Korean J. Math. **29** (2021), No. 2, pp. 435–443
http://dx.doi.org/10.11568/kjm.2021.29.2.435

ON AN OPERATOR PRESERVING POLYNOMIAL INEQUALITIES

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Abstract. In this paper, we consider an operator $N : \mathcal{P}_n \to \mathcal{P}_n$ on the space of polynomials $\mathcal{P}_n$ of degree at most n and establish some compact generalizations of Bernstein-type polynomial inequalities, which include several well known polynomial inequalities as special cases.

1. Introduction and Statement of results

Polynomial inequalities have been investigated for quite some time and have important applications in all those mathematical models whose solutions lead to the problem of valuing how large or small the maximum modulus of the derivative of an algebraic polynomial can be in terms of the maximum modulus and degree of that polynomial. Polynomial inequalities are also fundamental for the proofs of many inverse theorems in polynomial approximation theory, which is concerned with approximating unknown or complicated functions by polynomials. In the first place, this concept of best approximation was introduced in Mathematics mainly by the work of the famous mathematician Chebyshev(1821-1894), who studied some properties of polynomial with least deviation from given continuous function. He introduced the polynomials known today as Chebyshev polynomials of first kind, which appear prominently in various extremal problems concerned with polynomial (see [8]). Historically, the questions relating to approximations by polynomials have given rise to some of the interesting problems in Mathematics and engendered extensive research over the past millennium. This paper deals with Bernstein type inequities for polynomials. These inequalities have played a fundamental role in harmonic and complex Analysis, as well as in approximation theory [3, 4, 6] and in the study of random trigonometric series [5] [15, Chapter 6] or random Dirichlet series [13, Chapter 5]. One can also see their use in the theory of Banach spaces [10, pp. 20-21], and in Numerical Analysis. Having said that, the purpose of this paper is not to focus on applications but on certain objects of mathematical interest. The study of operators preserving inequalities between polynomials has been of some interest lately as they provide a unified approach of arriving at various polynomial inequalities and their generalizations simultaneously [14, pp. 538-565]. Actually it was this aim of simultaneously obtaining various polynomial inequalities and their generalizations that has

Received April 7, 2021. Revised June 10, 2021. Accepted June 11, 2021.
2010 Mathematics Subject Classification: 26D10, 41A17.
Key words and phrases: Polynomials, Operators, Inequalities in the complex domain.
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motivated our work in this paper. Before we state some of the fundamental results, we shall first define some symbols and notations which will be used throughout this paper.

We shall use $\mathcal{P}_n$ to denote the vector space of all polynomials of degree at most n over the field $\mathbb{C}$ of complex numbers. Note that for any $c \in \mathbb{C}$, the linear function taking $z \mapsto cz$, will be denoted by σ_c and the function which maps $z \mapsto z^n$ will be denoted by ψ. Further we shall use $\circ$ to denote the usual composition of functions. Thus, in this notation, for any complex function $F : \mathbb{C} \longmapsto \mathbb{C}$, the function $F \circ \sigma_R$ is defined as $F \circ \sigma_R(z) := F(\sigma_R(z)) = F(Rz)$, for $z \in \mathbb{C}$.
For $P \in \mathcal{P}_n$, we have

$$\max_{|z|=1} |P'(z)| \leq n \max_{|z|=1} |P(z)| \tag{1}$$

and

$$\max_{|z|=R>1} |P(z)| \leq R^n \max_{|z|=1} |P(z)| \tag{2}$$

Inequality (1) is due to S.Bernstein [3] (see also [8, 14]), whereas inequality (2) is a simple consequence of the maximum modulus principle [11, p.346]. Inequalities (1) and (2) were generalised by Aziz and Rather [2] by proving that if $P \in \mathcal{P}_n$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1, R > 1$ and $|z| \geq 1$,

$$\begin{aligned} &\left| P(Rz) - \alpha P(z) + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} P(z) \right| \\ &\qquad \leq \left| R^n - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| |z|^n \max_{|z|=1} |P(z)|. \end{aligned} \tag{3}$$

