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COMPARISON OF DISCRETE TIME INVENTORY SYSTEMS WITH POSITIVE SERVICE TIME AND LEAD TIME

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Abstract. This paper investigates two discrete time queueing inventory models with positive service time and lead time. Customers arrive according to a Bernoulli process and service time and lead time follow geometric distributions. The first model under discussion based on replenishment of order upto S policy where as the second model is based on order placement by a fixed quantity Q, where $Q = S - s$, whenever the inventory level falls to s. We analyse this queueing systems using the matrix geometric method and derive an explicit expression for the stability condition. We obtain the steady-state behaviour of these systems and several system performance measures. The influence of various parameters on the systems performance measures and comparison on the cost analysis are also discussed through numerical example.

1. Introduction

There is a growing research interest in discrete time queues mainly motivated by their applications in computer and communication systems because the basic time unit in these systems is a binary code (See [1]). Also the discrete time system can be used to approximate the continuous system. Discrete time queueing system has been found to be more appropriate in modelling computer systems and communication network. Recently, due to the fast progress of computer and telecommunication network technologies, the discrete time models have received more attention from researchers (see [15]). BISDN (Broadband Integrated Service Digital Network) has been of significant interest because it can provide a common interface for future communication needs including video, voice and data communication signals through high speed Local Area Network (LAN), on-demand video distribution and video telephony communications (see [15]). The Asynchronous Transfer Mode (ATM) is a key technology for accommodating such a wide area of services. In these systems, all the information is segmented into small packets, represented as cells. The time is slotted and in each slot the data units (packets) are transmitted. Applications in detail are discussed in the papers [1], [2] and in the books [3], [15]. By a discrete time analysis, we mean analysis in which the system is observed only at specific points in time which

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are equally spaced points on the time axis (see, [11], [16], [17]). That is, a system in which observation is made only at points of event occurrences such as arrivals or departures at specified points which are equally spaced and numbered sequentially as $0, 1, 2, \ldots$.

Lian and Liu [9] developed a discrete time inventory model with geometric inter demand times and constant life time. The (s, S) inventory system with positive lead time has been studied by several researchers (See, [4], [7], [8], [9], [10], [14]). Deepthi [5] have studied many discrete time inventory models with/withot positive time in her doctoral thesis. There exists a rich variety of different inventory models depending on the combination of different assumptions (see, [6], [12]). Some common assumptions are as follows. Continuous versus periodic review of the inventory, individual versus batch arrivals, different replenishment policies (fixed, order upto level S etc.), constant or random lead time etc. The inventory model in discussion is based on replenishment of order up to S policy.

In this paper, we analyze two discrete time (s, S) inventory models with positive service time and lead time where s is the reorder level and S is the maximum inventory level permitted. These models differ by their respective replenishment policies. Model 1 is based on replenishment of order upto S policy. That is whenever the inventory level reaches s, an order is placed to bring the level to S, where the replanishment quantity is $S-i$ when the inventory level is i, $0 \leq i \leq s$ just before the replenishment. Model 2 is based on order placement by a fixed quantity Q, where $Q = S - s$, whenever the inventory level falls to s. Here we assumed that S is greater than $2s$ to avoid perpetual reordering. The decision of the order size is according to a discrete probability function. In all these models we assume that demands are according to a Bernoulli process. Service times and lead times are geometrically distributed. We can construct a multidimensional Markov chain to model the joint queue length and inventory process to obtain a product form solution for these models.

This paper is organized as follows :- In section 2, we present the mathematical formulation of the model 1, steady-state anlaysis of the system. We also analyze the computation of steady-state probabilities of the system state and derive some performance measures. Section 3 discuss mathematical formulation of the model 2 and steady-state anlaysis of the system, its stability condition and some key performance measures. In section 4, we obtain a cost function for the models. Finally some numerical results are given in section 5.

2. Mathematical formulation of model 1

We consider a $Geo/Geo/1/(s, S)$ inventory system with positive lead time in which demands arrive according to a Bernoulli process with parameter p. The demand quantity at an epoch is for one unit of the item with probability p and is 0 with probability $\overline{p} = 1 - p$. Thus a demand takes place at a slot boundary with probability p and no demand with probability $\overline{p}$. We assume that customers are not allowed to join in the system when the inventory level is zero. The service time and lead time for replenishment of inventory follow independent geometric distributions with parameters q and r, respectively. Denote the complimentary probabilities as $\overline{q} = 1 - q$ and $\overline{r} = 1 - r$.

It is assumed that all inventory activities (demand arrival, replenishment, departure) take place around the slot boundaries. We assume that a departure or replenishment occurs in the interval (m^-, m) and an arrival in (m, m^+). Whenever the inventory level falls to s, an order is placed to bring the level to S. It requires a random amount of time for the fulfillment of orders placed and the inventory level can be reduced to zero during this period due to demand. The lead time takes at least one time slot to complete, hence an order can not be received at the epoch it is placed.

Let N_m denote the number of customers in the system and I_m, the inventory level at m^+. We denote the joint queue length and inventory process by $\{(N_m, I_m) : m \in N\}$. Then $\chi_1 = \{(N_m, I_m) : m \in N\}$ is a Markov Chain whose state space is $E = \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots s, s+1, \ldots S\}$.

The state space of the Markov chain is partitioned into levels defined as $\hat{i} = \{(i, 0), (i, 1), \ldots\ (i, s), (i, s+1), \ldots (i, S)\}$. The one step transition probability matrix P of the Markov chain χ_1 is given by

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & & & \\ P_{10} & P_{11} & P_{12} & & \\ & P_{10} & P_{11} & P_{12} & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \tag{1}$$

where each entry is a square matrix of order $S+1$.

