-calculus which appeared as a connection between mathematics and physics. It has a lot of applications in different mathematical areas such as number theory, combinatorics, orthogonal polynomials, basic hypergeometric functions, and other disciplines such as quantum theory, mechanics, and the theory of relativity [18]-[20], [22], [24]. Apparently, Euler is the founder of this branch of mathematics, where the parameter q is used in Newton's work of infinite series. Later, Jackson was the first to develop q-calculus that is known as "without limits calculus" in a systematic way [18]. In 1908-1909, Jackson defined the general q-integral and q-difference operator [22]. In 1969, Agarwal [7] described the q-fractional derivative for the first time . In 1966-1967, Al-Salam [8] introduced a q-analogs of the Riemann-Liouville fractional integral operator and q-fractional integral operator. In 2004, Rajkovic gave a definition of the Riemann-type q-integral which was the generalization of Jackson q-integral. In 2013, Tariboon introduced ${}_{\varepsilon_1}D_q$-difference operator [34].

Many integral inequalities well known in classical analysis such as Hölder inequality, Simpson's inequality, Newton's inequality, Hermite-Hadamard inequality and Ostrowski inequality, Cauchy-Bunyakovsky-Schwarz, Gruss, Gruss- Cebysev, and other integral inequalities have been proved and applied in the setup of q-calculus using classical convexity. Many mathematicians have done studies in q-calculus analysis, the interested reader can check [1]- [6], [12]- [15], [23], [25], [26], [28]- [30], [32], [35].

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In techniques of numerical integration and numerical estimation, Simpson's rules are well-known. T. Simpson (1710-1761) is the developer of this well-known technique. This method is also considered Kepler's law because J. Kepler used a similar approximation about 100 years ago. Simpson's rule includes the three-point Newton-Cotes quadrature rule, so estimations based on three steps quadratic kernel is sometimes called as Newton type results. Note that,

1: Simpson's 1/3 formula is given as

$$\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\Phi(\mu)\,d\mu \approx \frac{1}{6}\left[\Phi(\varepsilon_1)+4\Phi\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)+\Phi(\varepsilon_2)\right].$$

2: Simpson's 3/8 formula is given as follows

$$\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\Phi(\mu)\,d\mu$$
$$\approx \frac{1}{8}\left[\Phi(\varepsilon_1)+3\Phi\left(\frac{2\varepsilon_1+\varepsilon_2}{3}\right)+3\Phi\left(\frac{\varepsilon_1+2\varepsilon_2}{3}\right)+\Phi(\varepsilon_2)\right].$$

There are a large number of estimations related to these quadrature rules in the literature, one of them is the following estimation known as Simpson's inequality:

Theorem 1. *Let $\Phi:[\varepsilon_1,\varepsilon_2]\to\mathbb{R}$ be a four times continuously differentiable function on $(\varepsilon_1,\varepsilon_2)$, and*

$$\left\|\Phi^{(4)}\right\|_\infty=\sup_{\mu\in(\varepsilon_1,\varepsilon_2)}\left|\Phi^{(4)}(\mu)\right|<\infty.$$

Then, we have the following inequality

$$\left|\frac{1}{3}\left[\frac{\Phi(\varepsilon_1)+\Phi(\varepsilon_2)}{2}+2\Phi\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)\right]-\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\Phi(\mu)\,d\mu\right|$$
$$\leq \frac{1}{2880}\left\|\Phi^{(4)}\right\|_\infty(\varepsilon_2-\varepsilon_1)^4.$$

In recent years, many authors have considered Simpson's type inequalities for various classes of functions. Convex analysis provide effective and strong methods for solving a great number of problems which arise different branches in pure and applied mathematics. Some mathematicians have worked on Simpson's and Newton's type results for convex mappings. For example, Dragomir et al. [16] presented new Simpson's type results and their applications to quadrature formula in numerical integration. Some Simpson's type inequalities for s-convex functions are deduced by Alomari et al. [9]. Afterwards, Sarikaya et al. [33] observed the variants of Simpson's type inequalities based on convexity. Noor et al. [27], [31] provided some Newton's type inequalities for harmonic convex and p-harmonic convex functions. Furthermore, some Newton's type inequalities for functions whose local fractional derivatives are generalized convex were obtained by Iftikhar et al. [21].

The main objective of this paper is to study Newton's and Simpson's type inequalities for preinvex functions by using the notions of quantum calculus.

2. Preliminaries and Definitions of q-Calculus

The basic notions and findings which are needed in the sequel to prove our crucial results are reviewed in this section. Throughout this paper, we assume that $\varepsilon_1<\varepsilon_2$

and $0 < q < 1$. Let ω be a nonempty closed set in $\mathbb{R}^n$, $\Phi : \omega \to \mathbb{R}$ a continuous function and $\eta(.,.) : \omega \times \omega \to \mathbb{R}^n$ be a continuous bifunction.

DEFINITION 1. [15] A set ω is said to be invex set with respect to bifunction $\eta(.,.)$ if

$$\varepsilon_2 + t\eta(\varepsilon_1, \varepsilon_2) \in \omega, \forall\, \varepsilon_1, \varepsilon_2 \in \omega,\, t \in [0,1].$$

The invex set ω is also known as η-connected set.

DEFINITION 2. [15] A mapping Φ is said to be preinvex with respect to an arbitrary bifunction $\eta(.,.)$ if the following inequality holds:

$$\Phi(\varepsilon_2 + t\eta(\varepsilon_1, \varepsilon_2)) \le t\Phi(\varepsilon_1) + (1-t)\Phi(\varepsilon_2), \forall\, \varepsilon_1, \varepsilon_2 \in \omega,\, t \in [0,1].$$

The function Φ is called preconcave if $-\Phi$ is preinvex.

REMARK 1. If we set $\eta(\varepsilon_1, \varepsilon_2) = \varepsilon_1 - \varepsilon_2$, then the definition of preinvex functions reduces to the definition of a convex functions given below;

$$\Phi(\varepsilon_2 + t(\varepsilon_1 - \varepsilon_2)) \le t\Phi(\varepsilon_1) + (1-t)\Phi(\varepsilon_2), \forall\, \varepsilon_1, \varepsilon_2 \in \omega,\, t \in [0,1].$$

Now we present some well known concepts and theorems for q- derivative and q-integral of a function Φ on $[\varepsilon_1, \varepsilon_2]$.

DEFINITION 3. [24] For a function $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}$, the q_{ε_1}- derivative of Φ at $\mu \in [\varepsilon_1, \varepsilon_2]$ is characterized by the expression

$$_{\varepsilon_1}D_q\Phi(\mu) = \frac{\Phi(\mu) - \Phi(q\mu + (1-q)\varepsilon_1)}{(1-q)(\mu - \varepsilon_1)},\ \mu \ne \varepsilon_1. \tag{2.1}$$

If $\mu = \varepsilon_1$, we define $_{\varepsilon_1}D_q f(\mu) = \lim_{\mu \to \varepsilon_1} {}_{\varepsilon_1}D_q f(\mu)$ if it exists and it is finite.

