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A GENERALIZED APPROACH TOWARDS NORMALITY FOR TOPOLOGICAL SPACES

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Abstract. A uniform study towards normality is provided for topological spaces. Following Császár, γ-normality and $\gamma(\theta)$-normality are introduced and investigated. For $\gamma \in \Gamma_{13}$, γ-normality is found to satisfy Urysohn's lemma and provide partition of unity. Several existing variants of normality such as θ-normality, Δ-normality etc. are shown to be particular cases of $\gamma(\theta)$-normality. In this process, γ-regularity and $\gamma(\theta)$-regularity are introduced and studied. Several important characterizations of all these notions are provided.

1. Introduction

Normal spaces occupy a special place in the study of topological structures, as they are rich sources of continuous functions. Several important variants of normality are available in the literature such as θ-normality [7], Δ-normality [3] etc. Here arises a natural question: Does there exist a common approach to study these different variants of normality? In this paper, we have provided an answer to this question in affirmative. We have used monotonic mapping approach for this purpose. Following Császár [2], we have introduced γ-normality and $\gamma(\theta)$-normality for topological spaces. We have also introduced γ-regularity and $\gamma(\theta)$-regularity in the process. Some important characterizations of all these notions are provided. It is found that for $\gamma = int$, γ-normality and $\gamma(\theta)$-normality reduce to normality and θ-normality respectively. On the other hand, for $\gamma = cl\ int$, $\gamma(\theta)$-normality reduces to Δ-normality. Further it is proved that for $\gamma \in \Gamma_{13}$, γ-normal spaces satisfy Urysohn's lemma and provide partition of unity as well. Thus this study, while providing a uniform study of normality, has also brought to light a wide variety of topological spaces which are rich sources of continuous functions.

2. Basic Definitions and Preliminaries

Á. Császár [2] has used a map $\gamma : P(X) \longrightarrow P(X)$, where $P(X)$ is the power set of X, as his main tool for developing a generalized form of topological space. The

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map γ possesses the property of monotonicity, which says that, if $A \subseteq B$ then $\gamma(A) \subseteq \gamma(B)$.
The collection of all such mappings on X is denoted by $\Gamma(X)$, or simply by Γ.
Some interesting subfamilies of Γ, possessing some additional properties, are
Γ_0 : $\gamma(\emptyset) = \emptyset$;
Γ_1 : $\gamma(X) = X$;
Γ_2 : $\gamma^2(A) = \gamma(A)$ $\forall A \subseteq X$;
Γ_+ : $A \subseteq \gamma(A)$ $\forall A \subseteq X$;
Γ_- : $\gamma(A) \subseteq A$ $\forall A \subseteq X$.

Definition 2.1. [2] Consider a non empty set X and a map $\gamma \in \Gamma(X)$. We say that a subset A of X is *γ-open* if $A \subseteq \gamma(A)$.

Definition 2.2. Let (X, τ) be a topological space. A subset A of (X, τ) is called
(*i*) *semi open* [8] if $A \subseteq cl\ int(A)$;
(*ii*) *α-open* [10] if $A \subseteq int\ cl\ int(A)$;
(*iii*) *pre open* [9] if $A \subseteq int\ cl(A)$;
(*iv*) *β-open* [1] if $A \subseteq cl\ int\ cl(A)$.
(*v*) *regular open* set (*resp. regular closed* set) [5] if $A = int\ cl(A)$ (resp. $A = cl\ int(A)$).

Now, suppose X is a topological space and $\gamma \in \Gamma(X)$.
If we take $\gamma = int$, the interior operator of X, then the γ-open sets are precisely the open sets of X. In the same way, if we consider γ equal to *int cl, cl int, int cl int, cl int cl* then we get pre open, semi open, α-open, β-open sets respectively.

Definition 2.3. [2] Let A be a subset of X and γ be a monotonic mapping on X. Then the union of all γ-open sets contained in A is called the *γ-interior* of A, and is denoted by $i_\gamma(A)$.

Proposition 2.4. *[2] A subset A of X is γ-open if and only if $A = i_\gamma(A)$ if and only if A is i_γ-open.*

Definition 2.5. [2] A subset A of X is called *γ-closed* if $X \setminus A$ is γ-open.