In the above matrix P_{00} denotes the probability of transitions among states within level 0; P_{01} is those from level 0 to level 1. The transitions from level i to level $i+1$ are represented by elements of the matrix P_{12}, those from level i to $i-1$ by those of P_{10} and transitions within the level i are represented by that in P_{11}. Entries of these matrices are

$$[P_{00}]_{ij} = \begin{cases} \overline{r}, & j = i, \quad i = 0 \\ \overline{p}\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\ \overline{p}, & j = i, \quad i = s+1, s+2, \ldots, S \\ \overline{p}r, & j = S, \quad i = 0, 1, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[P_{01}]_{ij} = \begin{cases} p\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\ p, & j = i, \quad i = s+1, s+2, \ldots, S \\ p\,r, & j = S, \quad i = 0, 1, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[P_{10}]_{ij} = \begin{cases} q\,\overline{r}, & j = i-1, \quad i = 1 \\ \overline{p}q\overline{r}, & j = i-1, \quad i = 2, 3, \ldots, s \\ \overline{p}q, & j = i-1, \quad i = s+1, s+2, \ldots, S \\ \overline{p}qr, & j = S-1, \quad i = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[P_{12}]_{ij} = \begin{cases} p\overline{q}\,\overline{r}, & j = i \quad i = 1, 2, \ldots, s \\ p, & j = i, \quad i = s+1, s+2, \ldots, S \\ p\,r, & j = S \quad i = 0 \\ p\overline{q}r, & j = S, \quad i = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[P_{11}]_{ij} = \begin{cases} \overline{r}, & j = i, \quad i = 0 \\ \overline{p}\,\overline{q}\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\ \overline{p}\,\overline{q}, & j = i, \quad i = s+1, s+2, \ldots, S \\ p\ q\overline{r}, & j = i-1, \quad i = 2, 3, \ldots, s \\ p\ \ q\,r, & j = S-1, \quad i = 1, 2, \ldots, s \\ \overline{p}r, & j = S, \quad i = 0 \\ \overline{p}\,\overline{q}r, & j = S, \quad i = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

where $\overline{p} = 1 - p$, $\overline{q} = 1 - q$, $\overline{r} = 1 - r$.

2.1. Stability Condition. For determining the stability condition for the system, consider the transition matrix $A = P_{10} + P_{11} + P_{12}$ given by

$$[A]_{ij} = \begin{cases} \overline{r}, & j = i, \quad i = 0 \\ q\,\overline{r}, & j = i-1, \quad i = 1, 2, \ldots, s \\ \overline{p}q, & j = i-1, \quad i = s+1, s+2, \ldots, S \\ \overline{q}\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\ p + \overline{p}\,\overline{q}, & j = i, \quad i = s+1, s+2, \ldots, S \\ r, & j = S, \quad i = 0 \\ q\,r, & j = S-1, \quad i = 1, 2, \ldots, s \\ \overline{q}r, & j = S, \quad i = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

The Markov chain χ_1 is stable (see, Neuts [13]) if and only if $\boldsymbol{\pi} P_{12}\,\boldsymbol{e} \ < \boldsymbol{\pi} P_{10}\,\boldsymbol{e}$ where $\boldsymbol{\pi}$ is the stationary probability vector of A satisfying $\boldsymbol{\pi} A = \boldsymbol{\pi}$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$, where $\boldsymbol{e}$ is a column vector of 1's of appropriate order.

Write $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots \pi_s, \ldots, \pi_S)$. Then $\boldsymbol{\pi} A = \boldsymbol{\pi}$ gives

$$\pi_j = \begin{cases} \frac{(1-\overline{r})(1-\overline{q}\,\overline{r})^{j-1}}{(q\,\overline{r})^j}\pi_0, & j = 1, 2, \ldots, s \\ \frac{(1-\overline{r})(1-\overline{q}\,\overline{r})^{j-1}}{\overline{p}\,q(q\,\overline{r})^{j-1}}\pi_0, & j = s+1 \end{cases}$$

$\pi_{s+1} = \pi_{s+2} = \cdots = \pi_{S-1}$;

$\pi_S = \frac{(1-\overline{r})[q(q\,\overline{r})^s + \overline{q}(1-\overline{q}\,\overline{r})^s]}{\overline{p}\,q(q\,\overline{r})^s}\pi_0.$

Further, $\boldsymbol{\pi}\,\boldsymbol{e} = 1$ gives

$\pi_0 = \frac{\overline{p}\,q(q\,\overline{r})^s}{(1-\overline{q}\,\overline{r})^s[\overline{p}\,q + (S-s-1)r + r\,\overline{q}] + rq(q\,\overline{r})^s}$

It follows that

$$\boldsymbol{\pi} P_{12}\boldsymbol{e} = \left\{ p\,\overline{q}\frac{(1-\overline{q}\,\overline{r})^s - (q\,\overline{r})^s}{(q\,\overline{r})^s} + \frac{pr(S-s-1)(1-\overline{q}\,\overline{r})^s}{\overline{p}\,q(q\,\overline{r})^s} + pr + \frac{prq}{\overline{p}q} + \frac{pr\,\overline{q}(1-\overline{q}\,\overline{r})^s}{\overline{p}\,q(q\,\overline{r})^s} \right\} \pi_0$$

and

$$\boldsymbol{\pi} P_{10}\boldsymbol{e} = \left\{ \overline{p}q\frac{(1-\overline{q}\,\overline{r})^s - (q\,\overline{r})^s}{(q\,\overline{r})^s} + \frac{r(S-s-1)(1-\overline{q}\,\overline{r})^s}{(q\,\overline{r})^s} - \overline{p}r + q + \frac{r\,\overline{q}(1-\overline{q}\,\overline{r})^s}{(q\,\overline{r})^s} \right\} \pi_0.$$

We obtain the following theorem.