Restricting ourselves to the class of polynomials $P \in \mathcal{P}_n$ having no zero in $|z| < 1$, then the inequalities (1) and (2) can be significantly improved and respectively replaced by

$$\max_{|z|=1} |P'(z)| \leq \frac{n}{2} \max_{|z|=1} |P(z)| \tag{4}$$

and

$$\max_{|z|=R>1} |P(z)| \leq \frac{R^n + 1}{2} \max_{|z|=1} |P(z)| \tag{5}$$

Inequality (5) was conjectured by Erdös and verified by Lax [7], while as inequality (7) is due to Ankeny and Rivlin [1]. Aziz and Rather [2] used (3) to prove that if $P \in \mathcal{P}_n$ having no zero in $|z| < 1$, then for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > 1$,

(6)
$$\begin{aligned} \left| P(Rz) - \alpha P(z) + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} P(z) \right| &\leq \frac{1}{2} \left[\left| R^n - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| |z|^n \right. \\ &\left. + \left| 1 - \alpha + \beta \left\{ \left(\frac{R+1}{2} \right)^n - |\alpha| \right\} \right| \right] \max_{|z|=1} |P(z)|. \end{aligned}$$

Inequality (6) is a compact generalization of inequalities (4) and (5).

Recently Rather et al. [12] introduced an operator $N : \mathcal{P}_n \to \mathcal{P}_n$. That is, the operator N carries polynomials $P \in \mathcal{P}_n$ into

$$N[P](z) := \sum_{i=0}^{k} \lambda_i \left(\frac{nz}{2}\right)^i \frac{P^{(i)}(z)}{i!}, \qquad k \leq n,$$

where λ_i, $i \in \{0, 1, 2, ...k\}$, are fixed complex numbers chosen in such a way that all the zeros of

$$g(z) = \sum_{i=0}^{k} \binom{n}{i} \lambda_i z^i,$$

lie in the half plane $|z| \leq |z - \frac{n}{2}|$ and established certain sharp results concerning the upper-bound of $|N[P]|$ for $|z| \geq 1$ and among other things they [12] generalised inequalities (1) and (4), by proving the following results:

Theorem 1.1. *If all the zeros of polynomial $f(z)$ of degree n lie in $|z| \leq 1$ and $P \in \mathcal{P}_n$ such that*

$$|P(z)| \leq |f(z)| \qquad \text{for} \quad |z| = 1,$$

then

$$|N[P](z)| \leq |N[f](z)| \qquad for \quad |z| \geq 1. \tag{7}$$

The result is sharp and equality in (7) *holds for $P(z) = e^{i\alpha} f(z)$, $\alpha \in \mathbb{R}$.*

Theorem 1.2. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then*

$$|N[P](z)| \leq \frac{1}{2}\Big(|N[\psi_n](z)| + |\lambda_0|\Big) \max_{|z|=1} |P(z)| \qquad for \quad |z| \geq 1, \tag{8}$$

the result is best possible and equality in (8) *holds for $P(z) = az^n + b$, $|a| = |b| \neq 0$.*

In this paper, we prove some more general results concerning the operator $N : \mathcal{P}_n \to \mathcal{P}_n$ preserving inequalities between polynomials, which in turn yields compact generalizations of some well known polynomial inequalities. We begin by proving the following result:

Theorem 1.3. *If all the zeros of polynomial $f(z)$ of degree n lie in $|z| \leq 1$ and $P \in \mathcal{P}_n$ such that*

$$|P(z)| \leq |f(z)| \qquad \text{for} \quad |z| = 1,$$

then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$ and $|z| \geq 1$,

(9)
$$|N[P \circ \sigma_R](z) + \phi(\alpha, \beta, R, r) N[P \circ \sigma_r](z)| \leq |N[f \circ \sigma_R](z) + \phi(\alpha, \beta, R, r) N[f \circ \sigma_r](z)|,$$

$$\text{where} \qquad \phi(\alpha, \beta, R, r) = \beta \left\{ \left(\frac{R+1}{r+1}\right)^n - |\alpha| \right\} - \alpha. \tag{10}$$

The result is sharp and equality in (9) *holds for $P(z) = e^{ib} f(z)$, $b \in \mathbb{R}$.*

Remark 1.4. For $\alpha = \beta = 0$ and $R = r = 1$, Theorem 1.3 reduces to Theorem 1.1.