Definition 2.6. [2] The intersection of all γ-closed sets containing A is called *γ-closure* of A and is denoted by $c_\gamma(A)$.

It can be shown that $c_\gamma(A)$ is the smallest γ-closed set containing A.
Another operator called γ^* is defined, with the help of γ in the following way:

Definition 2.7. [2] For any $A \subseteq X$ and $\gamma \in \Gamma(X)$, we define

$$\gamma^*(A) = X \setminus (\gamma(X \setminus A)).$$

Proposition 2.8. *[2] A subset A of X is γ^*-closed if and only if $\gamma(A) \subseteq A$.*

Definition 2.9. [2] Let X be a topological space. For $\gamma \in \Gamma(X)$, we say $\gamma \in \Gamma_3 \subseteq \Gamma(X)$, if for any open set G and $A \subseteq X$, we have

$$G \cap \gamma(A) \subseteq \gamma(G \cap A).$$

The subclass $\Gamma_{13} \subseteq \Gamma$ is defined as $\Gamma_{13} = \Gamma_1 \cap \Gamma_3$.

Proposition 2.10. *[2] If $\gamma \in \Gamma_{13}$, then every open set is γ-open.*

Definition 2.11. A topological space X is called *γ-Hausdorff space* if for each pair of distinct points x_1 and x_2 of X, there exist disjoint γ-open neighbourhoods U_1 and U_2 of x_1 and x_2, respectively.

Definition 2.12. [11] Let X be a topological space and let $A \subseteq X$. A point $x \in X$ is in *θ-closure* of A if every closed neighbourhood of x intersects A. The θ-closure of A is denoted by $cl_\theta(A)$. The set A is called *θ-closed* if $A = cl_\theta A$.

The complement of a θ-closed set is called *θ-open* set.

Definition 2.13. [11] Let X be a topological space and let $A \subseteq X$. A point $x \in X$ is in *δ-closure* of A if every regular open neighbourhood of x intersects A. The δ-closure of A is denoted by $cl_\delta(A)$. The set A is called *δ-closed* if $A = cl_\delta(A)$.

The complement of a δ-closed set is called *δ-open* set.

Definition 2.14. A topological space X is said to be

1. [7] *θ-normal* if every pair of disjoint closed sets one of which is θ-closed are contained in disjoint open sets;
2. [3] *Δ-normal* if every pair of disjoint closed sets one of which is δ-closed are contained in disjoint open sets.

Through out this paper, $\gamma \in \Gamma$ always means $\gamma \in \Gamma_3$, unless explicitly stated.

3. γ-regularity and $\gamma(\theta)$-regularity

Definition 3.1. Let X be a topological space and $\gamma \in \Gamma_3$. Let $A \subseteq X$. Then a point $x \in X$ is in *$\gamma(\theta)$-closure* of A if $c_\gamma(U) \cap A \neq \emptyset$ for all open neighbourhood U of x.

The $\gamma(\theta)$-closure of A is denoted by $c_{\gamma(\theta)}(A)$. The set A is called *$\gamma(\theta)$-closed* if $A = c_{\gamma(\theta)}(A)$.
The complement of $\gamma(\theta)$-closed set is *$\gamma(\theta)$-open* set.

Theorem 3.2. *A subset $A \subseteq X$ is $\gamma(\theta)$-open, where $\gamma \in \Gamma$ if and only if for each $x \in A$, there is an open neighbourhood U such that $x \in U \subseteq c_\gamma(U) \subseteq A$.*

Definition 3.3. Let (X, τ) be a topological space and $\gamma \in \Gamma_3$. Then X is said to be *$\gamma(\theta)$-regular* (resp. *γ-regular*) if for each pair consisting of a point x and $\gamma(\theta)$-closed (resp. γ-closed) set A disjoint from x, there exist disjoint open sets containing x and A respectively.