THEOREM 2.1. *The system χ_1 is stable if and only if*

$$\frac{pr+(\overline{p}q)^2-p\overline{p}\,\overline{r}}{pqr+(\overline{p}q)^2+\overline{p}qr(S-s-1)+\overline{p}q\overline{q}(r-p)+pr(S-s)}\cdot\frac{q(q\,\overline{r})^s}{(1-\overline{q}\,\overline{r})^s}<1 \tag{2}$$

2.2. Steady-state analysis of model 1. Assume that the stability condition (2) is satisfied and $\boldsymbol{x}$ be the steady-state probability vector of the transition probability matrix $\mathbf{P}$ given in (1). That is,

$$\boldsymbol{x}\mathbf{P}=\boldsymbol{x};\quad \boldsymbol{x}\,\boldsymbol{e}=1 \tag{3}$$

Partition this vector as $\boldsymbol{x}=(\boldsymbol{x}_0,\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots)$. Then we see that, $\boldsymbol{x}$ has the matrix geometric form $\boldsymbol{x}_n=\boldsymbol{x}_1R^{n-1}, n\geq 2$ (see, Neuts [13]) where R is the minimal solution of the matrix quadratic equation $P_{12}+RP_{11}+R^2P_{10}=R$.

$\boldsymbol{x}\mathbf{P}=\boldsymbol{x}$ leads us to

$$\begin{aligned}\boldsymbol{x}_0\,P_{00}+\boldsymbol{x}_1\,P_{10}&=\boldsymbol{x}_0\\ \boldsymbol{x}_0\,P_{01}+\boldsymbol{x}_1\,P_{11}+\boldsymbol{x}_2\,P_{10}&=\boldsymbol{x}_1\\ \boldsymbol{x}_{n-1}\,P_{12}+\boldsymbol{x}_n\,P_{11}+\boldsymbol{x}_{n+1}\,P_{10}&=\boldsymbol{x}_n,\quad n\geq 2\end{aligned} \tag{4}$$

The normalizing condition of (3) gives

$$\boldsymbol{x}_0\,\boldsymbol{e}+\boldsymbol{x}_1\,(I-R)^{-1}\boldsymbol{e}=1. \tag{5}$$

The rate matrix R can be obtained using the successive iterative method $R(n+1):=(P_{12}+R(n)^2P_{10})(I-P_{11})^{-1}$, with $R(0)=0$ and $R(n)$ is the value of R at the n^{th} iteration. The iteration is usually stopped when $|R(n)-R(n+1)|_{ij}<\epsilon,\forall i,j$.

For finding the steady-state probability vector of the process $\chi_1=\{(N_m,I_m):m\in N\}$, consider the system where service time is negligible and where no customer joins when inventory is out of stock (see, [8]). This means that if the item is available at the epoch of demand, then it would be immediately delivered. As a consequence the customer need not have to wait. Hence the system has only inventory and is of finite state space.

The corresponding Markov chain is designated as $\hat{\chi}_1=\{I_m:m\in N\}$ where I_m denote the inventory level. The state space of the process is given by $\hat{E}=\{0,1,2,\ldots,S\}$. The transition probability matrix corresponding to $\hat{\chi}_1$ is given by

$$[\hat{P}]_{ij}=\begin{cases}\overline{r}, & j=i, & i=0\\ p\,\overline{r}, & j=i-1, & i=1,2,\ldots,s\\ p, & j=i-1, & i=s+1,s+2,\ldots,S\\ \overline{p}\,\overline{r}, & j=i, & i=1,2,\ldots,s\\ \overline{p}, & j=i, & i=s+1,s+2,\ldots,S\\ r, & j=S, & i=0,1,\ldots,s\\ 0, & \text{otherwise}\end{cases}$$

Let $\hat{\boldsymbol{\pi}}=(\hat{\pi}_0,\hat{\pi}_1,\ldots,\hat{\pi}_S)$ be the steady-state probability vector of the process $\hat{\chi}_1$. Then $\hat{\boldsymbol{\pi}}\hat{P}=\hat{\boldsymbol{\pi}}$ and $\hat{\boldsymbol{\pi}}\boldsymbol{e}=1$.

It can be seen that

$$\hat{\pi}_j=\begin{cases}\frac{(1-\overline{r})(1-\overline{p}\,\overline{r})^{j-1}}{(p\,\overline{r})^j}\hat{\pi}_0, & j=1,2,\ldots,s\\ \frac{(1-\overline{r})(1-\overline{p}\,\overline{r})^{j-1}}{p(p\,\overline{r})^{j-1}}\hat{\pi}_0, & j=s+1\end{cases}$$

$\hat{\pi}_{s+1}=\hat{\pi}_{s+2}=\cdots=\hat{\pi}_S.$

Also $\hat{\boldsymbol{\pi}}\boldsymbol{e} = 1$ gives
$\hat{\pi}_0 = \frac{p(p\bar{r})^s}{(1-\bar{p}\ \bar{r})^s[p+(S-s)r]}$