DEFINITION 4. [34] Let $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}$ be a function. Then, the q_{ε_1}-definite integral on $[\varepsilon_1, \varepsilon_2]$ is defined by

$$\begin{aligned} &\int_{\varepsilon_1}^{\mu} \Phi(s)\ {}_{\varepsilon_1}d_q s \\ &= (1-q)(\mu - \varepsilon_1)\sum_{n=0}^{\infty} q^n \Phi(q^n\mu + (1-q^n)\varepsilon_1),\ \ \mu \in [\varepsilon_1, \varepsilon_2]. \end{aligned} \tag{2.2}$$

REMARK 2. If $\varepsilon_1 = 0$ in (2.2), then $\int_0^{\mu} \Phi(s)\ {}_0d_q s = \int_0^{\mu} \Phi(s)\ d_q s$, where $\int_0^{\mu} \Phi(s)\ d_q s$ is the familiar q-definite integral (see, [24]) on $[0, \mu]$ defined by

$$\int_0^{\mu} \Phi(s)\ {}_0d_q s = \int_0^{\mu} \Phi(s)\ d_q s = (1-q)\mu\sum_{n=0}^{\infty} q^n \Phi(q^n\mu). \tag{2.3}$$

DEFINITION 5. If $c \in (\varepsilon_1, \mu)$, then the q- definite integral on $[c, \mu]$ is expressed as

$$\int_c^{\mu} \Phi(s)\ {}_{\varepsilon_1}d_q s = \int_{\varepsilon_1}^{\mu} \Phi(s)\ {}_{\varepsilon_1}d_q s - \int_{\varepsilon_1}^{c} \Phi(s)\ {}_{\varepsilon_1}d_q s\ . \tag{2.4}$$

Alp et al. [10] proved the following q-Hermite-Hadamard inequality:

THEOREM 2. *(q_{ε_1}-Hermite-Hadamard inequality) Let $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}$ be a convex differentiable function on $[\varepsilon_1, \varepsilon_2]$ and $0 < q < 1$. Then we have*

$$\Phi\left(\frac{q\varepsilon_1+\varepsilon_2}{1+q}\right) \leq \frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2} \Phi(\mu) \ {}_{\varepsilon_1}d_q\mu \ \leq \frac{q\Phi(\varepsilon_1)+\Phi(\varepsilon_2)}{1+q}.$$

On the other hand, Bermudo et al. [11] gave the following new definitions of quantum integral and derivative. In the same paper authors proved a new variant of quantum Hermite-Hadamard type inequality linked with their newly defined quantum integral:

DEFINITION 6. [11] Let $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}$ be a function. Then, the q^{ε_2}-definite integral on $[\varepsilon_1, \varepsilon_2]$ is given by

$$\begin{aligned}\int_{\varepsilon_1}^{\varepsilon_2} \Phi(\mu) \ {}^{\varepsilon_2}d_q\mu &= (1-q)(\varepsilon_2-\varepsilon_1)\sum_{n=0}^{\infty} q^n\Phi(q^n\varepsilon_1+(1-q^n)\varepsilon_2) \\ &= (\varepsilon_2-\varepsilon_1)\int_0^1 \Phi(s\varepsilon_1+(1-s)\varepsilon_2) \ d_qs \ .\end{aligned}$$

DEFINITION 7. [11] Let $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}$ be a function. The q^{ε_2}-derivative of Φ at $\mu \in [\varepsilon_1, \varepsilon_2]$ is given by

$${}^{\varepsilon_2}D_q\Phi(\mu) = \frac{\Phi(q\mu+(1-q)\varepsilon_2)-\Phi(\mu)}{(1-q)(\varepsilon_2-\mu)}, \ \mu \neq \varepsilon_2.$$

THEOREM 3. [11] *(q^{ε_2}-Hermite-Hadamard inequality) If $\Phi : [\varepsilon_1, \varepsilon_2] \to \mathbb{R}$ is a convex differentiable function on $[\varepsilon_1, \varepsilon_2]$ and $0 < q < 1$. Then, q^{ε_2}-Hermite-Hadamard inequalities are given as follows:*

$$\Phi\left(\frac{\varepsilon_1+q\varepsilon_2}{1+q}\right) \leq \frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2} \Phi(\mu) \ {}^{\varepsilon_2}d_q\mu \ \leq \frac{\Phi(\varepsilon_1)+q\Phi(\varepsilon_2)}{1+q}. \tag{2.5}$$

Let us set the following notations:

$$[n]_q = \begin{cases} \frac{q^n-1}{q-1} = \sum\limits_{i=0}^{n-1} q^i, & n \in N \\ \frac{q^n-1}{q-1}, & n \in C \end{cases},$$

and

$$(1-s)_q^n = (s,q)_n = \prod_{i=0}^{n-1}\left(1-q^is\right). \tag{2.6}$$

LEMMA 1. *[10] For $\alpha \in \mathbb{R}\backslash\{-1\}$, the following formula holds:*

$$\int_{\varepsilon_1}^{\mu} (s-\varepsilon_1)^{\alpha} \ {}_{\varepsilon_1}d_qs \ = \frac{(\mu-\varepsilon_1)^{\alpha+1}}{[\alpha+1]_q}. \tag{2.7}$$

3. Quantum Integral Identities

In this section, we will prove two equalities which will help us to obtain our main results.

LEMMA 2. *Let $\Phi : I = [\varepsilon_2 + \eta(\varepsilon_1, \varepsilon_2), \varepsilon_2] \to \mathbb{R}$ be a differentiable function on I°(interior of I) with $-\eta(\varepsilon_1, \varepsilon_2) = \eta(\varepsilon_2, \varepsilon_1) > 0$. Then the following identity holds for q^{ε_2}-integrals:*

$$\begin{aligned}(3.1)\qquad & \frac{1}{\eta(\varepsilon_2, \varepsilon_1)} \int_{\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}^{\varepsilon_2} \Phi(\mu)\ {}^{\varepsilon_2}d_q\mu \\ & -\frac{1}{6}\left[\Phi(\varepsilon_2 + \eta(\varepsilon_1, \varepsilon_2)) + 4\Phi\left(\frac{2\varepsilon_2 + \eta(\varepsilon_1, \varepsilon_2)}{2}\right) + \Phi(\varepsilon_2)\right] \\ = \ & \eta(\varepsilon_2, \varepsilon_1) \int_0^1 \varpi_q(t)\ {}^{\varepsilon_2}D_q\Phi(\varepsilon_2 + t\eta(\varepsilon_1, \varepsilon_2))\, d_qt\end{aligned}$$

where

$$\varpi_q(t) = \begin{cases} qt - \frac{1}{6}, & \text{if } 0 \le t < \frac{1}{2}, \\ qt - \frac{5}{6}, & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Proof. Using the basic properties of q-integral and definition of $\varpi_q(t)$, we have

$$\begin{aligned}(3.2)\qquad & \int_0^1 \varpi_q(t)\ {}^{\varepsilon_2}D_q\Phi(\varepsilon_2 + t\eta(\varepsilon_1, \varepsilon_2))\, d_qt \\ = \ & \frac{2}{3}\int_0^{\frac{1}{2}} {}^{\varepsilon_2}D_q\Phi(\varepsilon_2 + t\eta(\varepsilon_1, \varepsilon_2))\, d_qt + \int_0^1 qt\ {}^{\varepsilon_2}D_q\Phi(\varepsilon_2 + t\eta(\varepsilon_1, \varepsilon_2))\, d_qt \\ & -\frac{5}{6}\int_0^1 {}^{\varepsilon_2}D_q\Phi(\varepsilon_2 + t\eta(\varepsilon_1, \varepsilon_2))\, d_qt.\end{aligned}$$