Taking $f(z) = Mz^n$ where $M = \max_{|z|=1} |P(z)|$ in Theorem 1.3, we obtain the following result.

Corollary 1.5. *If $P \in \mathcal{P}_n$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R > r \geq 1$ and $|z| \geq 1$,*

$$|N[P \circ \sigma_R](z) + \phi(\alpha, \beta, R, r)N[P \circ \sigma_r](z)| \leq |R^n + r^n\phi(\alpha, \beta, R, r)||N[\psi](z)| \max_{|z|=1} |P(z)|, \tag{11}$$

where $\phi(\alpha, \beta, R, r)$ is defined in (10). The result is sharp and equality in (11) holds for $P(z) = az^n, a \neq 0, a \in \mathbb{R}$.

Remark 1.6. Taking $\lambda_i = 0 \ \forall\ i < k$ and $\lambda_k \neq 0$ in Corollary 1.5, it follows that if $P \in \mathcal{P}_n$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R > r \geq 1$ and $|z| \geq 1$,

$$|R^k P^{(k)}(Rz) + \phi(\alpha, \beta, R, r)r^k P^{(k)}(rz)| \leq \frac{n!}{(n-k)!}|R^n + r^n\phi(\alpha, \beta, R, r)||z|^{n-k} \max_{|z|=1} |P(z)|. \tag{12}$$

For different choices of k, α, β, R, r, the above inequality yield many interesting results. In particular, for $k = 0,\ r = 1$, inequality (12) reduces to the inequality (3) which includes inequalities (1) and (2) as special cases.

Further setting $P(z) = mz^n$ in Theorem 1.3 where $m = \min_{|z|=1} |f(z)|$, we obtain the following result.

Corollary 1.7. *If all the zeros of polynomial $f(z)$ of degree n lie in $|z| \leq 1$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R > r \geq 1$ and $|z| \geq 1$, we have*

$$|N[f \circ \sigma_R](z) + \phi(\alpha, \beta, R, r)N[f \circ \sigma_r](z)| \geq |R^n + r^n\phi(\alpha, \beta, R, r)||N[\psi](z)|m,$$

where $\phi(\alpha, \beta, R, r)$ is defined in (10). The result is sharp as shown by polynomial $f(z) = az^n, a \neq 0, a \in \mathbb{R}$.

Remark 1.8. Similarly as in the case of Remark 1.6, Corollary 1.7 implies that if all the zeros of polynomial $f(z)$ of degree n lie in $|z| \leq 1$, then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R > r \geq 1$ and $|z| \geq 1$, we have

$$|R^k f^{(k)}(Rz) + \phi(\alpha, \beta, R, r)r^k f^{(k)}(rz)| \geq \frac{n!}{(n-k)!}|R^n + \phi(\alpha, \beta, R, r)r^n||z|^{n-k}m.$$

Next we present the following result for the class of polynomials having no zero inside the unit circle $|z| = 1$.

Theorem 1.9. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$,then for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R > r \geq 1$ and $|z| \geq 1$,*

$$\begin{aligned} &|N[P \circ \sigma_R](z) + \phi(\alpha, \beta, R, r)N[P \circ \sigma_r](z)| \\ &\qquad \leq \frac{1}{2}\Big(|R^n + r^n\phi(\alpha, \beta, R, r)||N[\psi](z)| + |1 + \phi(\alpha, \beta, R, r)||\lambda_0|\Big)M, \end{aligned} \tag{13}$$

where $M = \max_{|z|=1} |P(z)|$ and $\phi(\alpha, \beta, R, r)$ is defined in (10). The result is best possible and equality in (13) holds for $P(z) = az^n + b, |a| = |b| \neq 0$.

Remark 1.10. For $\alpha = \beta = 0$ and $R = r = 1$, Theorem 1.9 reduces to Theorem 1.2.

Now taking $\lambda_i = 0 \ \forall\ i < k$ and $\lambda_k \neq 0$ in Theorem 1.9, we get the following result.