Now using $\hat{\boldsymbol{\pi}}$, we shall find the steady-state probability vector of χ_1 given in (1). Let $\boldsymbol{x}_0 = \rho\,\hat{\boldsymbol{\pi}}$ and $\boldsymbol{x}_n = \rho\left(\frac{p}{\bar{p}q}\right)^n\hat{\boldsymbol{\pi}}$, for $n \geq 1$, where ρ is a constant to be determined. This will satisfy the above equations (4). For,

$$\begin{aligned}
& \boldsymbol{x}_{n-1}P_{12} + \boldsymbol{x}_n P_{11} + \boldsymbol{x}_{n+1}P_{10} \\
= \ & \boldsymbol{x}_{n-1}P_{12} + \boldsymbol{x}_n\left[P_{00} - \frac{\bar{p}q}{p}P_{12}\right] + \boldsymbol{x}_{n+1}P_{10} \\
= \ & \rho\left(\frac{p}{\bar{p}q}\right)^{n-1}\hat{\boldsymbol{\pi}}P_{12} + \rho\left(\frac{p}{\bar{p}q}\right)^{n}\hat{\boldsymbol{\pi}}\left[P_{00} - \frac{\bar{p}q}{p}P_{12}\right] + \rho\left(\frac{p}{\bar{p}q}\right)^{n+1}\hat{\boldsymbol{\pi}}P_{10} \\
= \ & \rho\left(\frac{p}{\bar{p}q}\right)^{n}\hat{\boldsymbol{\pi}}\ \left[P_{00} + \frac{p}{\bar{p}q}P_{10}\right] \\
= \ & \rho\left(\frac{p}{\bar{p}q}\right)^{n}\hat{\boldsymbol{\pi}} = \boldsymbol{x}_n
\end{aligned}$$

Using normalised condition, it follows that, $\rho = 1 - \frac{p}{\bar{p}q}$
This leads to the following result.

THEOREM 2.2. *Under the necessary and sufficient condition that $p < \bar{p}q$, the steady-state probability vector of the process χ_1 with transition probability matrix P is given by $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \ldots)$ where $\mathbf{x}_0 = \rho\,\hat{\boldsymbol{\pi}}$ and $\mathbf{x}_n = \rho\left(\frac{p}{\bar{p}q}\right)^n\hat{\boldsymbol{\pi}}$, for $n \geq 1$; $\rho = 1 - \frac{p}{\bar{p}q}$ and the finite probability vector $\hat{\boldsymbol{\pi}}$ is given by $\hat{\boldsymbol{\pi}} = (\hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_S)$ where*

$$\hat{\pi}_j = \begin{cases} \frac{(1-\bar{r})(1-\bar{p}\ \bar{r})^{j-1}}{(p\bar{r})^j}\pi_0, & j = 1, 2, \ldots, s \\ \frac{(1-\bar{r})(1-\bar{p}\ \bar{r})^{s}}{p(p\bar{r})^s}\pi_0, & j = s+1, s+2, \ldots, S \end{cases}$$

and

$\pi_0 = \frac{\bar{p}\,q(q\,\bar{r})^s}{(1-\bar{q}\,\bar{r})^s[\bar{p}\,q+(S-s-1)r+r\,\bar{q}]+rq(q\,\bar{r})^s}$

2.3. System performance measures of model 1. In this section, we derive some important performance measures and specific probabilities descriptions for the inventory system.
Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ be the steady-state probability vector and $\boldsymbol{x}_n$, $n \geq 0$ is further partitioned as $\boldsymbol{x}_n = (x_{n0}, x_{n1}, \ldots, x_{nS})$.

1. Expected number of customers in the system :
$$\mu_{cust} = \sum_{n=0}^{\infty} n\,\boldsymbol{x}_n\,\boldsymbol{e}$$
2. Expected inventory level :
$$\mu_{inv} = \sum_{n=0}^{\infty}\sum_{m=1}^{S} m\,x_{nm}$$
3. Expected reorder rate :
$$\mu_{reord} = q\sum_{n=0}^{\infty} x_{n,s+1}$$

4. Expected replenishment rate :
$$\mu_{repl} = r\sum_{n=0}^{\infty}\sum_{m=0}^{s} x_{nm}$$
5. Probability that the system is idle : $P_{idle} = \sum_{n=0}^{\infty} x_{n0}$
6. Expected loss rate of customers :
$$\mu_{loss} = p\sum_{n=0}^{\infty} x_{n0}$$
7. Expected number of customers waiting in the system when the inventory level is zero :
$$\mu_{CW} = \sum_{n=0}^{\infty} n\, x_{n0}$$
8. Expected rate, of departure after completing service :
$$\mathrm{ED} = q\sum_{n=1}^{\infty}\sum_{m=1}^{S} x_{nm}$$

3. Mathematical formulation of model 2 and its analysis

In this section, we consider a discrete time (s, S) inventory system with positive lead time in which demands arrive according to a Bernoulli process with parameter p. The service time and lead time follow geometric distributions with parameters q and r respectively. Whenever the inventory level falls to s, place an order for replenishment by a fixed quantity Q, where $Q = S - s$. Here S denote the maximum inventory level and s is the reorder level. There is a positive lead time for replenishment. We assume that no customer joins when the inventory level is zero. Those who are already present in the system do not renege and exactly one item is demanded by each customer.