From Definition 7, we have

$${}^{\varepsilon_2}D_q\Phi(\varepsilon_2 + t\eta(\varepsilon_1, \varepsilon_2)) = \frac{\Phi(\varepsilon_2 + tq\eta(\varepsilon_1, \varepsilon_2)) - \Phi(\varepsilon_2 + t\eta(\varepsilon_1, \varepsilon_2))}{(1-q)\, t\eta(\varepsilon_2, \varepsilon_1)}.$$

We now compute the integrals on the right side of (3.2). Using Definition 6, we obtain that

$$\begin{aligned}(3.3)\qquad & \int_0^{\frac{1}{2}} {}^{\varepsilon_2}D_q\Phi(\varepsilon_2 + t\eta(\varepsilon_1, \varepsilon_2))\, d_qt \\ = \ & \int_0^{\frac{1}{2}} \frac{\Phi(\varepsilon_2 + tq\eta(\varepsilon_1, \varepsilon_2)) - \Phi(\varepsilon_2 + t\eta(\varepsilon_1, \varepsilon_2))}{(1-q)\, t\eta(\varepsilon_2, \varepsilon_1)} d_qt \\ = \ & \frac{1}{\eta(\varepsilon_2, \varepsilon_1)}\left[\sum_{n=0}^{\infty} \Phi\left(\varepsilon_2 + \frac{q^{n+1}}{2}\eta(\varepsilon_1, \varepsilon_2)\right) - \sum_{n=0}^{\infty} \Phi\left(\varepsilon_2 + \frac{q^n}{2}\eta(\varepsilon_1, \varepsilon_2)\right)\right] \\ = \ & \frac{1}{\eta(\varepsilon_2, \varepsilon_1)}\left[\Phi(\varepsilon_2) - \Phi\left(\frac{2\varepsilon_2 + \eta(\varepsilon_1, \varepsilon_2)}{2}\right)\right],\end{aligned}$$

$$\begin{aligned}(3.4)\qquad & \int_0^1 {}^{\varepsilon_2}D_q\Phi\left(\varepsilon_2+t\eta\left(\varepsilon_1,\varepsilon_2\right)\right)d_qt \\ = & \frac{1}{\eta\left(\varepsilon_2,\varepsilon_1\right)}\left[\Phi\left(\varepsilon_2\right)-\Phi\left(\varepsilon_2+\eta\left(\varepsilon_1,\varepsilon_2\right)\right)\right]\end{aligned}$$

and

$$\begin{aligned}(3.5)\qquad & \int_0^1 qt\ {}^{\varepsilon_2}D_q\Phi\left(\varepsilon_2+t\eta\left(\varepsilon_1,\varepsilon_2\right)\right)d_qt \\ = & \int_0^1 q\frac{\Phi\left(\varepsilon_2+tq\eta\left(\varepsilon_1,\varepsilon_2\right)\right)-\Phi\left(\varepsilon_2+t\eta\left(\varepsilon_1,\varepsilon_2\right)\right)}{(1-q)\,\eta\left(\varepsilon_2,\varepsilon_1\right)}d_qt \\ = & \frac{1}{\eta\left(\varepsilon_2,\varepsilon_1\right)}\left[(1-q)\sum_{n=0}^{\infty}q^n\Phi\left(\varepsilon_2+q^n\eta\left(\varepsilon_1,\varepsilon_2\right)\right)-\Phi\left(\varepsilon_2+\eta\left(\varepsilon_1,\varepsilon_2\right)\right)\right] \\ = & \frac{1}{\eta\left(\varepsilon_2,\varepsilon_1\right)}\left[\frac{1}{\eta\left(\varepsilon_2,\varepsilon_1\right)}\int_{\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}^{\varepsilon_2}\Phi\left(\mu\right)\ {}^{\varepsilon_2}d_q\mu-\Phi\left(\varepsilon_2+\eta\left(\varepsilon_1,\varepsilon_2\right)\right)\right].\end{aligned}$$

Finally, by substituting (3.3)-(3.5) in (3.2) and multiplying the resultant equality by $\eta\left(\varepsilon_2,\varepsilon_1\right)$, we obtain the required identity which completes the proof. □

Remark 3. If we set $\eta\left(\varepsilon_2,\varepsilon_1\right)=\varepsilon_2-\varepsilon_1$ and $\eta\left(\varepsilon_1,\varepsilon_2\right)=\varepsilon_1-\varepsilon_2$ in Lemma 2, then Lemma 2 reduces to [14, Lemma 2].

Lemma 3. *Let $\Phi : I=\left[\varepsilon_2+\eta\left(\varepsilon_1,\varepsilon_2\right),\varepsilon_2\right]\to\mathbb{R}$ be a differentiable function on I° (interior of I) with $-\eta\left(\varepsilon_1,\varepsilon_2\right)=\eta\left(\varepsilon_2,\varepsilon_1\right)>0$. Then the following identity holds for q^{ε_2}-integrals:*

$$\begin{aligned} & \frac{1}{\eta\left(\varepsilon_2,\varepsilon_1\right)}\int_{\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}^{\varepsilon_2}\Phi\left(\mu\right)\ {}^{\varepsilon_2}d_q\mu \\ & -\frac{1}{8}\left[\Phi\left(\varepsilon_2+\eta\left(\varepsilon_1,\varepsilon_2\right)\right)+3\Phi\left(\frac{3\varepsilon_2+\eta\left(\varepsilon_1,\varepsilon_2\right)}{3}\right)\right. \\ & \left.+3\Phi\left(\frac{3\varepsilon_2+2\eta\left(\varepsilon_1,\varepsilon_2\right)}{3}\right)+\Phi\left(\varepsilon_2\right)\right] \\ = & \ \eta\left(\varepsilon_2,\varepsilon_1\right)\int_0^1\Pi_q\left(t\right)\ {}^{\varepsilon_2}D_q\left(\varepsilon_2+t\eta\left(\varepsilon_1,\varepsilon_2\right)\right)d_qt\end{aligned}$$

where

$$\Pi_q\left(t\right)=\begin{cases} qt-\frac{1}{8}, & \text{if } 0\le t<\frac{1}{3}, \\ qt-\frac{1}{2}, & \text{if } \frac{1}{3}\le t<\frac{2}{3}, \\ qt-\frac{7}{8}, & \text{if } \frac{2}{3}\le t\le 1.\end{cases}$$

Proof. By the fundamental properties of q-integrals and definition of $\Pi_q(t)$, we obtain that

$$\begin{aligned} &\int_0^1 \Pi_q(t)\ {}^{\varepsilon_2}D_q(\varepsilon_2 + t\eta(\varepsilon_1,\varepsilon_2))\,d_qt \\ = &\ \frac{3}{8}\int_0^{\frac{1}{3}} {}^{\varepsilon_2}D_q(\varepsilon_2 + t\eta(\varepsilon_1,\varepsilon_2))\,d_qt + \frac{3}{8}\int_0^{\frac{2}{3}} {}^{\varepsilon_2}D_q(\varepsilon_2 + t\eta(\varepsilon_1,\varepsilon_2))\,d_qt \\ &+ \int_0^1 \left(qt - \frac{7}{8}\right) {}^{\varepsilon_2}D_q(\varepsilon_2 + t\eta(\varepsilon_1,\varepsilon_2))\,d_qt. \end{aligned}$$

Following arguments similar to those in the proof of Lemma 2, the required identity can be proved. □

REMARK 4. If we set $\eta(\varepsilon_2,\varepsilon_1) = \varepsilon_2 - \varepsilon_1$ and $\eta(\varepsilon_1,\varepsilon_2) = \varepsilon_1 - \varepsilon_2$ in Lemma 3, then Lemma 3 becomes [14, Lemma 3].