Remark 3.4. If $\gamma \in \Gamma_{13}$, then $cl(A) \subseteq c_{\gamma(\theta)}(A)$, therefore every $\gamma(\theta)$-closed set is closed. Hence

$$\gamma\text{-regular} \Longrightarrow \text{regular} \Longrightarrow \gamma(\theta)\text{-regular}$$

But the converse is not true in general. Here is an example:

Example 3.5. Let $X = \mathbb{R}$ be the set of real numbers with usual topology $\mathcal{U}$. Consider $\gamma = cl\ int$ mapping. Then $\gamma \in \Gamma_{13}$. If we take $A = (0,1) \cup \{\pi\}$ and a point 1 which is disjoint from A. Here A is γ-closed because $\gamma^*(A) = [cl(int)]^*(A) = int(cl(A)) = (0,1) \subseteq A$. Here $(X,\mathcal{U})$ is regular but not γ-regular as disjoint pair A and 1 cannot be separated by disjoint pair of open sets in $(X,\mathcal{U})$.

In the next theorem, we provide a characterization for $\gamma(\theta)$-regular and γ-regular spaces.

Theorem 3.6. *Let (X, τ) be a topological space. Then X is $\gamma(\theta)$-regular (resp. γ-regular) if and only if for a given point x of X and a $\gamma(\theta)$-open (resp. γ-open) neighbourhood U of x, there is an open neighbourhood V of x such that $cl(V) \subseteq U$.*

Proof. Suppose that X is $\gamma(\theta)$-regular (resp. γ-regular) and suppose x and a $\gamma(\theta)$-open (resp. γ-open) neighbourhood U of x are given. Then $B = X \setminus U$, is a $\gamma(\theta)$-closed (resp. γ-closed) set disjoint from x. By the given hypothesis, there exist disjoint open sets V and W containing x and B respectively, that is, $x \in V$ and $B \subseteq W$. Then $cl(V)$ is disjoint from B and hence $x \in V \subseteq cl(V) \subseteq X \setminus W \subseteq X \setminus B \subseteq U$, that is $cl(V) \subseteq U$.
Conversely, suppose that a point x and a $\gamma(\theta)$-closed (resp. γ-closed) set B not containing x are given. Then $U = X \setminus B$ is a $\gamma(\theta)$-open (resp. γ-open) set containing x. Therefore there exists a neighbourhood V of x such that $cl(V) \subseteq U$. Then the open sets V and $X \setminus cl(V)$ are disjoint open sets containing x and B respectively. Thus (X, τ) is $\gamma(\theta)$-regular (resp. γ-regular). □

Lemma 3.7. [4] *A topological space X is regular if and only if every closed set in X is θ-closed.*

Therefore, if a topological space X is regular and $\gamma \in \Gamma_{13}$ then it is $\gamma(\theta)$-regular and θ-regular also.

4. γ-normality and $\gamma(\theta)$-normality

Definition 4.1. Let (X, τ) be a topological space and $\gamma \in \Gamma$. Then X is said to be *$\gamma(\theta)$-normal* (resp. *γ-normal*) if every pair of disjoint sets one of which is closed and the other is $\gamma(\theta)$-closed (resp. γ-closed) are contained in disjoint open sets.

For $\gamma \in \Gamma_{13}$, we have $cl(A) \subseteq c_{\gamma(\theta)}(A) \subseteq cl_\theta(A)$. Hence

$$\gamma\text{-normality} \Longrightarrow \text{normality} \Longrightarrow \gamma(\theta)\text{-normality} \Longrightarrow \theta\text{-normality}$$

But converse is not true. Here are the examples:

Example 4.2. Let $X = \{a, b, c, d\}$ with topology τ on X generated by $\mathcal{S} = \{\{a, b\}, \{b, c\}, \{d\}\}$ as a subbase. If we take $\gamma =$ interior operator, then X is $\gamma(\theta)$-normal. But X is not normal as disjoint sets $\{a\}$ and $\{c\}$ can not be separated by disjoint pair of open sets.

Example 4.3. Let $X = \mathbb{N}$, the set of natural numbers with the co-finite topology τ on it. Then for any $A \subseteq X$, $c_{\gamma(\theta)}(A) = cl_\theta(A) = X$, for $\gamma = int$. Hence (X, τ) is $\gamma(\theta)$-normal. But clearly, X is not normal.

