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STABILITY OF THE JENSEN TYPE FUNCTIONAL EQUATION IN BANACH ALGEBRAS: A FIXED POINT APPROACH

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Abstract. Using fixed point methods, we prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras and of derivations on Banach algebras for the following Jensen type functional equation:

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=f(x).$$

1. Introduction and preliminaries

The stability problem of functional equations was originated from a question of Ulam [30] concerning the stability of group homomorphisms: Let $(G_1, \star)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exist a $\delta(\varepsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x \star y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \varepsilon$$

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for all $x \in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x \star y) = H(x) \diamond H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

Hyers [8] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon$$

for all $x, y \in X$ and some $\varepsilon \geq 0$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leq \varepsilon$$

for all $x \in X$.

Th.M. Rassias [20] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

THEOREM 1.1. (Th.M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ε and p are constants with $\varepsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\varepsilon}{2 - 2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

The above inequality (1.1) that was introduced for the first time by Th.M. Rassias [20] for the proof of the stability of the linear mapping between Banach spaces has provided a lot of influence in the development of what is now known as *generalized Hyers-Ulam stability* or as *Hyers-Ulam-Rassias stability* of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [7] extended the Hyers-Ulam stability by proving the following theorem in the spirit of Th.M. Rassias' approach.

THEOREM 1.2. [7] *Let $f : E \to E'$ be a mapping for which there exists a function $\varphi : E \times E' \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x,y) := \sum_{j=0}^{\infty} 2^{-j}\varphi(2^j x, 2^j y) \quad < \quad \infty,$$
$$\|f(x+y) - f(x) - f(y)\| \quad \le \quad \varphi(x,y)$$

for all $x, y \in E$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \le \frac{1}{2}\widetilde{\varphi}(x,x)$$

for all $x \in E$.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1, 2, 4, 5, 10, 11, 13, 14, 15, 16, 17, 18, 19, 21, 22, 23, 24, 25, 26, 27, 28, 29]).

We recall the following theorem by Diaz and Margolis. The reader is referred to the book of D.H. Hyers, G. Isac and Th.M. Rassias [9] for an extensive account of fixed point theory with several applications.

THEOREM 1.3. [6] *Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) *$d(J^n x, J^{n+1}x) < \infty$ for all $n \ge n_0$;*

(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;*

(4) *$d(y, y^*) \le \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

This paper is organized as follows: In Section 2, using the fixed point method, we prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras for the Jensen type functional equation.

In Section 3, using the fixed point method, we prove the generalized Hyers-Ulam stability of derivations on Banach algebras for the Jensen type functional equation.

In 1996, G. Isac and Th.M. Rassias [12] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

Throughout this paper, assume that A is a real Banach algebra with norm $\|\cdot\|_A$ and that B is a real Banach algebra with norm $\|\cdot\|_B$.

2. Stability of homomorphisms in Banach algebras

For a given mapping $f : A \to B$, we define

$$Df(x,y) := f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - f(x)$$

for all $x, y \in A$.

Note that an $\mathbb{R}$-linear mapping $H : A \to B$ is called a *homomorphism* in Banach algebras if H satisfies $H(xy) = H(x)H(y)$ for all $x, y \in A$.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x, y \in X$;

(3) $d(x,z) \le d(x,y) + d(y,z)$ for all $x, y, z \in X$.

We prove the generalized Hyers-Ulam stability of homomorphisms in Banach algebras for the functional equation $Df(x,y) = 0$.

Theorem 2.1. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^2 \to [0, \infty)$ such that*

$$\|Df(x,y)\|_B \le \varphi(x,y), \tag{2.1}$$

$$\|f(xy) - f(x)f(y)\|_B \le \varphi(x,y) \tag{2.2}$$

for all $x, y \in A$. If for each $x \in A$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$ and if there exists an $L < 1$ such that $\varphi(x,y) \le 2L\varphi(\frac{x}{2}, \frac{y}{2})$ for all $x, y \in A$, then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \le \frac{L}{1-L}\varphi(x,0) \tag{2.3}$$

for all $x \in A$.

Proof. Consider the set

$$X := \{g : A \to B\}$$

and introduce the *generalized metric* on X:

$$d(g,h) = \inf\{C \in \mathbb{R}_+ : \|g(x) - h(x)\|_B \le C\varphi(x,0) \text{ for all } x \in A\}.$$

It is easy to show that (X, d) is complete.