Corollary 1.11. *If $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$,then for every $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R > r \geq 1,\ 1 \leq k \leq n$ and $|z| \geq 1$,*

$$|R^k P^{(k)}(Rz) + \phi(\alpha, \beta, R, r) r^k P^{(k)}(rz)| \leq \frac{n!}{2(n-k)!}\left(|R^n + \phi(\alpha, \beta, R, r) r^n||z|^{n-k}\right) M,$$

which includes inequality (4) *as a special case.*

Remark 1.12. Further for $k = 0, r = 1$, inequality (13) reduces to inequality (6) including (5) as a special case.

2. Lemmas

For the proof of these theorems, we need the following lemmas.

Lemma 2.1. *If $P \in \mathcal{P}_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then for every $R > r \geq 1$ and $|z| = 1$,*

$$|P(Rz)| \geq \left(\frac{R+1}{r+1}\right)^n |P(rz)|.$$

Proof. Since all the zeros of $P(z)$ lie in $|z| \leq 1$, we write

$$P(z) = C \prod_{j=1}^{n} \left(z - r_j e^{i\theta_j}\right),$$

where $r_j \leq 1$. Now for $0 \leq \theta < 2\pi$, $R > r \geq 1$, one can easily show that

$$\left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{re^{i\theta} - r_j e^{i\theta_j}}\right| \geq \left(\frac{R + r_j}{r + r_j}\right), \text{for } j = 1, 2, \cdots, n.$$

Hence for $0 \leq \theta < 2\pi$ and $R > r \geq 1$,

$$\begin{aligned} \left|\frac{P(Re^{i\theta})}{P(re^{i\theta})}\right| &= \prod_{j=1}^{n} \left|\frac{Re^{i\theta} - r_j e^{i\theta_j}}{re^{i\theta} - r_j e^{i\theta_j}}\right|, \\ &\geq \left(\frac{R + r_j}{r + r_j}\right)^n, \\ &\geq \left(\frac{R+1}{r+1}\right)^n. \end{aligned}$$

This implies for $|z| = 1$ and $R > r \geq 1$,

$$|P(Rz)| \geq \left(\frac{R+1}{r+1}\right)^n |P(rz)|,$$

which completes the proof of Lemma 2.1. □

Next lemma is due to N. A. Rather et al. [12]

Lemma 2.2. *If all the zeros of polynomial $F(z)$ of degree n lie in $|z| \leq r$ and if all the zeros of the polynomial*

$$g(z) = \sum_{i=0}^{k} \lambda_i \binom{n}{i} z^i, \quad k \leq n$$

lie in $|z| \leq s|z-t|$, $s>0$, *then the polynomial*

$$h(z)=\sum_{i=0}^{k}\lambda_i F^{(i)}(z)\frac{(tz)^i}{i!}$$

has all its zeros in $|z| \leq r\ \max(1,s)$.

LEMMA 2.3. *If* $P \in \mathcal{P}_n$ *has all its zeros in* $|z|\geq 1$, *then for every* $\alpha,\beta\in\mathbb{C}$ *with* $|\alpha|\leq 1, |\beta|\leq 1,\ R>r\geq 1$ *and* $|z|\geq 1$,

(14)

$$|N[P\circ\sigma_R](z)+\phi(\alpha,\beta,R,r)N[P\circ\sigma_r](z)| \leq |N[P^*\circ\sigma_R](z)+\phi(\alpha,\beta,R,r)N[P^*\circ\sigma_r](z)|,$$

where $P^*(z)=z^n\overline{P(1/\overline{z})}$ and $\phi(\alpha,\beta,R,r)$ *is defined in* (10).

Proof. Since $P\in\mathcal{P}_n$ has all its zeros in $|z|\geq 1$, $P^*\in\mathbb{P}_n$ and has all its zeros in $|z|\leq 1$. Applying Theorem 1.3 with $f(z)$ replaced by $P^*(z)$, it follows that for every $\alpha,\beta\in\mathbb{C}$ with $|\alpha|\leq 1,|\beta|\leq 1$, $R>r\geq 1$ and $|z|\geq 1$,

$$|N[P\circ\sigma_R](z)+\phi(\alpha,\beta,R,r)N[P\circ\sigma_r](z)| \leq |N[P^*\circ\sigma_R](z)+\phi(\alpha,\beta,R,r)N[P^*\circ\sigma_r](z)|.$$

This completes the proof of Lemma 2.3. □

Finally we prove the following lemma.