4. Simpson's type inequalities for quantum Integrals

In this section, we present some new Simpson's type inequalities for preinvex functions by using the Lemma 2.

THEOREM 4. *We assume that the conditions of Lemma 2 hold. If $|{}^{\varepsilon_2}D_q\Phi|$ is preinvex and integrable on I, then the following inequality holds for q^{ε_2}-integrals:*

$$\begin{aligned} &\left|\frac{1}{\eta(\varepsilon_2,\varepsilon_1)}\int_{\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}^{\varepsilon_2} \Phi(\mu)\ {}^{\varepsilon_2}d_q\mu \right.\\ &\left. -\frac{1}{6}\left[\Phi(\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)) + 4\Phi\left(\frac{2\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}{2}\right) + \Phi(\varepsilon_2)\right]\right| \\ \leq &\ \eta(\varepsilon_2,\varepsilon_1)\left[A_1(q) + A_3(q)\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_1)\right| + (A_2(q)+A_4(q))\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_2)\right|\right] \end{aligned}$$

where $A_i, i = 1,2,3,4$ are defined by

$$A_1(q) = \int_0^{\frac{1}{2}} \left|qt - \frac{1}{6}\right| t\,d_qt = \begin{cases} \frac{1-2q-2q^2}{24(1+q)(1+q+q^2)}, & \text{if } 0 < q < \frac{1}{3} \\ \frac{18q^2+18q-7}{216(1+q)(1+q+q^2)}, & \text{if } \frac{1}{3} \leq q < 1, \end{cases}$$

$$A_2(q) = \int_0^{\frac{1}{2}} \left|qt - \frac{1}{6}\right| (1-t)\,d_qt = \begin{cases} \frac{1-4q^3}{24(1+q)(1+q+q^2)}, & \text{if } 0 < q < \frac{1}{3} \\ \frac{36q^3+12q^2+12q+1}{216(1+q)(1+q+q^2)}, & \text{if } \frac{1}{3} \leq q < 1, \end{cases}$$

$$A_3(q) = \int_{\frac{1}{2}}^1 \left|qt - \frac{5}{6}\right| t\,d_qt = \begin{cases} \frac{15-6q-6q^2}{24(1+q)(1+q+q^2)}, & \text{if } 0 < q < \frac{5}{6} \\ \frac{18q^2+18q+25}{216(1+q)(1+q+q^2)}, & \text{if } \frac{5}{6} \leq q < 1, \end{cases}$$

$$A_4(q) = \int_{\frac{1}{2}}^1 \left|qt - \frac{5}{6}\right| (1-t)\,d_qt = \begin{cases} \frac{-5+8q+8q^2-8q^3}{24(1+q)(1+q+q^2)}, & \text{if } 0 < q < \frac{5}{6} \\ \frac{12q^2+12q+5}{216(1+q)(1+q+q^2)}, & \text{if } \frac{5}{6} \leq q < 1. \end{cases}$$

Proof. On taking modulus on the right hand side of an identity in Lemma 2 and using the properties of modulus , we obtain that

$$\begin{aligned}(4.1)\qquad &\left|\frac{1}{\eta(\varepsilon_2,\varepsilon_1)}\int_{\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}^{\varepsilon_2}\Phi(\mu)\ {}^{\varepsilon_2}d_q\mu\right.\\ &\left.-\frac{1}{6}\left[\Phi(\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2))+4\Phi\left(\frac{2\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}{2}\right)+\Phi(\varepsilon_2)\right]\right|\\ \leq\ & \eta(\varepsilon_2,\varepsilon_1)\left[\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_2+t\eta(\varepsilon_1,\varepsilon_2))\right|\ d_qt\right.\\ &\left.+\int_{\frac{1}{2}}^{1}\left|qt-\frac{5}{6}\right|\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_2+t\eta(\varepsilon_1,\varepsilon_2))\right|\ d_qt\right].\end{aligned}$$

Since $\left|{}^{\varepsilon_2}D_q\Phi\right|$ is preinvex function, we have

$$\begin{aligned}(4.2)\qquad &\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|\left|{}^{\varepsilon_2}D_q\Phi(t\varepsilon_1+(1-t)\varepsilon_2)\right|d_qt\\ \leq\ & \left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_1)\right|\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|td_qt+\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_2)\right|\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|(1-t)\,d_qt\\ =\ & A_1(q)\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_1)\right|+A_2(q)\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_2)\right|\end{aligned}$$

and

$$\begin{aligned}(4.3)\qquad &\int_{\frac{1}{2}}^{1}\left|qt-\frac{5}{6}\right|\left|{}^{\varepsilon_2}D_q\Phi(t\varepsilon_1+(1-t)\varepsilon_2)\right|d_qt\\ \leq\ & \left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_1)\right|\int_{\frac{1}{2}}^{1}\left|qt-\frac{5}{6}\right|td_qt+\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_2)\right|\int_{\frac{1}{2}}^{1}\left|qt-\frac{5}{6}\right|(1-t)\,d_qt\\ =\ & A_3(q)\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_1)\right|+A_4(q)\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_2)\right|.\end{aligned}$$

Finally, substituting (4.2) and (4.3) in (4.1), we obtain the desired inequality which completes the proof. □

Corollary 1. *In Theorem 4, if we take limit $q\to 1^-$, then we have*

$$\begin{aligned}&\left|\frac{1}{\eta(\varepsilon_2,\varepsilon_1)}\int_{\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}^{\varepsilon_2}\Phi(\mu)\ d\mu\right.\\ &\left.-\frac{1}{6}\left[\Phi(\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2))+4\Phi\left(\frac{2\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}{2}\right)+\Phi(\varepsilon_2)\right]\right|\\ \leq\ & \frac{5\eta(\varepsilon_2,\varepsilon_1)}{72}\left[\left|\Phi'(\varepsilon_1)\right|+\left|\Phi'(\varepsilon_2)\right|\right]\end{aligned}$$

which can be viewed as a special case of inequality derived in [17].

Therefore, we can deduce the following result for convex functions

Remark 5. If we set $\eta(\varepsilon_2,\varepsilon_1)=\varepsilon_2-\varepsilon_1$ and $\eta(\varepsilon_1,\varepsilon_2)=\varepsilon_1-\varepsilon_2$ in Theorem 4, then Theorem 4 reduces to [14, Theorem 4].

Remark 6. If we set $\eta(\varepsilon_2,\varepsilon_1)=\varepsilon_2-\varepsilon_1$, $\eta(\varepsilon_1,\varepsilon_2)=\varepsilon_1-\varepsilon_2$, and $q\to 1^-$ in Theorem 4, then Theorem 4 reduces to [9, Corollary 1].

REMARK 7. In Theorem 4, if $\eta(\varepsilon_2,\varepsilon_1)=\varepsilon_2-\varepsilon_1$, $\eta(\varepsilon_1,\varepsilon_2)=\varepsilon_1-\varepsilon_2$, $\Phi(\varepsilon_1)=\Phi\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)=\Phi(\varepsilon_2)$, and $q\to 1^-$, then Theorem 4 reduces to [9, Corollary 3].

The corresponding version of the Simpson's inequality for powers in terms of the first q-derivative is incorporated in the following result.