Now we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := \frac{1}{2}g(2x)$$

for all $x \in A$.

By Theorem 3.1 of [3],

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in X$.

Letting $y = 0$ in (2.1), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\|_B \le \varphi(x,0)$$

for all $x \in A$. So

$$\left\|f(x) - \frac{1}{2}f(2x)\right\|_B \le \frac{1}{2}\varphi(2x,0) \le L\varphi(x,0)$$

for all $x \in A$. Hence $d(f, Jf) \le L$.

By Theorem 1.3, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J, i.e.,

$$H(2x) = 2H(x) \tag{2.4}$$

for all $x \in A$. The mapping H is a unique fixed point of J in the set

$$Y = \{g \in X : d(f,g) < \infty\}.$$

This implies that H is a unique mapping satisfying (2.4) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_B \le C\varphi(x,0)$$

for all $x \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{2^n} = H(x) \tag{2.5}$$

for all $x \in A$.

(3) $d(f,H) \le \frac{1}{1-L} d(f,Jf)$, which implies the inequality

$$d(f,H) \le \frac{L}{1-L}.$$

This implies that the inequality (2.3) holds.

One can easily show that

$$\lim_{j\to\infty} \frac{1}{2^j}\varphi(2^j x, 2^j y) = 0 \tag{2.6}$$

for all $x, y \in A$. It follows from (2.1), (2.5) and (2.6) that

$$\begin{aligned}
&\left\| H\left(\frac{x+y}{2}\right) + H\left(\frac{x-y}{2}\right) - H(x)\right\|_B \\
&= \lim_{n\to\infty} \frac{1}{2^n} \| f(2^{n-1}(x+y)) + f(2^{n-1}(x-y)) - f(2^n x)\|_B \\
&\le \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y) = 0
\end{aligned}$$

for all $x, y \in A$. So

$$H\left(\frac{x+y}{2}\right) + H\left(\frac{x-y}{2}\right) = H(x)$$

for all $x, y \in A$. Letting $z = \frac{x+y}{2}$ and $w = \frac{x-y}{2}$ in the above equation, we get

$$H(z) + H(w) = H(z+w)$$

for all $z, w \in A$. So the mapping $H : A \to B$ is Cauchy additive, i.e., $H(z+w) = H(z) + H(w)$ for all $z, w \in A$.

By the same reasoning as in the proof of Theorem 1.1 [20], one can show that the mapping $H : A \to B$ is $\mathbb{R}$-linear.

It follows from (2.2) that

$$\begin{aligned}
\|H(xy) - H(x)H(y)\|_B &= \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n xy) - f(2^n x) f(2^n y)\|_B \\
&\le \lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n x, 2^n y) \le \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y) = 0
\end{aligned}$$

for all $x, y \in A$. So

$$H(xy) = H(x)H(y)$$

for all $x, y \in A$.

Thus $H : A \to B$ is a homomorphism satisfying (2.3), as desired. □

Corollary 2.2. *Let $r < 1$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping such that*

$$\|Df(x,y)\|_B \leq \theta(\|x\|_A^r + \|y\|_A^r), \tag{2.7}$$

$$\|f(xy) - f(x)f(y)\|_B \leq \theta(\|x\|_A^r + \|y\|_A^r) \tag{2.8}$$

for all $x, y \in A$. If for each $x \in A$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{2^r\theta}{2 - 2^r}\|x\|_A^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.1 by taking

$$\varphi(x,y) := \theta(\|x\|_A^r + \|y\|_A^r)$$

for all $x, y \in A$. Then we can choose $L = 2^{r-1}$ and we get the desired result. □

Theorem 2.3. *Let $f : A \to B$ be a mapping for which there exists a function $\varphi : A^2 \to [0,\infty)$ satisfying* (2.1) *and* (2.2)*. If for each $x \in A$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$ and if there exists an $L < 1$ such that $\varphi\big(\frac{x}{2}, \frac{y}{2}\big) \leq \frac{L}{4}\varphi(x,y)$ for all $x, y \in A$, then there exists a unique homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \leq \frac{1}{1-L}\varphi(x,0) \tag{2.9}$$

for all $x \in A$.

Proof. Consider the complete generalized metric space (X,d) given in the proof of Theorem 2.1.