LEMMA 2.4. *If* $P\in\mathcal{P}_n$, *then for every* $\alpha,\beta\in\mathbb{C}$ *with* $|\alpha|\leq 1,|\beta|\leq 1,\ R>r\geq 1$ *and* $|z|\geq 1$,

(15)

$$\begin{aligned}|N[P\circ\sigma_R](z)+\phi(\alpha,\beta,R,r)N[P\circ\sigma_r](z)|+|N[P^*\circ\sigma_R](z)+\phi(\alpha,\beta,R,r)N[P^*\circ\sigma_r](z)|\\ \leq \big(|R^n+r^n\phi(\alpha,\beta,R,r)||N[\psi](z)|+|1+\phi(\alpha,\beta,R,r)||\lambda_0|\big)M,\end{aligned}$$

where $M=\max_{|z|=1}|P(z)|$, $P^*(z)=z^n\overline{P(1/\overline{z})}$ and $\phi(\alpha,\beta,R,r)$ *is defined in* (10).

Proof. Since $|P(z)|\leq M$ for $|z|=1$, by Rouche's Theorem it follows that for $\mu\in\mathbb{C},\ |\mu|>1,\ F(z)=P(z)-\mu M$ does not vanish in $|z|<1$. Applying Lemma 2.3 to $F(z)$, it follows that for every $\alpha,\beta\in\mathbb{C}$ with $|\alpha|\leq 1,|\beta|\leq 1,\ R>r\geq 1$ and $|z|\geq 1$,

$$|N[F\circ\sigma_R](z)+\phi(\alpha,\beta,R,r)N[F\circ\sigma_r](z)| \leq |N[F^*\circ\sigma_R](z)+\phi(\alpha,\beta,R,r)N[F^*\circ\sigma_r](z)|.$$

where $F^*(z)=z^n\overline{F(1/\overline{z})}=P^*(z)-\overline{\mu}z^nM$. Hence for $\alpha,\beta,\mu\in\mathbb{C}$ with $|\mu|>1,\ |\alpha|\leq 1,\ |\beta|\leq 1,\ R>r\geq 1$ and $|z|\geq 1$, we obtain

(16)

$$\begin{aligned}&|N[P\circ\sigma_R](z)+\phi(\alpha,\beta,R,r)N[P\circ\sigma_r](z)-\mu(1+\phi(\alpha,\beta,R,r))\lambda_0 M|\\ &\leq |N[P^*\circ\sigma_R](z)+\phi(\alpha,\beta,R,r)N[P^*\circ\sigma_r](z)-\overline{\mu}(R^n+r^n\phi(\alpha,\beta,R,r))N[\psi](z)M|.\end{aligned}$$

Choosing argument of μ in the right hand side of inequality (16) such that

$$\begin{aligned}&|N[P^*\circ\sigma_R](z)+\phi(\alpha,\beta,R,r)N[P^*\circ\sigma_r](z)-\overline{\mu}(R^n+r^n\phi(\alpha,\beta,R,r))N[\psi](z)M|\\ &=|\mu||R^n+r^n\phi(\alpha,\beta,R,r)||N[\psi](z)|M-|N[P^*\circ\sigma_R](z)+\phi(\alpha,\beta,R,r)N[P^*\circ\sigma_r](z)|\end{aligned}$$

which is possible by inequality (11), therefore from inequality (16) we have for $\alpha,\beta,\mu\in\mathbb{C}$ with $|\mu|>1,\ |\alpha|\leq 1,\ |\beta|\leq 1,\ R>r\geq 1$ and $|z|\geq 1$,

$$\begin{aligned}|N[P\circ\sigma_R](z)+\phi(\alpha,\beta,R,r)N[P\circ\sigma_r](z)|+|N[P^*\circ\sigma_R](z)+\phi(\alpha,\beta,R,r)N[P^*\circ\sigma_r](z)|\\ \leq |\mu|\big(|R^n+r^n\phi(\alpha,\beta,R,r)||N[\psi](z)|+|1+\phi(\alpha,\beta,R,r)||\lambda_0|\big)M.\end{aligned}$$

Letting $|\mu| \to 1$, we get inequality (15). This completes the proof of Lemma 2.4. $\square$