THEOREM 5. *We assume that the assumptions of Lemma 2 hold. If* $\left|{}^{\varepsilon_2}D_q\Phi\right|^r$ *is preinvex and integrable on* I *where* $r>1$ *with* $\frac{1}{r}+\frac{1}{s}=1$, *then we have*

$$\begin{aligned}(4.4)\qquad &\left|\frac{1}{\eta(\varepsilon_2,\varepsilon_1)}\int_{\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}^{\varepsilon_2}\Phi(\mu)\ {}^{\varepsilon_2}d_q\mu\right.\\ &\left.-\frac{1}{6}\left[\Phi(\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2))+4\Phi\left(\frac{2\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}{2}\right)+\Phi(\varepsilon_2)\right]\right|\\ \leq\ &\frac{1}{6}\eta(\varepsilon_2,\varepsilon_1)\left[2^{1-\frac{1}{s}}\right.\\ &\times\left(\frac{1}{4(1+q)}\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_1)\right|^r+\frac{2q+1}{4(1+q)}\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_2)\right|^r\right)^{\frac{1}{r}}\\ &+\left(5^s-2^{s-1}\right)^{\frac{1}{s}}\\ &\left.\times\left(\frac{3}{4(1+q)}\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_1)\right|^r+\frac{2q-1}{4(1+q)}\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_2)\right|^r\right)^{\frac{1}{r}}\right].\end{aligned}$$

Proof. Applying Hölder's inequality on the first right integral of (4.1) and using the fact that $\left|{}^{\varepsilon_2}D_q\Phi\right|^r$ is preinvex function, we have

$$\begin{aligned}(4.5)\qquad &\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_2+t\eta(\varepsilon_1,\varepsilon_2))\right|d_qt\\ \leq\ &\left(\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|^s d_qt\right)^{\frac{1}{s}}\\ &\times\left(\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_1)\right|^r\int_0^{\frac{1}{2}}td_qt+\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_2)\right|^r\int_0^{\frac{1}{2}}(1-t)\,d_qt\right)^{\frac{1}{r}}.\end{aligned}$$

Computing the integrals that appear on the right side of (4.5)

$$\begin{aligned}\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|^s d_qt &= (1-q)\frac{1}{2}\sum_{n=0}^{\infty}q^n\left|\frac{q^{n+1}}{2}-\frac{1}{6}\right|^s\\ &\leq (1-q)\frac{1}{2}\sum_{n=0}^{\infty}q^n\left|\frac{1}{2}-\frac{1}{6}\right|^s\\ &\leq (1-q)\frac{1}{2}\frac{1}{(1-q)}\frac{1}{3^s}\\ &\leq \frac{1}{2.3^s}\end{aligned}$$

$$\int_0^{\frac{1}{2}}td_qt = (1-q)\frac{1}{2}\sum_{n=0}^{\infty}\frac{q^{2n}}{2}=\frac{1}{4(1+q)}$$

and

$$\int_0^{\frac{1}{2}} (1-t)\, d_qt = \frac{1+2q}{4(1+q)}.$$

So, we have

$$\begin{aligned}&\int_0^{\frac{1}{2}} \left|qt - \frac{1}{6}\right| \left|{}^{\varepsilon_2}D_q\Phi\left(\varepsilon_2 + t\eta\left(\varepsilon_1, \varepsilon_2\right)\right)\right| \; d_qt \\ \leq\; & \left(\frac{1}{2.3^s}\right)^{\frac{1}{s}} \left[\frac{1}{4(1+q)} \left|{}^{\varepsilon_2}D_q\Phi\left(\varepsilon_1\right)\right|^r + \frac{1+2q}{4(1+q)} \left|{}^{\varepsilon_2}D_q\Phi\left(\varepsilon_2\right)\right|^r\right]^{\frac{1}{r}}.\end{aligned}$$

Using the similar operations to the second integral on the right side of (4.1), we obtain that

$$\begin{aligned}&\int_{\frac{1}{2}}^1 \left|qt - \frac{5}{6}\right| \left|{}^{\varepsilon_2}D_q\Phi\left(\varepsilon_2 + t\eta\left(\varepsilon_1, \varepsilon_2\right)\right)\right| d_qt \\ \leq\; & \left(\frac{5^s - 2^{s-1}}{6^s}\right)^{\frac{1}{s}} \left(\frac{3}{4(1+q)} \left|{}^{\varepsilon_2}D_q\Phi\left(\varepsilon_1\right)\right|^r + \frac{2q-1}{4(1+q)} \left|{}^{\varepsilon_2}D_q\Phi\left(\varepsilon_2\right)\right|^r\right)^{\frac{1}{r}}.\end{aligned}$$

Thus, the desired inequality can be easily obtained. □

REMARK 8. If we set $\eta\left(\varepsilon_2, \varepsilon_1\right) = \varepsilon_2 - \varepsilon_1$ and $\eta\left(\varepsilon_1, \varepsilon_2\right) = \varepsilon_1 - \varepsilon_2$ in Theorem 5, then Theorem 5 reduces to [14, Theorem 5].

Another version of the Simpson's inequality for powers in terms of the first q-derivative is obtained as follows:

THEOREM 6. *Suppose that the assumptions of Lemma 2 hold. If* $\left|{}^{\varepsilon_2}D_q\Phi\right|^r$ *is preinvex and integrable on* I *where* $r \geq 1$, *then we have*

$$\begin{aligned}(4.6)\qquad &\left|\frac{1}{\eta\left(\varepsilon_2, \varepsilon_1\right)} \int_{\varepsilon_2+\eta\left(\varepsilon_1, \varepsilon_2\right)}^{\varepsilon_2} \Phi(\mu) \;{}^{\varepsilon_2}d_q\mu \right. \\ &\left. -\frac{1}{6}\left[\Phi\left(\varepsilon_2 + \eta\left(\varepsilon_1, \varepsilon_2\right)\right) + 4\Phi\left(\frac{2\varepsilon_2 + \eta\left(\varepsilon_1, \varepsilon_2\right)}{2}\right) + \Phi\left(\varepsilon_2\right)\right]\right| \\ \leq\; & \eta\left(\varepsilon_2, \varepsilon_1\right)\left(A_5(q)\right)^{1-\frac{1}{r}} \left[A_1(q) \left|{}^{\varepsilon_2}D_q\Phi\left(\varepsilon_1\right)\right|^r + A_2(q) \left|{}^{\varepsilon_2}D_q\Phi\left(\varepsilon_2\right)\right|^r\right]^{\frac{1}{r}} \\ &+\eta\left(\varepsilon_2, \varepsilon_1\right)\left(A_6(q)\right)^{1-\frac{1}{r}} \left[A_3(q) \left|{}^{\varepsilon_2}D_q\Phi\left(\varepsilon_1\right)\right|^r + A_4(q) \left|{}^{\varepsilon_2}D_q\Phi\left(\varepsilon_2\right)\right|^r\right]^{\frac{1}{r}}\end{aligned}$$

where $A_i, i = 1, 2, 3, 4$ *are defined as in Theorem 4. Furthermore,* A_5 *and* A_6 *are defined by*

$$\begin{aligned}A_5(q) &= \int_0^{\frac{1}{2}} \left|qt - \frac{1}{6}\right| d_qt = \begin{cases} \frac{1-2q}{12(1+q)}, & \text{if} \quad 0 < q < \frac{1}{3} \\ \frac{6q-1}{36(1+q)}, & \text{if} \quad \frac{1}{3} \leq q < 1, \end{cases} \\ A_6(q) &= \int_{\frac{1}{2}}^1 \left|qt - \frac{5}{6}\right| d_qt = \begin{cases} \frac{5-4q}{12(1+q)}, & \text{if} \quad 0 < q < \frac{5}{6} \\ \frac{5}{36(1+q)}, & \text{if} \quad \frac{5}{6} \leq q < 1. \end{cases}\end{aligned}$$