We denote the joint queue length and inventory process by $\chi_2 = \{(N_m, I_m) : m \in N\}$ where N_m denotes the number of customers in the system and I_m denotes the inventory level at time m^+. Then χ_2 provides a Markov Chain whose state space is $E = \{0, 1, \ldots\} \times \{0, 1, \ldots, s, s+1, \ldots, Q, Q+1, \ldots, S\}$.
The one step transition probability matrix of the process is given by

$$\mathbf{P} = \begin{bmatrix} P_{00} & P_{01} & & & \\ P_{10} & P_{11} & P_{12} & & \\ & P_{10} & P_{11} & P_{12} & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

where each entry is a square matrix of order $S+1$ are given by

$$[P_{00}]_{ij} = \begin{cases} \overline{r}, & j = 0, \quad i = 0 \\ \overline{p}\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\ \overline{p}, & j = i, \quad i = s+1, s+2, \ldots, S \\ \overline{p}r, & j = Q+i, \quad i = 0, 1, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[P_{01}]_{ij} = \begin{cases} p\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\ p, & j = i, \quad i = s+1, s+2, \ldots, S \\ p\,r, & j = Q + i, \quad i = 0, 1, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

For $i \geq 1$, the transitions from level i to level $i+1$, transitions within the level i and transitions from level i to level $i-1$ are represented by the matrices P_{12}, P_{11} and P_{10} respectively, and are given by

$$[P_{12}]_{ij} = \begin{cases} p\overline{q}\,\overline{r}, & j = i \quad i = 1, 2, \ldots, s \\ p, & j = i, \quad i = s+1, s+2, \ldots, S \\ p\,r, & j = Q \quad i = 0 \\ p\overline{q}r, & j = Q + i, \quad i = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[P_{11}]_{ij} = \begin{cases} \overline{r}, & j = 0, \quad i = 0 \\ \overline{p}\,\overline{q}\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\ \overline{p}\,\overline{q}, & j = i, \quad i = s+1, s+2, \ldots, S \\ p\ q\overline{r}, & j = i - 1, \quad i = 2, 3, \ldots, s \\ p\ \ q\,r, & j = Q - 1 + i, \quad i = 1, 2, \ldots, s \\ \overline{p}r, & j = Q, \quad i = 0 \\ \overline{p}\,\overline{q}r, & j = Q + i, \quad i = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[P_{10}]_{ij} = \begin{cases} q\,\overline{r}, & j = 0, \quad i = 1 \\ \overline{p}q\overline{r}, & j = i - 1, \quad i = 2, 3, \ldots, s \\ \overline{p}q, & j = i - 1, \quad i = s+1, s+2, \ldots, S \\ \overline{p}qr, & j = Q - 1 + i, \quad i = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

where, $\overline{p} = 1 - p$, $\overline{q} = 1 - q$, $\overline{r} = 1 - r$

3.1. Stability Condition of model 2. To obtain the stability condition for the model 2, consider the transition matrix $A = P_{10} + P_{11} + P_{12}$ by

$$[A]_{ij} = \begin{cases} \overline{r}, & j = 0, \quad i = 0 \\ \overline{q}\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\ q\,\overline{r}, & j = i - 1, \quad i = 1, 2, \ldots, s \\ \overline{p}q, & j = i - 1, \quad i = s+1, s+2, \ldots, S \\ p + \overline{p}\,\overline{q}, & j = i, \quad i = s+1, s+2, \ldots, S \\ r, & j = Q, \quad i = 0 \\ \overline{q}\,r, & j = Q + i, \quad i = 1, 2, \ldots, s \\ q\,r, & j = Q - 1, \quad i = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

The Markov chain χ_2 is stable if and only if the left drift rate is higher than the rate of drift to the right. That is, $\boldsymbol{\pi} P_{12}\boldsymbol{e} < \boldsymbol{\pi} P_{10}\boldsymbol{e}$ (see, Neut's [13]) where $\boldsymbol{\pi}$ is the stationary probability vector of A satisfying $\boldsymbol{\pi} A = \boldsymbol{\pi}$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$, where $\boldsymbol{e}$ is a column vector of 1's of appropriate order. If we partition the probability vector $\boldsymbol{\pi} = (\pi_0, \ldots, \pi_s, \pi_{s+1}, \ldots, \pi_Q, \ldots, \pi_S)$.
Then,

$$
\begin{aligned}
\pi_1 &= \left(\frac{1-\overline{r}}{q\,\overline{r}}\right)\pi_0\\
\pi_2 &= \frac{(1-\overline{r})(1-\overline{q}\,\overline{r})}{(q\,\overline{r})^2}\pi_0\\
&\vdots\\
\pi_s &= \frac{(1-\overline{r})(1-\overline{q}\,\overline{r})^{s-1}}{(q\,\overline{r})^s}\pi_0\\
\pi_i &= \frac{(1-\overline{r})(1-\overline{q}\,\overline{r})^{s}}{\overline{p}q(q\overline{r})^s}\pi_0, \quad i=s+1,s+2,\ldots,Q\\
\pi_{Q+1} &= \left(\frac{(1-\overline{r})(1-\overline{q}\,\overline{r})^{s}}{\overline{p}\,q\,(q\,\overline{r})^s} - \frac{qr}{\overline{p}\,q\,(q\,\overline{r})}\right)\pi_0\\
&\vdots\\
\pi_{S-2} &= \frac{r(1-\overline{r})(1-\overline{q}\,\overline{r})^{s-3}}{\overline{p}\,q(q\,\overline{r})^{s-2}}\left[\overline{q}+\frac{1-\overline{q}\,\overline{r}}{q\,\overline{r}}+\frac{(1-\overline{q}\,\overline{r})^2}{(q\,\overline{r})^2}\right]\pi_0\\
\pi_{S-1} &= \frac{r(1-\overline{r})(1-\overline{q}\,\overline{r})^{s-2}}{\overline{p}\,q(q\,\overline{r})^{s-1}}\left[\overline{q}+\frac{1-\overline{q}\,\overline{r}}{q\,\overline{r}}\right]\pi_0\\
\pi_S &= \frac{r(1-\overline{r})(1-\overline{q}\,\overline{r})^{s-1}}{\overline{p}\,q(q\,\overline{r})^{s}}\,\overline{q}\pi_0
\end{aligned}
$$

and $\boldsymbol{\pi}\boldsymbol{e}=1$ gives $\pi_0 = \dfrac{\overline{p}\,q(q\,\overline{r})^s}{(1-\overline{q}\,\overline{r})^s[Qr-pq]+q(q\,\overline{r})^s}$