Similarly, in Example 3.5, $(X, \mathcal{U})$ is a normal space but not γ-normal for $\gamma = cl\ int$. As $A = (0, 1) \cup \{\pi\}$ is γ-closed and $B = \{1\}$ is closed in $(X, \mathcal{U})$. But there does not exist any disjoint pair of open sets which can separate them.

Also it is clear that if (X, τ) is $\gamma(\theta)$-normal (resp. γ-normal) T_1 topological space, then it is $\gamma(\theta)$-regular (resp. γ-regular) space.
Here it may be mentioned that there exists many other monotonic mappings apart from *int*, *cl* etc. which belong to the family Γ_{13} in a topological space. Because of

this fact, we get newer families of γ-normal and $\gamma(\theta)$-normal spaces, not encountered earlier. Below we provide an example of such mapping:

DEFINITION 4.4. [6] Let (X, τ) be a topological space and $A \subseteq X$. Then *minimal cover* of A, denoted by $\mathcal{C}_m(A)$, is defined as

$$\mathcal{C}_m(A) = \bigcap\{U : U \in \tau, A \subseteq U\}.$$

EXAMPLE 4.5. Consider a space X with a topology τ. Consider a mapping $\gamma = int(\mathcal{C}_m)$. Here $int \in \Gamma_{13}$ and $\mathcal{C}_m \in \Gamma_{13}$ as well. Therefore $int(\mathcal{C}_m) \in \Gamma_{13}$. As for any open set G and any subset $A \subseteq X$, we have $G \cap \mathcal{C}_m(A) = \mathcal{C}_m(G \cap A)$, because $\mathcal{C}_m(A \cap B) = \mathcal{C}_m(A) \cap \mathcal{C}_m(B)$ for any $A, B \subseteq X$ and $\mathcal{C}_m(G) = G$, for all $G \in \tau$ and $\mathcal{C}_m(X) = X$ as well. Hence $\mathcal{C}_m \in \Gamma_{13}$.

In our next theorem, we provide few characterizations of $\gamma(\theta)$-normal (resp. γ-normal) spaces.

THEOREM 4.6. *For a topological space (X, τ), the following are equivalent:*

(a) *X is $\gamma(\theta)$-normal (resp. γ-normal);*
(b) *for every closed set A and every $\gamma(\theta)$-open (resp. γ-open) set B containing A, there exists an open set U such that $A \subseteq U \subseteq cl(U) \subseteq B$;*
(c) *for every $\gamma(\theta)$-closed (resp. γ-closed) set A and every open set B containing A, there exists an open set U such that $A \subseteq U \subseteq cl(U) \subseteq B$;*
(d) *for every pair consisting of disjoint sets A and B, one of which is closed and other is $\gamma(\theta)$-closed (resp. γ-closed), there exist open sets U and V such that $A \subseteq U$, $B \subseteq V$ and $cl(U) \cap cl(V) = \emptyset$.*

Proof. **(a) ⇒ (b):** Let A be a closed set and B be any $\gamma(\theta)$-open (resp. γ-open) set containing A. Then $X \setminus B$ is a $\gamma(\theta)$-closed (resp. γ-closed) set disjoint from A. Therefore there exist disjoint open sets U and V such that $A \subseteq U$, $X \setminus B \subseteq V$ and $U \cap V = \emptyset$. Hence, we have $A \subseteq U \subseteq cl(U) \subseteq X \setminus V \subseteq B$, that is, $A \subseteq U \subseteq cl(U) \subseteq B$.
(b) ⇒ (c): Let B be any open set and A be any $\gamma(\theta)$-closed (resp. γ-closed) set contained in B. Then $X \setminus A$ is a $\gamma(\theta)$-open (resp. γ-open) set containing a closed set $X \setminus B$. Thus, by the given hypothesis, there exists an open set U such that $X \setminus B \subseteq U \subseteq cl(U) \subseteq X \setminus A$. Therefore $A \subseteq X \setminus cl(U) \subseteq X \setminus U \subseteq B$. We take $X \setminus cl(U) = V$, which is open in X. Hence $A \subseteq V \subseteq cl(V) \subseteq B$.
(c) ⇒ (d): Let A be a $\gamma(\theta)$-closed (resp. γ-closed) set and B be a closed set such that $A \cap B = \emptyset$. Then $A \subseteq X \setminus B$. The $\gamma(\theta)$-closed (resp. γ-closed) set A contained in an open set $X \setminus B$. Thus, there exists an open set M such that $A \subseteq M \subseteq cl(M) \subseteq X \setminus B$. Again M is an open set containing the $\gamma(\theta)$-closed (resp. γ-closed) set A, therefore there exists another open set U, such that $A \subseteq U \subseteq cl(U) \subseteq M$. We take $X \setminus cl(M) = V$, then V is open in X. Then we have, $A \subseteq U$, $B \subseteq V$ and $cl(U) \cap cl(V) = \emptyset$.
(d) ⇒ (a): Obvious from the definition of $\gamma(\theta)$-normal (resp. γ-normal) space. □