3. Proof of theorems

Proof of Theorem 1.3. By hypothesis $f(z)$ is a polynomial of degree n having all zeros in $|z| \le 1$ and $P \in \mathcal{P}_n$ such that

$$|P(z)| \le |f(z)| \qquad \text{for} \quad |z| = 1. \tag{17}$$

If z_ν is a zero of $f(z)$ of multiplicity s_ν on the unit circle $|z| = 1$, then it is evident from (17) that z_ν is also a zero of $P(z)$ of multiplicity at least s_ν. If $P(z)/f(z)$ is a constant, then inequality (9) is obvious. We now assume that $P(z)/f(z)$ is not a constant, so that by the maximum modulus principle, it follows that

$$|P(z)| < |f(z)| \ \text{for} \ |z| > 1 \ .$$

Let $g(z) = \prod\limits_{z_\nu \in \Omega} (z - z_\nu)^{s_\nu}$ where $\Omega = \{z_\nu \in \mathbb{C} : f(z_\nu) = 0 \wedge |z_\nu| = 1\}$. Then again from (17), we have

$$\left|\frac{P(z)}{g(z)}\right| \le \left|\frac{f(z)}{g(z)}\right| \quad \text{for} \quad |z| = 1.$$

By Rouche's theorem for every $\gamma \in \mathbb{C}$ with $|\gamma| > 1$, the polynomial $G(z) = \frac{P(z) - \gamma f(z)}{g(z)}$ has all its $n - \sum s_v$ zeros in $|z| < 1$. Since the polynomial $g(z)$ has $\sum s_v$ zeros on $|z| = 1$, the polynomial $h(z) = G(z)g(z) = P(z) - \gamma f(z)$ has all the n zeros in $|z| \le 1$ with at least one zero in $|z| < 1$, so that we can write

$$h(z) = (z - te^{i\delta})H(z)$$

where $t < 1$ and $H(z)$ is a polynomial of degree $n - 1$ having all its zeros in $|z| \le 1$. Applying Lemma 2.1 to the polynomial $H(z)$, we obtain for every $R > r \ge 1$ and $0 \le \theta < 2\pi$,

$$\begin{aligned}
|h(Re^{i\theta})| &= |Re^{i\theta} - te^{i\delta}||H(Re^{i\theta})| \\
&\ge |Re^{i\theta} - te^{i\delta}| \left(\frac{R+1}{r+1}\right)^{n-1} |H(re^{i\theta})|, \\
&= \left(\frac{R+1}{r+1}\right)^{n-1} \frac{|Re^{i\theta} - te^{i\delta}|}{|re^{i\theta} - te^{i\delta}|} |(re^{i\theta} - te^{i\delta})H(re^{i\theta})|, \\
&\ge \left(\frac{R+1}{r+1}\right)^{n-1} \left(\frac{R+t}{r+t}\right) |h(re^{i\theta})|.
\end{aligned}$$

This implies for $R > r \ge 1$ and $0 \le \theta < 2\pi$,

$$\left(\frac{r+t}{R+t}\right) |h(Re^{i\theta})| \ge \left(\frac{R+1}{r+1}\right)^{n-1} |h(re^{i\theta})|. \tag{18}$$

Since $R > r \ge 1 > t$ so that $h(Re^{i\theta}) \ne 0$ for $0 \le \theta < 2\pi$ and $\frac{r+1}{R+1} > \frac{r+t}{R+t}$, from inequality (18), we obtain

$$|h(Re^{i\theta})| > \left(\frac{R+1}{k+1}\right)^{n} |h(re^{i\theta})|, \quad R > r \ge 1 \ \text{ and } \ 0 \le \theta < 2\pi. \tag{19}$$

Equivalently,

$$|h(Rz)| > \left(\frac{R+1}{r+1}\right)^n |h(rz)| \qquad \text{for} \quad |z|=1 \quad \text{and} \quad R>r\geq 1. \tag{20}$$

Hence for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and $R > r \geq 1$, we have

$$\begin{aligned}|h(Rz)-\alpha h(rz)| &\geq |h(Rz)| - |\alpha|\,|h(rz)| \\ &> \left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\} |h(rz)|, \ \text{ for } \ |z|=1.\end{aligned} \tag{21}$$