Now we consider the linear mapping $J : X \to X$ such that

$$Jg(x) := 2g\left(\frac{x}{2}\right)$$

for all $x \in A$.

By Theorem 3.1 of [3],

$$d(Jg, Jh) \leq Ld(g,h)$$

for all $g, h \in X$.

Letting $y = 0$ in (2.1), we get

$$\left\|2f\left(\frac{x}{2}\right) - f(x)\right\|_B \leq \varphi(x,0)$$

for all $x \in A$. So

$$\left\| f(x) - 2f\left(\frac{x}{2}\right) \right\|_B \leq \varphi(x, 0) \leq \frac{L}{2}\varphi(2x, 0)$$

for all $x \in A$. Hence $d(f, Jf) \leq 1$.

By Theorem 1.3, there exists a mapping $H : A \to B$ such that

(1) H is a fixed point of J. This implies that H is a unique mapping satisfying (2.4) such that there exists $C \in (0, \infty)$ satisfying

$$\|H(x) - f(x)\|_B \leq C\varphi(x, 0)$$

for all $x \in A$.

(2) $d(J^n f, H) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right) = H(x)$$

for all $x \in A$.

(3) $d(f, H) \leq \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, H) \leq \frac{1}{1-L}.$$

This implies that the inequality (2.9) holds.

One can easily show that

$$\lim_{j\to\infty} 4^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) = 0$$

for all $x, y \in A$. By (2.1), we see that

$$\begin{aligned}
&\left\| H\left(\frac{x+y}{2}\right) + H\left(\frac{x-y}{2}\right) - H(x) \right\|_B \\
&= \lim_{n\to\infty} 2^n \left\| f\left(\frac{x+y}{2^{n+1}}\right) + f\left(\frac{x-y}{2^{n+1}}\right) - f\left(\frac{x}{2^n}\right) \right\|_B \\
&\leq \lim_{n\to\infty} 2^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) \leq \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0
\end{aligned}$$

for all $x, y \in A$.

By the proof of Theorem 2.1, the mapping $H : A \to B$ is Cauchy additive.

By the same reasoning as in the proof of Theorem 1.1 [20], one can show that the mapping $H : A \to B$ is $\mathbb{R}$-linear.

It follows from (2.2) that

$$\|H(xy) - H(x)H(y)\|_B = \lim_{n\to\infty} 4^n \left\| f\left(\frac{xy}{4^n}\right) - f\left(\frac{x}{2^n}\right) f\left(\frac{y}{2^n}\right)\right\|_B$$
$$\leq \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0$$

for all $x, y \in A$. So

$$H(xy) = H(x)H(y)$$

for all $x, y \in A$.

Thus $H : A \to B$ is a homomorphism satisfying (2.9), as desired. □

COROLLARY 2.4. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A \to B$ be a mapping satisfying* (2.7) *and* (2.8). *If for each $x \in A$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique homomorphism $H : A \to B$ such that*

$$\|f(x) - H(x)\|_B \leq \frac{2^r\theta}{2^r - 4}\|x\|_A^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x, y) := \theta(\|x\|_A^r + \|y\|_A^r)$$

for all $x, y \in A$. Then we can choose $L = 2^{2-r}$ and we get the desired result. □

3. Stability of derivations on Banach algebras

Note that an $\mathbb{R}$-linear mapping $\delta : A \to A$ is called a *derivation* on A if δ satisfies $\delta(xy) = \delta(x)y + x\delta(y)$ for all $x, y \in A$.

We prove the generalized Hyers-Ulam stability of derivations on Banach algebras for the functional equation $Df(x, y) = 0$.

THEOREM 3.1. *Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^2 \to [0, \infty)$ such that*

$$\|Df(x, y)\|_A \leq \varphi(x, y), \tag{3.1}$$
$$\|f(xy) - f(x)y - xf(y)\|_A \leq \varphi(x, y) \tag{3.2}$$

for all $x, y \in A$*. If there exists an* $L < 1$ *such that* $\varphi(x,y) \le 2L\varphi(\frac{x}{2}, \frac{y}{2})$ *for all* $x, y \in A$*. If for each* $x \in A$ *the mapping* $f(tx)$ *is continuous in* $t \in \mathbb{R}$*, then there exists a unique derivation* $\delta : A \to A$ *such that*

$$\|f(x) - \delta(x)\|_A \le \frac{L}{1-L}\varphi(x,0) \tag{3.3}$$

for all $x \in A$*.*

Proof. By the same reasoning as the proof of Theorem 2.1, there exists a unique $\mathbb{R}$-linear mapping $\delta : A \to A$ satisfying (3.3). The mapping $\delta : A \to A$ is given by

$$\delta(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

for all $x \in A$.