In the light of Lemma 3.7, it is immediate that in the class of regular spaces, $\gamma(\theta)$-normality and θ-normality coincide with normality whenever $\gamma \in \Gamma_{13}$.

LEMMA 4.7. *If a topological space X is γ-Hausdorff then every singleton in X is $\gamma(\theta)$-closed.*

Proof. Let X be a γ-Hausdorff space. Then, we have to show that every $\{a\} \subseteq X$ is $\gamma(\theta)$-closed, that is $X \setminus \{a\}$ is $\gamma(\theta)$-open. Consider $x \in X \setminus \{a\}$. Then $x \neq a$, since x and a are distinct point. Therefore, there exist disjoint γ-open sets U and V such that $x \in U$ and $a \in V$. Then $x \in U \subseteq X \setminus V \subseteq X \setminus \{a\}$. Thus, $x \in U \subseteq c_\gamma(U) \subseteq X \setminus V \subseteq X \setminus \{a\}$, because $X \setminus V$ is γ-closed. Hence $X \setminus \{a\}$ is $\gamma(\theta)$-open. Therefore $\{a\}$ is $\gamma(\theta)$-closed. □

THEOREM 4.8. *Every γ-normal Hausdorff space, where $\gamma \in \Gamma_{13}$ is regular.*

Proof. Let (X, τ) be a γ-normal Hausdorff space and $\gamma \in \Gamma_{13}$. Since X is a Hausdorff space. By Lemma 4.7, we have every singleton is θ-closed, that is $cl_\theta(\{x\}) = \{x\}$. Therefore every singleton is γ-closed as well. Let x and A be a disjoint pair of a point and a closed set. Thus by γ-normality, there exists a pair of disjoint open sets U and V such that $x \in U$ and $A \subseteq V$. Hence X is regular. □

THEOREM 4.9. *For a Hausdorff space X and $\gamma \in \Gamma_{13}$, the following are equivalent:*

(i) *X is normal;*
(ii) *X is $\gamma(\theta)$-normal;*
(iii) *X is θ-normal.*

Proof. $(i) \implies (ii) \implies (iii)$ is obvious from the definition of $\gamma(\theta)$-closed set and θ-closed.

$(iii) \implies (i)$ Let X be a θ-normal Hausdorff space. Then by Lemma 4.7 every singleton in X is θ-closed. Therefore by θ-normality of the space X, every closed set in X and a point disjoint from it are contained in disjoint open sets. Thus X is regular and by Lemma 3.7, every closed set in X is θ-closed. Hence every pair of disjoint closed sets in X are separated by disjoint open sets. □

Our next theorem shows that paracompactness is a sufficient condition for $\gamma(\theta)$-regularity (resp. γ-regularity) to imply $\gamma(\theta)$-normality (resp. γ-normality).