Proof. Applying power mean inequality on the first right integral of (4.1) and using the fact that $\left|{}^{\varepsilon_2}D_q\Phi\right|^r$ is preinvex function, we have

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|\left|{}^{\varepsilon_2}D_q\Phi\left(\varepsilon_2+t\eta\left(\varepsilon_1,\varepsilon_2\right)\right)\right|d_qt\\
\leq\ &\left(\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|d_qt\right)^{1-\frac{1}{r}}\left(\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|\left|{}^{\varepsilon_2}D_q\Phi\left(\varepsilon_2+t\eta\left(\varepsilon_1,\varepsilon_2\right)\right)\right|^r d_qt\right)^{\frac{1}{r}}\\
\leq\ &\left(\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|d_qt\right)^{1-\frac{1}{r}}\\
&\times\left[\left|{}^{\varepsilon_2}D_q\Phi\left(\varepsilon_1\right)\right|^r\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|td_qt+\left|{}^{\varepsilon_2}D_q\Phi\left(\varepsilon_2\right)\right|^r\int_0^{\frac{1}{2}}\left|qt-\frac{1}{6}\right|(1-t)\,d_qt\right]^{\frac{1}{r}}\\
=\ &\left(A_5\left(q\right)\right)^{1-\frac{1}{r}}\left[A_1\left(q\right)\left|{}^{\varepsilon_2}D_q\Phi\left(\varepsilon_1\right)\right|^r+A_2\left(q\right)\left|{}^{\varepsilon_2}D_q\Phi\left(\varepsilon_2\right)\right|^r\right]^{\frac{1}{r}}.
\end{aligned}$$

If we use the same operations to the second integral on the right side of (4.1), we can compute that

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1}\left|qt-\frac{5}{6}\right|\left|{}^{\varepsilon_2}D_q\Phi\left(t\varepsilon_1+(1-t)\,\varepsilon_2\right)\right|d_qt\\
\leq\ &\left(A_6\left(q\right)\right)^{1-\frac{1}{r}}\left[A_3\left(q\right)\left|{}^{\varepsilon_2}D_q\Phi\left(\varepsilon_1\right)\right|^r+A_4\left(q\right)\left|{}^{\varepsilon_2}D_q\Phi\left(\varepsilon_2\right)\right|^r\right]^{\frac{1}{r}}.
\end{aligned}$$

Thus, the required inequality can be easily proved. □

COROLLARY 2. *If we take limit $q\to 1^-$ in Theorem 6, then we have following inequality*

$$\begin{aligned}
&\left|\frac{1}{\eta\left(\varepsilon_2,\varepsilon_1\right)}\int_{\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}^{\varepsilon_2}\Phi\left(\mu\right)\,d\mu\right.\\
&\left.-\frac{1}{6}\left[\Phi\left(\varepsilon_2+\eta\left(\varepsilon_1,\varepsilon_2\right)\right)+4\Phi\left(\frac{2\varepsilon_2+\eta\left(\varepsilon_1,\varepsilon_2\right)}{2}\right)+\Phi\left(\varepsilon_2\right)\right]\right|\\
\leq\ &\frac{5^{1-\frac{1}{r}}}{72}\eta\left(\varepsilon_2,\varepsilon_1\right)\left[\left(\frac{29}{18}\left|\Phi'\left(\varepsilon_1\right)\right|^r+\frac{61}{18}\left|\Phi'\left(\varepsilon_2\right)\right|^r\right)^{\frac{1}{r}}+\left(\frac{61}{18}\left|\Phi'\left(\varepsilon_1\right)\right|^r+\frac{29}{18}\left|\Phi'\left(\varepsilon_2\right)\right|^r\right)^{\frac{1}{r}}\right]
\end{aligned}$$

which can be viewed as a special case of inequality derived in [17].

REMARK 9. If we set $\eta\left(\varepsilon_2,\varepsilon_1\right)=\varepsilon_2-\varepsilon_1$ and $\eta\left(\varepsilon_1,\varepsilon_2\right)=\varepsilon_1-\varepsilon_2$ in Theorem 6, then Theorem 6 reduces to [14, Theorem 6].

REMARK 10. In Theorem 6, if we take $\eta\left(\varepsilon_2,\varepsilon_1\right)=\varepsilon_2-\varepsilon_1$, $\eta\left(\varepsilon_1,\varepsilon_2\right)=\varepsilon_1-\varepsilon_2$, and $q\to 1^-$, then we have following inequality

$$\begin{aligned}
&\left|\frac{1}{6}\left[\Phi\left(\varepsilon_1\right)+4\Phi\left(\frac{\varepsilon_1+\varepsilon_2}{2}\right)+\Phi\left(\varepsilon_2\right)\right]-\frac{1}{\varepsilon_2-\varepsilon_1}\int_{\varepsilon_1}^{\varepsilon_2}\Phi\left(\mu\right)\,d\mu\right|\\
\leq\ &\frac{5^{1-\frac{1}{r}}}{72}\left(\varepsilon_2-\varepsilon_1\right)\left[\left(\frac{29}{18}\left|\Phi'\left(\varepsilon_1\right)\right|^r+\frac{61}{18}\left|\Phi'\left(\varepsilon_2\right)\right|^r\right)^{\frac{1}{r}}\right.\\
&\left.+\left(\frac{61}{18}\left|\Phi'\left(\varepsilon_1\right)\right|^r+\frac{29}{18}\left|\Phi'\left(\varepsilon_2\right)\right|^r\right)^{\frac{1}{r}}\right]
\end{aligned}$$

which can be proved as a special case of inequality derived in [9].

5. Newton's type inequalities for quantum integrals

In this section, we prove some Newton's type inequalities for preinvex functions using the Lemma 3.

THEOREM 7. *We assume that the assumptions of Lemma 3 hold. If $|{}^{\varepsilon_2}D_q\Phi|$ is preinvex and integrable on I, then the following inequality holds for q^{ε_2}-integrals:*

$$\begin{aligned}
(5.1)\qquad & \left|\frac{1}{\eta(\varepsilon_2,\varepsilon_1)}\int_{\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}^{\varepsilon_2}\Phi(\mu)\ {}^{\varepsilon_2}d_q\mu \right.\\
& -\frac{1}{8}\left[\Phi(\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2))+3\Phi\left(\frac{3\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}{3}\right)\right.\\
& \left.\left.+3\Phi\left(\frac{3\varepsilon_2+2\eta(\varepsilon_1,\varepsilon_2)}{3}\right)+\Phi(\varepsilon_2)\right]\right|\\
\leq\ & \eta(\varepsilon_2,\varepsilon_1)\left[(\Psi_1(q)+\Psi_3(q)+\Psi_5(q))\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_1)\right|\right.\\
& \left.+(\Psi_2(q)+\Psi_4(q)+\Psi_6(q))\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_2)\right|\right]
\end{aligned}$$

where

$$\Psi_1(q)=\int_0^{\frac{1}{3}}\left|qt-\frac{1}{8}\right|t\,d_qt=\begin{cases}\frac{3-5q-5q^2}{216(1+q)(1+q+q^2)} & 0<q<\frac{3}{8}\\[2mm] \frac{160q^2+160q-69}{6912(1+q)(1+q+q^2)} & \frac{3}{8}<q<1,\end{cases}$$