It follows that

$$\boldsymbol{\pi}P_{12}\boldsymbol{e} = \left[-pq\tfrac{(1-\overline{q}\,\overline{r})^s}{(q\,\overline{r})^s} - p\,\overline{q} + pr\right]\pi_0 + p$$

$$\boldsymbol{\pi}P_{10}\boldsymbol{e} = [pr-\overline{p}q]\,\pi_0 + \overline{p}q$$

Thus we proved the theorem,

THEOREM 3.1. *The system χ_2 is stable if and only if*

$$\frac{\overline{p}\,q[(q-p)(q\,\overline{r})^s - pq(1-\overline{q}\,\overline{r})^s]}{(1-\overline{q}\,\overline{r})^s[Qr-pq]+q(q\,\overline{r})^s} < \overline{p}\,q - p \tag{6}$$

3.2. Steady-state analysis of model 2. Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ be the steady-state probability vector of the Markov process χ_2 satisfying $\boldsymbol{x}\mathbf{P} = \boldsymbol{x}$ and $\boldsymbol{x}\,\boldsymbol{e} = 1$. Then $\boldsymbol{x}_n$ has the matrix geometric form $\boldsymbol{x}_n = \boldsymbol{x}_1 R^{n-1}, n \geq 2$ where R is the minimal solution of the matrix quadratic equation $P_{12} + RP_{11} + R^2P_{10} = R$. The vectors $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ can be obtained by solving the matrix equations.

$$\boldsymbol{x}_0 P_{00} + \boldsymbol{x}_1 P_{10} = \boldsymbol{x}_0 \tag{7}$$

$$\boldsymbol{x}_0 P_{01} + \boldsymbol{x}_1\left(P_{11} + RP_{10}\right) = \boldsymbol{x}_1 \tag{8}$$

and the normalizing condition

$$\boldsymbol{x}_0\boldsymbol{e} + \boldsymbol{x}_1(I-R)^{-1}\boldsymbol{e} = 1 \tag{9}$$

From the above equations, to determine $\boldsymbol{x}$, we have to compute the rate matrix R (see, Neuts [13]) and we solved numerically. In some special cases we can compute the rate matrix R explicitly.

3.3. System performance measures of model 2. Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ be the steady-state probability vector and $\boldsymbol{x}_n$, $n \geq 0$, be partitioned as $\boldsymbol{x}_n = (x_{n0}, x_{n1}, \ldots, x_{nS})$. We have then the following measures for evaluating performance of the system.

1. Expected number of customers in the system :
$$\mu_{cust} = \sum_{n=0}^{\infty} n\, \boldsymbol{x}_n \boldsymbol{e}$$
2. Expected inventory level :
$$\mu_{inv} = \sum_{n=0}^{\infty} \sum_{m=1}^{S} m\, x_{nm}$$
3. Expected reorder rate :
$$\mu_{reord} = q \sum_{n=1}^{\infty} x_{n,s+1}$$
4. Expected replenishment rate :
$$\mu_{repl} = r \sum_{n=0}^{\infty} \sum_{n=0}^{s} x_{nm}$$
5. Probability that the system is idle : $P_{idle} = \sum_{n=0}^{\infty} x_{n0}$
6. Expected loss rate of fresh arrivals :
$$\mu_{loss} = p \sum_{n=0}^{\infty} x_{n0}$$
7. Expected number of customers waiting in the system when the inventory level is zero :
$$\mu_{CW} = \sum_{n=1}^{\infty} n\, x_{n0}$$
8. Expected rate of departure after completing service :
$$ED = q \sum_{n=1}^{\infty} \sum_{m=1}^{S} x_{nm}$$

4. Cost Analysis

In this section we discuss optimization problem of the systems under study. We define the following costs for our model.
Denote c_0 - fixed ordering cost,
c_1 - procurement cost/ unit,
c_2 - holding cost of inventory /unit/unit time,
c_3 - holding cost of customers/unit/unit time,
c_4 - cost due to the loss of customers /unit/unit time,
Using the above cost, we obtain the long run expected total cost function of these

models. The Expected Total Cost, for Model 1,

$$ETC_1 = \left[c_0 + \sum_{i=0}^{s} r\,(S-i)\,c_1\right] \mu_{reord} + c_2\,\mu_{inv} + c_3\,\mu_{CW} + c_4\,\mu_{loss} \tag{10}$$

for Model 2,

$$ETC_2 = [c_0 + Q\,c_1]\,\mu_{reord} + c_2\,\mu_{inv} + c_3\,\mu_{CW} + c_4\,\mu_{loss} \tag{11}$$

5. Numerical illustration

In this section, we present some numerical results that show the system performance measures with variations in values of underlying parameters. We assume that stability condition hold for both the models.