THEOREM 4.10. *Every paracompact $\gamma(\theta)$-regular (resp. γ-regular) space is $\gamma(\theta)$-normal (resp. γ-normal).*

Proof. Let A and B be disjoint sets such that one of them, say A is closed and the other one, say B, is $\gamma(\theta)$-closed (resp. γ-closed). Then $A \subseteq X \setminus B$ and $X \setminus B$ a $\gamma(\theta)$-open (resp. γ-open). Thus for each $a \in A$, there exists an open set U_a such that $a \in U_a \subseteq cl(U_a) \subseteq X \setminus B$, as X is $\gamma(\theta)$-regular (resp. γ-regular) space. Thus the collection $\mathcal{U} = \{U_a : a \in A\} \cup \{X \setminus A\}$ is an open cover of X. Then by the given hypothesis, there is a locally finite open refinement $\mathcal{V}$ of $\mathcal{U}$.
Let $\mathcal{D}$ denote the sub collection of $\mathcal{V}$ consisting of those members which intersects A. Then $\mathcal{D}$ covers A. Further if $D \in \mathcal{D}$, then $cl(D)$ is disjoint from B and intersects A. Let $V = \bigcup\{D : D \in \mathcal{D}\}$. Then V is an open set of X containing A. Since $D \in \mathcal{D}$ which implies that $D = U_{a'}$ for some $a' \in A$, with $cl(U_{a'}) \subseteq X \setminus B$, we have $cl(V) = \bigcup\{cl(D) : D \in \mathcal{D}\}$ is disjoint from B. Thus $U = \bigcup\{U_a : a \in A\}$ and $X \setminus cl(V)$ are disjoint open sets containing A and B respectively. Hence X is $\gamma(\theta)$-normal (resp. γ-normal).

□

5. Some more results on $\gamma(\theta)$-normal and γ-normal spaces

For a given topological space, several important weaker forms of open sets can be realized as γ-open sets. For example, semi open sets, pre-open sets, α-open sets, β-open sets etc. are γ-open sets for $\gamma = cl\ int$, $int\ cl$, $int\ cl\ int$, $cl\ int\ cl$ etc. respectively. Thus γ-normality accommodates all these generalized form of open sets. Many such variations of γ also satisfying the condition that $\gamma \in \Gamma_{13}$. In the following, we show that all these spaces are rich sources of continuous functions. To be more specific, we demonstrate that γ-normal spaces for $\gamma \in \Gamma_{13}$ satisfy Urysohn's lemma. They also provide partition of unity. Similarly, $\gamma(\theta)$-normal spaces are also found to exhibit some interesting properties.

THEOREM 5.1. *Let (X, τ) be a topological space and $\gamma \in \Gamma_{13}$. Then the following are equivalent:*

(a) *for each pair of disjoint subsets A and B of X, one of which is closed and the other one is γ-closed, there exists a continuous function f on X to $[0,1]$ (resp. $[a,b]$ for any real number a, b, $a < b$), such that $f(A) = \{0\}$ and $f(B) = \{1\}$ (resp. $f(A) = \{a\}$, and $f(B) = \{b\}$);*

(b) *X is γ-normal.*

Proof. **(a) $\Rightarrow$ (b):** Let A and B be two disjoint subsets of X, one of which is closed and the other is γ-closed. Then, there exists a continuous function f on X such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Now the set $U = \{x : f(x) < 1/2\}$ and $V = \{x : f(x) > 1/2\}$ are disjoint open sets such that $A \subseteq U$ and $B \subseteq V$. Hence X is γ-normal space.
(b) $\Rightarrow$ (a): Let C_0 be a γ-closed set and C_1 be a closed set in X such that $C_0 \cap C_1 = \emptyset$. Let P be the set of all dyadic rational numbers in $[0,1]$. We shall define for each p in P, an open set U_p of X, in such a way that whenever $p < q$, we have $cl(U_p) \subset U_q$. First we define $U_1 = X \setminus C_1$, an open set such that $C_0 \subseteq U_1$. Since X is γ-normal space, there exists an open set $U_{1/2}$ such that $C_0 \subseteq U_{1/2} \subseteq cl(U_{1/2}) \subseteq U_1$. Similarly, there exists open set $U_{1/4}$ such that $C_0 \subseteq U_{1/4} \subseteq cl(U_{1/4}) \subseteq U_{1/2} \subseteq cl(U_{1/2}) \subseteq U_1$. Since every closed set is γ-closed and $cl(U_{1/2})$ is closed and hence it is γ-closed. Therefore, there exists an open set $U_{3/4}$ such that $cl(U_{1/2}) \subseteq U_{3/4} \subseteq cl(U_{3/4}) \subseteq U_1$. That is, $C_0 \subseteq U_{1/4} \subseteq cl(U_{1/4}) \subseteq U_{1/2} \subseteq cl(U_{1/2}) \subseteq U_{3/4} \subseteq cl(U_{3/4}) \subseteq U_1$. Continuing the process, we define U_r, for each $r \in P$ such that $C_0 \subseteq U_r \subseteq cl(U_r) \subseteq U_1$ and $cl(U_r) \subseteq U_s$ whenever $r < s$, for $r, s \in P$.
Let us define $\mathbf{Q}(x)$ to be the set of those dyadic rational numbers p such that the corresponding open sets U_p contain x:

$$\mathbf{Q}(x) = \{p \mid x \in U_p\}$$

Now we define a function $f : X \to [0,1]$ as

$$f(x) = \inf \mathbf{Q}(x) = \inf\{p \mid x \in U_p\}$$

Clearly, $f(C_0) = \{0\}$ and $f(C_1) = \{1\}$. Now, we show that f is the desired continuous function. For a given point $x_0 \in X$ and an open interval (c,d) in $[0,1]$ containing the point $f(x_0)$, we wish to find a neighbourhood U of x_0 such that $f(U) \subseteq (c,d)$.
Let us choose rational numbers p and q such that $c < p < f(x_0) < q < d$, the $U = U_q \setminus cl(U_p)$ is the desired open neighbourhood of x_0.

Hence f is a continuous function on X to $[0,1]$ such that $f(C_0) = \{0\}$ and $f(C_1) = \{1\}$. □

Definition 5.2. For a topological space (X, τ), a cover $\mathcal{U} = \{U_\alpha : \alpha \in \wedge\}$ of X consisting of $\gamma(\theta)$-open sets (resp. γ-open sets) is said to be *$\gamma(\theta)$-shrinkable* (*resp. γ-shrinkable*) if there exists an open cover $\mathcal{V} = \{V_\alpha : \alpha \in \wedge\}$ of X such that $cl(V_\alpha) \subseteq U_\alpha$ for each $\alpha \in \wedge$.

Theorem 5.3. *Every point finite γ-open cover of X, where $\gamma \in \Gamma_{13}$, is γ-shrinkable if X is γ-normal.*

Proof. Suppose X is γ-normal, where $\gamma \in \Gamma_{13}$ and let $\mathcal{U} = \{U_\alpha : \alpha \in \wedge\}$ be a point finite γ-open cover of X. Without loss of generality, we assume that $\wedge$ is a well ordered set and 1 is its first element.
Consider $F_1 = X \setminus \bigcup_{\alpha>1} U_\alpha$. Then F_1 is a γ-closed subset of X such that $F_1 \subseteq U_1$. Therefore, by Theorem 4.6 there exists an open set V_1 such that $F_1 \subseteq V_1 \subseteq cl(V_1) \subseteq U_1$. Thus $\{V_1, U_2, U_3, \ldots\}$ covers X, as $F_1 \subseteq V_1$. Continuing like this, suppose V_β has been defined for each $\beta < \alpha$, we define

$$F_\alpha = X \setminus \bigcup_{\beta<\alpha} V_\beta \setminus \bigcup_{\delta>\alpha} U_\delta.$$

Then F_α is a γ-closed set because every open set is γ-open whenever $\gamma \in \Gamma_{13}$. Therefore we have a γ-closed set F_α contained in a open set U_α, that is $F_\alpha \subseteq U_\alpha$. So by Theorem 4.6, there exists an open set V_α such that $F_\alpha \subseteq V_\alpha \subseteq cl(V_\alpha) \subseteq U_\alpha$.
Then $\mathcal{V} = \{V_\alpha : \alpha \in \wedge\}$ is a γ-shrinking of $\mathcal{U}$, provided it covers X. Let $x \in X$. Then x belongs to only finitely many members of $\mathcal{U}$, say $U_{\alpha_1}, U_{\alpha_2}, \ldots U_{\alpha_k}$. Then choose $\alpha = \max\{\alpha_1, \alpha_2, \ldots, \alpha_k\}$. Now $x \notin U_\delta$ for $\delta > \alpha$ and hence if $x \notin V_\beta$ for $\beta < \alpha$, then $x \in F_\alpha \subseteq V_\alpha$. Thus in any case $x \in V_\beta$ for some $\beta \leq \alpha$. Therefore $\mathcal{V}$ covers X and hence $\mathcal{V}$ is a γ-shrinking of $\mathcal{U}$. □