$$\Psi_2(q)=\int_0^{\frac{1}{3}}\left|qt-\frac{1}{8}\right|(1-t)\,d_qt=\begin{cases}\frac{6-q-q^2-15q^3}{216(1+q)(1+q+q^2)} & 0<q<\frac{3}{8}\\[2mm] \frac{480q^3+248q^2+248q-3}{6912(1+q)(1+q+q^2)} & \frac{3}{8}<q<1,\end{cases}$$

$$\Psi_3(q)=\int_{\frac{1}{3}}^{\frac{2}{3}}\left|qt-\frac{1}{2}\right|t\,d_qt=\begin{cases}\frac{9-5q-5q^2}{54(1+q)(1+q+q^2)} & 0<q<\frac{3}{4}\\[2mm] \frac{6q^2+6q-3}{108(1+q)(1+q+q^2)} & \frac{3}{4}<q<1,\end{cases}$$

$$\Psi_4(q)=\int_{\frac{1}{3}}^{\frac{2}{3}}\left|qt-\frac{1}{2}\right|(1-t)\,d_qt=\begin{cases}\frac{5q+5q^2-9q^3}{54(1+q)(1+q+q^2)} & 0<q<\frac{3}{4}\\[2mm] \frac{6q^3+3}{108(1+q)(1+q+q^2)} & \frac{3}{4}<q<1,\end{cases}$$

$$\Psi_5(q)=\int_{\frac{2}{3}}^{1}\left|qt-\frac{7}{8}\right|t\,d_qt=\begin{cases}\frac{105-47q-47q^2}{216(1+q)(1+q+q^2)} & 0<q<\frac{7}{8}\\[2mm] \frac{224q^2+224q+525}{6912(1+q)(1+q+q^2)} & \frac{7}{8}<q<1,\end{cases}$$

$$\Psi_6(q)=\int_{\frac{2}{3}}^{1}\left|qt-\frac{7}{8}\right|(1-t)\,d_qt=\begin{cases}\frac{-42+53q+53q^2-57q^3}{216(1+q)(1+q+q^2)} & 0<q<\frac{7}{8}\\[2mm] \frac{-96q^3+184q^2+184q-21}{6912(1+q)(1+q+q^2)} & \frac{7}{8}<q<1.\end{cases}$$

Proof. Following arguments similar to those in the proof of Theorem 4 by taking into account the Lemma 3, the desired inequality (5.1) is attained. □

COROLLARY 3. *If we take $q \to 1^-$ in Theorem 7, then we have following inequality*

$$\begin{aligned} & \left| \frac{1}{\eta(\varepsilon_2,\varepsilon_1)} \int_{\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}^{\varepsilon_2} \Phi(\mu)\ d\mu - \frac{1}{8}\left[\Phi(\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)) + 3\Phi\left(\frac{3\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}{3}\right) \right.\right. \\ & \left.\left. +3\Phi\left(\frac{3\varepsilon_2+2\eta(\varepsilon_1,\varepsilon_2)}{3}\right) + \Phi(\varepsilon_2) \right]\right| \\ \leq\ & \frac{25\eta(\varepsilon_2,\varepsilon_1)}{576} \left[|\Phi'(\varepsilon_1)| + |\Phi'(\varepsilon_2)| \right] \end{aligned}$$

which can be viewed a special cases of inequality given in [17].

REMARK 11. If we set $\eta(\varepsilon_2,\varepsilon_1) = \varepsilon_2 - \varepsilon_1$ and $\eta(\varepsilon_1,\varepsilon_2) = \varepsilon_1 - \varepsilon_2$ in Theorem 7, then Theorem 7 reduces to [14, Theorem 7].

REMARK 12. If we set $\eta(\varepsilon_2,\varepsilon_1) = \varepsilon_2 - \varepsilon_1$, $\eta(\varepsilon_1,\varepsilon_2) = \varepsilon_1 - \varepsilon_2$, and $q \to 1^-$ in Theorem 7, then we have following inequality

$$\begin{aligned} & \left| \frac{1}{\varepsilon_2-\varepsilon_1} \int_{\varepsilon_1}^{\varepsilon_2} \Phi(\mu)\ d\mu - \frac{1}{8}\left[\Phi(\varepsilon_1) + 3\Phi\left(\frac{\varepsilon_1+2\varepsilon_2}{3}\right) + 3\Phi\left(\frac{2\varepsilon_1+\varepsilon_2}{3}\right) + \Phi(\varepsilon_2) \right]\right| \\ \leq\ & \frac{25(\varepsilon_2-\varepsilon_1)}{576} \left[|\Phi'(\varepsilon_1)| + |\Phi'(\varepsilon_2)| \right] \end{aligned}$$

which was derived as special case of an inequality proved in [21].

THEOREM 8. *We assume that the assumptions of Lemma 3 hold. If $\left|{}^{\varepsilon_2}D_q\Phi\right|^r$ is preinvex and integrable on I where $r > 1$ with $\frac{1}{r}+\frac{1}{s}=1$, then we have*

$$\begin{aligned} & \left| \frac{1}{\eta(\varepsilon_2,\varepsilon_1)} \int_{\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}^{\varepsilon_2} \Phi(\mu)\ {}^{\varepsilon_2}d_q\mu \right. \\ & -\frac{1}{8}\left[\Phi(\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)) + 3\Phi\left(\frac{3\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}{3}\right) \right. \\ & \left.\left. +3\Phi\left(\frac{3\varepsilon_2+2\eta(\varepsilon_1,\varepsilon_2)}{3}\right) + \Phi(\varepsilon_2) \right]\right| \\ \leq\ & \eta(\varepsilon_2,\varepsilon_1) \left[\left(\frac{5^s}{3.8^s}\right)^{\frac{1}{s}} \right. \\ & \times \left(\frac{1}{9(1+q)} \left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_1)\right|^r + \frac{3q+2}{9(1+q)} \left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_2)\right|^r \right)^{\frac{1}{r}} \\ & + \left(\frac{2.3^s-1}{3.6^s}\right)^{\frac{1}{s}} \\ & \times \left(\frac{3}{9(1+q)} \left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_1)\right|^r + \frac{3q}{9(1+q)} \left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_2)\right|^r \right)^{\frac{1}{r}} \\ & + \left(\frac{3.7^s-2}{3.8^s}\right)^{\frac{1}{s}} \\ & \left. \times \left(\frac{5}{9(1+q)} \left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_1)\right|^r + \frac{3q-2}{9(1+q)} \left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_2)\right|^r \right)^{\frac{1}{r}} \right]. \end{aligned} \tag{5.2}$$

Proof. If the techniques used in the proof of Theorem 5 are applied by taking into account the Lemma 3, the desired inequality (5.2) can be attained. □