Since $R+1 > r+1$, from inequality (20), we have

$$|h(rz)| < |h(Rz)| \ \text{ for } \ |z|=1.$$

Since all the zeros of $h(Rz)$ lie in $|z| \leq (1/R) < 1$, a direct application of Rouche's theorem shows that for every $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ the polynomial $h(Rz) - \alpha h(rz)$ has all its zeros in $|z| < 1$. Applying Rouche's theorem again, it follows from (21) that for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > r \geq 1$, all the zeros of the polynomial

$$\begin{aligned}T(z) &= h(Rz) - \alpha h(rz) + \beta\left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\} h(rz) \\ &= h(Rz) + \left[\beta\left\{\left(\frac{R+1}{r+1}\right)^n - |\alpha|\right\} - \alpha\right] h(rz) \\ &= P(Rz) - \gamma f(Rz) + \phi(\alpha,\beta,R,r)\left(P(rz) - \gamma f(rz)\right) \\ &= P(Rz) + \phi(\alpha,\beta,R,r)P(rz) - \gamma\left(f(Rz) + \phi(\alpha,\beta,R,r)f(rz)\right)\end{aligned}$$

lie in $|z| < 1$. Invoking Lemma 2.2 with $s = 1,\ t = n/2$ and noting that N is a linear operator, it follows that all the zeros of the polynomial

$$\begin{aligned}N[T](z) = {} & N[P\circ\sigma_R](z) + \phi(\alpha,\beta,R,r)N[P\circ\sigma_r](z) \\ & - \gamma\left(N[f\circ\sigma_R](z) + \phi(\alpha,\beta,R,r)N[f\circ\sigma_R](z)\right)\end{aligned} \tag{22}$$

lie in $|z| < 1$. This implies for $|z| \geq 1$ and $R > r \geq 1$,

(23)

$$|N[P\circ\sigma_R](z) + \phi(\alpha,\beta,R,r)N[P\circ\sigma_r](z)| \leq |N[f\circ\sigma_R](z) + \phi(\alpha,\beta,R,r)N[f\circ\sigma_r](z)|.$$

For if inequality (23) is not true, then there exists $z = z_0$ with $|z_0| \geq 1$ such that

$$|N[P\circ\sigma_R](z_0) + \phi(\alpha,\beta,R,r)N[P\circ\sigma_r](z_0)| > |N[f\circ\sigma_R](z_0) + \phi(\alpha,\beta,R,r)N[f\circ\sigma_R](z_0)|.$$

Taking $\gamma = \frac{N[P\circ\sigma_R](z_0)+\phi(\alpha,\beta,R,r)N[P\circ\sigma_r](z_0)}{N[f\circ\sigma_R](z_0)+\phi(\alpha,\beta,R,r)N[f\circ\sigma_R](z_0)}$, which is well defined complex number with $|\gamma| > 1$ and with this choice of γ, from (22) we obtain $N[T](z_0) = 0$ where $|z_0| \geq 1$. This contradicts the fact that all the zeros of $N[T](z)$ lie in $|z| < 1$. Thus establishes (23) and this completes the proof of Theorem 1.3. □

Proof of Theorem 1.9. Since $P \in \mathcal{P}_n$, having all its zeros in $|z| \geq 1$, from Lemma 2.3 and Lemma 2.4 it follows that for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1,\ |\beta| \leq 1,\ R > r \geq 1$ and $|z| \geq 1$,

$$\begin{aligned}&2|N[P\circ\sigma_R](z) + \phi(\alpha,\beta,R,r)N[P\circ\sigma_r](z)| \\ &\quad \leq |N[P\circ\sigma_R](z) + \phi(\alpha,\beta,R,r)N[P\circ\sigma_r](z)| + |N[P^*\circ\sigma_R](z) + \phi(\alpha,\beta,R,r)N[P^*\circ\sigma_r](z)| \\ &\quad \leq \Big(|R^n + r^n\phi(\alpha,\beta,R,r)||N[\psi](z)| + |1+\phi(\alpha,\beta,R,r)||\lambda_0|\Big)M,\end{aligned}$$

which is equivalent to inequality (13). This completes the proof of Theorem 1.9. □

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