It follows from (3.2) that

$$\begin{aligned}
&\|\delta(xy) - \delta(x)y - x\delta(y)\|_A \\
&\quad = \lim_{n\to\infty} \frac{1}{4^n}\|f(4^n xy) - f(2^n x)\cdot 2^n y - 2^n x f(2^n y)\|_A \\
&\quad \le \lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n x, 2^n y) \le \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y) = 0
\end{aligned}$$

for all $x, y \in A$. So

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for all $x, y \in A$. Thus $\delta : A \to A$ is a derivation satisfying (3.3). □

Corollary 3.2. *Let* $r < 1$ *and* θ *be nonnegative real numbers, and let* $f : A \to A$ *be a mapping such that*

$$\|Df(x,y)\|_A \le \theta(\|x\|_A^r + \|y\|_A^r), \tag{3.4}$$

$$\|f(xy) - f(x)y - xf(y)\|_A \le \theta(\|x\|_A^r + \|y\|_A^r) \tag{3.5}$$

for all $x, y \in A$*. If for each* $x \in A$ *the mapping* $f(tx)$ *is continuous in* $t \in \mathbb{R}$*, then there exists a unique derivation* $\delta : A \to A$ *such that*

$$\|f(x) - \delta(x)\|_A \le \frac{2^r\theta}{2-2^r}\|x\|_A^r$$

for all $x \in A$*.*

Proof. The proof follows from Theorem 3.1 by taking

$$\varphi(x,y) := \theta(\|x\|_A^r + \|y\|_A^r)$$

for all $x, y \in A$. Then we can choose $L = 2^{r-1}$ and we get the desired result. □

Theorem 3.3. *Let $f : A \to A$ be a mapping for which there exists a function $\varphi : A^2 \to [0, \infty)$ satisfying* (3.1) *and* (3.2)*. If there exists an $L < 1$ such that $\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \le \frac{L}{4}\varphi(x, y)$ for all $x, y \in A$. If for each $x \in A$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique derivation $\delta : A \to A$ such that*

$$\|f(x) - \delta(x)\|_A \le \frac{1}{1 - L}\varphi(x, 0) \tag{3.6}$$

for all $x \in A$.

Proof. By the same reasoning as the proof of Theorem 2.3, there exists a unique $\mathbb{R}$-linear mapping $\delta : A \to A$ satisfying (3.6). The mapping $\delta : A \to A$ is given by

$$\delta(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$.

It follows from (3.2) that

$$\begin{aligned}
&\|\delta(xy) - \delta(x)y - x\delta(y)\|_A \\
&\quad = \lim_{n\to\infty} 4^n \left\| f\left(\frac{xy}{4^n}\right) - f\left(\frac{x}{2^n}\right) \cdot \frac{y}{2^n} - \frac{x}{2^n} f\left(\frac{y}{2^n}\right)\right\|_A \\
&\quad \le \lim_{n\to\infty} 4^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0
\end{aligned}$$

for all $x, y \in A$. So

$$\delta(xy) = \delta(x)y + x\delta(y)$$

for all $x, y \in A$. Thus $\delta : A \to A$ is a derivation satisfying (3.6). □

Corollary 3.4. *Let $r > 2$ and θ be nonnegative real numbers, and let $f : A \to A$ be a mapping satisfying* (3.4) *and* (3.5)*. If for each $x \in A$ the mapping $f(tx)$ is continuous in $t \in \mathbb{R}$, then there exists a unique derivation $\delta : A \to A$ such that*

$$\|f(x) - \delta(x)\|_A \le \frac{2^r\theta}{2^r - 4}\|x\|_A^r$$

for all $x \in A$.

Proof. The proof follows from Theorem 3.3 by taking

$$\varphi(x, y) := \theta(\|x\|_A^r + \|y\|_A^r)$$

for all $x, y \in A$. Then we can choose $L = 2^{2-r}$ and we get the desired result. □

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