TABLE 1. Effect of p on various performance measures of two models : fix $s=8, S=25, q=0.8$

p	ρ		μ_{cust}		μ_{inv}	
	Model-1	Model-2	Model-1	Model-2	Model-1	Model-2
$r = 0.1$						
0.20	0.2648	0.2887	0.4315	0.4471	15.816	15.493
0.25	0.3559	0.3867	0.6542	0.6931	15.522	15.355
0.30	0.4614	0.4995	1.0003	1.0932	15.228	15.202
0.35	0.5844	0.6305	1.6193	1.8590	14.923	15.021
0.40	0.7296	0.7844	3.0634	3.9279	14.593	14.799
$r = 0.5$						
0.20	0.2942	0.2934	0.4485	0.4484	16.705	15.594
0.25	0.3937	0.3926	0.6971	0.6967	16.631	15.483
0.30	0.5081	0.5067	1.1046	1.1034	16.548	15.359
0.35	0.6407	0.6391	1.8967	1.8920	16.448	15.212
0.40	0.7961	0.7943	4.1207	4.0925	16.318	15.029
$r = 0.7$						
0.20	0.3020	0.2966	0.4509	0.4493	16.787	15.662
0.25	0.4035	0.3967	0.7038	0.6993	16.729	15.569
0.30	0.5199	0.5116	1.1237	1.1107	16.664	15.465
0.35	0.6545	0.6449	1.9589	1.9157	16.586	15.341
0.40	0.8121	0.8011	4.4437	4.2145	16.485	15.186

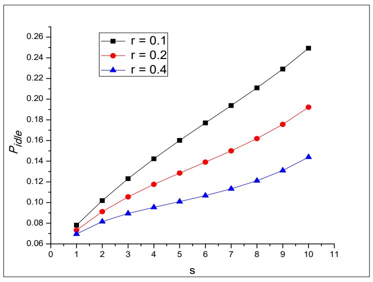
In table 1 and table 2, it is seen that in both the models, the traffic intensity and the expected number of customers increases with increase in arrival rate p for various values of r, consequently inventory level decreases. Also expected number of customers waiting in the system when inventory level is zero increases.

From table 3 and table 4 as the service rate q increases, expected number of customers decreases and consequently inventory level increases and expected reorder rate also increases. As the rate of leadtime for replenishment increases the inventory level increases, as expected. Also reorder rate increases and expected loss rate of customers decreases for various values of r.

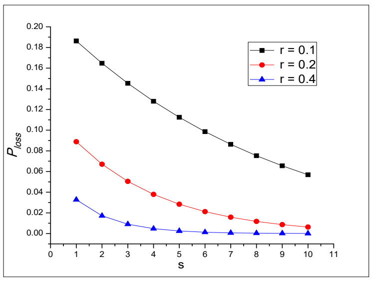
TABLE 2. Effect of p on various performance measures of two models when $s=8, S=25, q=0.8$

p	μ_{reord}		μ_{repl}		μ_{CW}	
	Model-1	Model-2	Model-1	Model-2	Model-1	Model-2
$r = 0.1$						
0.20	0.0422	0.0471	0.0036	0.0087	0.2436	0.2482
0.25	0.0410	0.0471	0.0036	0.0101	0.3194	0.3294
0.30	0.0398	0.0472	0.0035	0.0111	0.4015	0.4208
0.35	0.0387	0.0474	0.0032	0.0118	0.4901	0.5242
0.40	0.0381	0.0481	0.0030	0.0122	0.5856	0.6417
$r = 0.5$						
0.20	0.0464	0.0471	0.0094	0.0095	0.2486	0.2486
0.25	0.0462	0.0471	0.0110	0.0112	0.3302	0.3302
0.30	0.0460	0.0471	0.0123	0.0126	0.4224	0.4223
0.35	0.0459	0.0474	0.0133	0.0137	0.5272	0.5269
0.40	0.0461	0.0479	0.0139	0.0144	0.6472	0.6464
$r = 0.7$						
0.20	0.0470	0.0471	0.0114	0.0102	0.2492	0.2488
0.25	0.0470	0.0471	0.0142	0.0122	0.3315	0.3307
0.30	0.0470	0.0471	0.0168	0.0139	0.4250	0.4233
0.35	0.0471	0.0473	0.0194	0.0154	0.5319	0.5287
0.40	0.0473	0.0478	0.0218	0.0165	0.6552	0.6497

From table 5 and table 6, it is noticed that as replenishment rate r increases, the inventory level in both models increases. Note that the traffic intensity and expected number of customers also increases this is because of customers are leaving the system after completing the service.



(A) s versus P_{idle}



(B) s versus P_{loss}

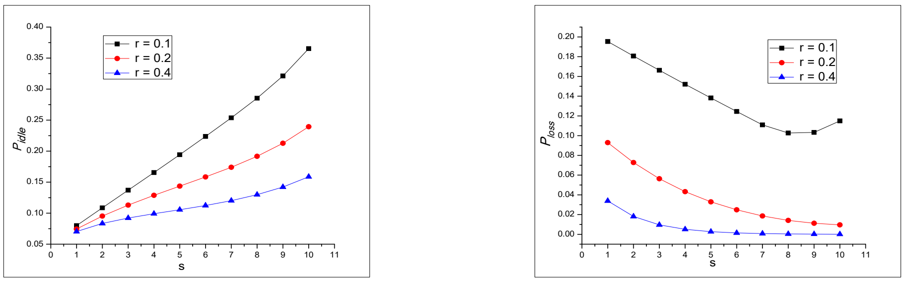
FIGURE 1. Idle probability and loss probability of Model-1 when $S = 15; p = 0.4; q = 0.7$

As seen from figure 1 and figure 2, in both the models, as the reorder level increases, customer loss probability decreases and also system idle probability increases.