In a similar way, we can prove the following theorem:

Theorem 5.4. *Every point finite $\gamma(\theta)$-open cover of X, where $\gamma \in \Gamma_{13}$, is $\gamma(\theta)$-shrinkable if X is $\gamma(\theta)$-normal.*

Our next result for γ-normal spaces is analogous to the well known result "Existence of Partition of Unity" for normal spaces.

Definition 5.5. Let $\{U_1, U_2, \ldots, U_n\}$ be a finite indexed open covering of the space X. An indexed family of continuous functions

$$\phi_i : X \to [0,1] \text{ for } i = 1, ..., n,$$

is said to be a partition of unity dominated by $\{U_i\}$ if:

(*i*) (support ϕ_i) $\subset U_i$ for each i.
(*ii*) $\sum_{i=1}^{n} \phi_i(x) = 1$ for each x.

Theorem 5.6. *Let $\{U_1, U_2, \ldots, U_n\}$ be a finite γ-open covering of a γ-normal space X, where $\gamma \in \Gamma_{13}$. Then there exists a partition of unity dominated by $\{U_i\}$.*

Proof. Suppose X is γ-normal, where $\gamma \in \Gamma_{13}$ and let $\mathcal{U} = \{U_\alpha : \alpha = 1 \ldots n\}$ be a finite γ-open covering of X. Then by Theorem 5.3, we have another cover $\mathcal{V} = \{V_\alpha : \alpha = 1 \ldots n\}$ such that $cl(V_i) \subseteq U_i$ for each i. Since every γ-normal

space is normal, we can choose an open covering $\{W_1, W_2, \ldots, W_n\}$ of X such that $cl(W_i) \subseteq V_i \subseteq cl(V_i) \subseteq U_i$ for each i, in view of Theorem 5.3. Thus by Urhsohn's lemma, for each i, there exists a continuous function $\psi_i : X \to [0, 1]$ such that, $\psi_i(cl(W_i)) = \{1\}$ and $\psi_i(X \setminus V_i) = \{0\}$.
Thus we have (support ψ_i) $\subseteq cl(V_i) \subseteq U_i$. Because the collection $\{W_i\}$ covers X, then the sum $\sum_{i=1}^{n} \psi_i(x) = \Psi(x)$ is positive for each x. Therefore, we may define, for each j, $\phi_j(x) = \dfrac{\psi_j(x)}{\Psi(x)}$. Then the collection $\phi_1, \phi_2, \ldots \phi_n$ form the desired partition of unity. □

6. Conclusion

If we take $\gamma =$ interior operator then γ-closure of a set is nothing but the closure of the set and γ-closed sets are nothing but the closed set. Hence, $\gamma(\theta)$-normality is nothing but θ-normality and γ-normality coincides with normality. Whereas if we consider $\gamma = cl\ int$, that is, closure interior operator, then γ-closure of a set is nothing but the semi closure of the set. Then $C_\gamma(A) = A \cup int(cl(A))$. Hence, for every open set U, $C_\gamma(U) = int(cl(U))$, which is a regular open set. Since every regular open set is an open set. Thus $\gamma(\theta)$-normal spaces coincide with Δ-normal spaces. Therefore, the investigations provided in this paper is a generalized approach towards normality. On the one hand, it presents the existing variants of normality, such as normality, Δ-normality, θ-normality etc. as particular cases in our study. On the other hand, it also establishes existence of several other variants of normality by taking $\gamma = int\ cl$, $cl\ int$, $int\ cl\ int$, $cl\ int\ cl$ etc. for a topological space.

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