COROLLARY 4. *In Theorem 8, if we take limit $q \to 1^-$, then we have following inequality*

$$\begin{aligned}
(5.3)\qquad & \left|\frac{1}{\eta(\varepsilon_2,\varepsilon_1)}\int_{\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}^{\varepsilon_2}\Phi(\mu)\ d\mu \right.\\
& -\frac{1}{8}\left[\Phi(\varepsilon_1+\eta(\varepsilon_1,\varepsilon_2))+3\Phi\left(\frac{3\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}{3}\right)\right.\\
& \left.\left.+3\Phi\left(\frac{3\varepsilon_2+2\eta(\varepsilon_1,\varepsilon_2)}{3}\right)+\Phi(\varepsilon_2)\right]\right|\\
\leq\ & \frac{\eta(\varepsilon_2,\varepsilon_1)}{3}\left[\frac{5}{8}\left(\frac{|\Phi'(\varepsilon_1)|^r+5|\Phi'(\varepsilon_2)|^r}{6}\right)^{\frac{1}{r}}\right.\\
& +\left(\frac{2.3^s-1}{6^s}\right)^{\frac{1}{s}}\left(\frac{|\Phi'(\varepsilon_1)|^r+|\Phi'(\varepsilon_2)|^r}{2}\right)^{\frac{1}{r}}\\
& \left.+\left(\frac{3.7^s-2}{8^s}\right)^{\frac{1}{s}}\left(\frac{5|\Phi'(\varepsilon_1)|^r+|\Phi'(\varepsilon_2)|^r}{6}\right)^{\frac{1}{r}}\right].
\end{aligned}$$

REMARK 13. If we set $\eta(\varepsilon_2,\varepsilon_1)=\varepsilon_2-\varepsilon_1$ and $\eta(\varepsilon_1,\varepsilon_2)=\varepsilon_1-\varepsilon_2$ in (5.3), then inequality (5.3) reduces to inequality presented in [14, Remark 4].

REMARK 14. If we set $\eta(\varepsilon_2,\varepsilon_1)=\varepsilon_2-\varepsilon_1$ and $\eta(\varepsilon_1,\varepsilon_2)=\varepsilon_1-\varepsilon_2$ in Theorem 8, then Theorem 8 reduces to [14, Theorem 8].

THEOREM 9. *Suppose that the assumptions of Lemma 3 hold. If $|{}^{\varepsilon_2}D_q\Phi|^r$ is preinvex and integrable on I where $r \geq 1$, then we have*

$$\begin{aligned}
(5.4)\qquad & \left|\frac{1}{\eta(\varepsilon_2,\varepsilon_1)}\int_{\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}^{\varepsilon_2}\Phi(\mu)\ {}^{\varepsilon_2}d_q\mu \right.\\
& -\frac{1}{8}\left[\Phi(\varepsilon_1+\eta(\varepsilon_1,\varepsilon_2))+3\Phi\left(\frac{3\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}{3}\right)\right.\\
& \left.\left.+3\Phi\left(\frac{3\varepsilon_2+2\eta(\varepsilon_1,\varepsilon_2)}{3}\right)+\Phi(\varepsilon_2)\right]\right|\\
\leq\ & \eta(\varepsilon_2,\varepsilon_1)(\Psi_7(q))^{1-\frac{1}{r}}\\
& \times\left[\Psi_1(q)\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_1)\right|^r+\Psi_2(q)\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_2)\right|^r\right]^{\frac{1}{r}}\\
& +\eta(\varepsilon_2,\varepsilon_1)(\Psi_8(q))^{1-\frac{1}{r}}\\
& \times\left[\Psi_3(q)\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_1)\right|^r+\Psi_4(q)\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_2)\right|^r\right]^{\frac{1}{r}}\\
& +\eta(\varepsilon_2,\varepsilon_1)(\Psi_9(q))^{1-\frac{1}{r}}\\
& \times\left[\Psi_5(q)\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_1)\right|^r+\Psi_6(q)\left|{}^{\varepsilon_2}D_q\Phi(\varepsilon_2)\right|^r\right]^{\frac{1}{r}}
\end{aligned}$$

where $\Psi_i : i = 1, 2, ...6$ *are defined as in Theorem 7 . Moreover,* Ψ_7, Ψ_8, Ψ_9 *are defined as*

$$\Psi_7(q) = \int_0^{\frac{1}{3}} \left| qt - \frac{1}{8} \right| d_q t = \begin{cases} \frac{3-5q}{72(1+q)} & 0 < q < \frac{3}{8} \\ \frac{20q-3}{288(1+q)} & \frac{3}{8} \le q < 1, \end{cases}$$

$$\Psi_8(q) = \int_{\frac{1}{3}}^{\frac{2}{3}} \left| qt - \frac{1}{2} \right| d_q t = \begin{cases} \frac{3-3q}{18(1+q)} & 0 < q < \frac{3}{4} \\ \frac{q}{18(1+q)} & \frac{3}{4} \le q < 1, \end{cases}$$

$$\Psi_9(q) = \int_{\frac{2}{3}}^{1} \left| qt - \frac{7}{8} \right| d_q t = \begin{cases} \frac{21-19q}{72(1+q)} & 0 < q < \frac{7}{8} \\ \frac{21-4q}{288(1+q)} & \frac{7}{8} \le q < 1. \end{cases}$$

Proof. The proof follows on the same lines used in the proof of Theorem 6 by taking into account the Lemma 3. □

Corollary 5. *In Theorem 9, if we take limit* $q \to 1^-$, *then we have following inequality*

$$\begin{aligned} & \left| \frac{1}{\eta(\varepsilon_2, \varepsilon_1)} \int_{\varepsilon_2+\eta(\varepsilon_1,\varepsilon_2)}^{\varepsilon_2} \Phi(\mu)\ d\mu \right. \\ & -\frac{1}{8} \left[\Phi(\varepsilon_1 + \eta(\varepsilon_1, \varepsilon_2)) + 3\Phi\left(\frac{3\varepsilon_2 + \eta(\varepsilon_1, \varepsilon_2)}{3} \right) \right. \\ & \left. \left. +3\Phi\left(\frac{3\varepsilon_2 + 2\eta(\varepsilon_1, \varepsilon_2)}{3} \right) + \Phi(\varepsilon_2) \right] \right| \\ \le\ & \frac{\eta(\varepsilon_2, \varepsilon_1)}{36} \left[\left(\frac{17}{16} \right)^{1-\frac{1}{r}} \left(\frac{251}{1152} |\Phi'(\varepsilon_1)|^r + \frac{973}{1152} |\Phi'(\varepsilon_2)|^r \right)^{\frac{1}{r}} \right. \\ & + \left(\frac{|\Phi'(\varepsilon_1)|^r + |\Phi'(\varepsilon_2)|^r}{2} \right)^{\frac{1}{r}} \\ & + \left(\frac{17}{16} \right)^{1-\frac{1}{r}} \left(\frac{973}{1152} |\Phi'(\varepsilon_1)|^r + \frac{251}{1152} |\Phi'(\varepsilon_2)|^r \right)^{\frac{1}{r}}. \end{aligned} \tag{5.5}$$

Remark 15. If we set $\eta(\varepsilon_2, \varepsilon_1) = \varepsilon_2 - \varepsilon_1$ and $\eta(\varepsilon_1, \varepsilon_2) = \varepsilon_1 - \varepsilon_2$ in (5.5), then inequality (5.5) reduces to inequality presented in [14, Remark 5].

Remark 16. If we set $\eta(\varepsilon_2, \varepsilon_1) = \varepsilon_2 - \varepsilon_1$ and $\eta(\varepsilon_1, \varepsilon_2) = \varepsilon_1 - \varepsilon_2$ in Theorem 9, then Theorem 9 reduces to [14, Theorem 9].

6. Concluding Remarks

In this paper, we proved some new inequalities of Simpson's and Newton's type for q-differentiable preinvex functions by using the notions of q^{ε_2}-integral. It is also shown that some classical results can be obtained by the results presented in the current research by taking limit $q \to 1^-$. It will be an interesting problem to prove similar inequalities for the functions of two variables.

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