In table 7, we compute the optimum inventory level S and expected total cost per unit time for the models by varying parameter S one at a time while keeping others fixed and find the most profitable one by comparing the costs. Again as the maximum inventory level S is increased, the cost function first decreases and then

TABLE 3. Effect of q on various performance measures of two models when $s=8, S=25, p=0.4$

q	ρ		μ_{cust}		μ_{inv}	
	Model-1	Model-2	Model-1	Model-2	Model-1	Model-2
$r = 0.1$						
0.70	0.8535	0.8251	6.4457	5.5031	14.537	11.775
0.75	0.7875	0.7572	4.1527	3.7032	14.568	11.809
0.80	0.7296	0.6976	3.0634	2.7896	14.593	11.841
0.85	0.6785	0.6448	2.4267	2.2371	14.613	11.869
0.90	0.6331	0.5978	2.0088	1.8668	14.628	11.895
$r = 0.5$						
0.70	0.9197	0.9183	11.8948	11.7086	16.283	14.993
0.75	0.8539	0.8524	6.1186	6.0627	16.302	15.012
0.80	0.7961	0.7943	4.1207	4.0925	16.318	15.030
0.85	0.7448	0.7427	3.1076	3.0900	16.333	15.045
0.90	0.6990	0.6967	2.4951	2.4828	16.346	15.059
$r = 0.7$						
0.70	0.9338	0.9337	14.3816	14.3637	16.463	15.429
0.75	0.8690	0.8689	6.7893	6.7847	16.474	15.441
0.80	0.8121	0.8120	4.4437	4.4415	16.485	15.451
0.85	0.7617	0.7616	3.3029	3.3015	16.494	15.460
0.90	0.7168	0.7166	2.6282	2.6273	16.502	15.468



(A) s versus P_{idle} (B) s versus P_{loss}

FIGURE 2. Idle probability and loss probability of Model-2 when $S = 15; p = 0.4; q = 0.7$

increases. From table 7, we observed that the expected total cost is minimum for model-2. Hence model-2 is more profitable. That is when the inventory level reaches s, for the first time, place an order for replenishment by a fixed quantity $Q = S - s$.

Conclusion

In this paper, we studied two discrete time queueing inventory models with positive service time and lead time. In model 1, when the inventory level i deplates to s, place an order upto S where the replenishment quantity is $S-i$. In model 2, replenishment of order upto S policy where order placement by a fixed quantity $Q = S-s$. The systems

TABLE 4. Effect of q on various performance measures of two models when $s=8, S=25, p=0.4$

q	μ_{reord}		μ_{repl}		μ_{CW}	
	Model-1	Model-2	Model-1	Model-2	Model-1	Model-2
$r = 0.1$						
0.70	0.0338	0.0421	0.0029	0.0036	0.5831	0.5664
0.75	0.0359	0.0446	0.0029	0.0036	0.5843	0.5680
0.80	0.0381	0.0471	0.0030	0.0037	0.5856	0.5695
0.85	0.0403	0.0497	0.0030	0.0037	0.5868	0.5709
0.90	0.0425	0.0523	0.0030	0.0037	0.5879	0.5723
$r = 0.5$						
0.70	0.0407	0.0422	0.0138	0.0144	0.6471	0.6464
0.75	0.0434	0.0451	0.0139	0.0144	0.6471	0.6464
0.80	0.0461	0.0479	0.0139	0.0144	0.6472	0.6464
0.85	0.0488	0.0507	0.0140	0.0145	0.6472	0.6465
0.90	0.0516	0.0536	0.0140	0.0145	0.6473	0.6465
$r = 0.7$						
0.70	0.0416	0.0418	0.0218	0.0219	0.6553	0.6552
0.75	0.0445	0.0447	0.0218	0.0219	0.6553	0.6552
0.80	0.0473	0.0475	0.0218	0.0219	0.6552	0.6552
0.85	0.0502	0.0504	0.0218	0.0219	0.6552	0.6552
0.90	0.0531	0.0533	0.0218	0.0219	0.6552	0.6552

TABLE 5. Effect of r on various performance measures of model 1 when $s=8, S=25, p=0.3, q=0.8$

r	ρ	μ_{cust}	μ_{inv}	μ_{reord}	μ_{repl}	μ_{CW}
0.2	0.6687	2.1891	16.0543	0.0326	0.0063	0.4144
0.3	0.6811	2.2670	16.3221	0.0337	0.0086	0.4187
0.6	0.6967	2.3677	16.5704	0.0349	0.0136	0.4233
0.7	0.6992	2.3844	16.6040	0.0351	0.0148	0.4240
0.8	0.7010	2.3974	16.6289	0.0352	0.0159	0.4246

TABLE 6. Effect of r on various performance measures of model 2 when $s=8, S=25, p=0.3, q=0.8$

r	ρ	μ_{cust}	μ_{inv}	μ_{reord}	μ_{repl}	μ_{CW}
0.2	0.6624	2.1638	14.3643	0.0357	0.0069	0.4131
0.3	0.6783	2.2550	14.8967	0.0356	0.0091	0.4182
0.6	0.6963	2.3656	15.4420	0.0355	0.0138	0.4232
0.7	0.6989	2.3832	15.5209	0.0354	0.0150	0.4240
0.8	0.7009	2.3968	15.5803	0.0354	0.0160	0.4245
0.9	0.7024	2.4075	15.6265	0.0354	0.0169	0.4250

are exhaustively analyzed. Using Matrix-geometric method we derived steady-state analysis of the model and obtained an explicit expression for the stability condition. Several performance measures are derived. The influence of various parameters on the

TABLE 7. Optimum value of S when $s=4, p=0.4,\ q=0.7, r=0.3$, $c_0 = \$50, c_1 = \$15, c_2 = \$0.2, c_3 = \$0.3, c_4 = \$0.5$

S	12	13	14	**15**	16	17	18	19	20	21
ETC_1	22.62	22.51	22.46	**22.45**	22.47	22.52	22.58	22.65	22.73	22.82
S	12	13	14	15	16	17	**18**	19	20	21
ETC_2	18.79	18.43	18.19	18.04	17.94	17.89	**17.87**	17.88	17.89	17.93

system performance are also investigated through numerical example. Cost analysis for the models are numerically investigated. We observed that the expected total cost is minimum for model-2 in comparison with model-1 for a fixed parameters of the models.

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