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ALMOST SPLITTING SETS S OF AN INTEGRAL DOMAIN D SUCH THAT D_S IS A PID

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Abstract. Let D be an integral domain, S be a multiplicative subset of D such that D_S is a PID, and $D[X]$ be the polynomial ring over D. We show that S is an almost splitting set in D if and only if every nonzero prime ideal of D disjoint from S contains a primary element. We use this result to give a simple proof of the known result that D is a UMT-domain and $Cl(D[X])$ is torsion if and only if each upper to zero in $D[X]$ contains a primary element.

1. Introduction

Let D be an integral domain with quotient field K, $D^* = D \setminus \{0\}$, S be a multiplicative subset of D, X be an indeterminate over D, and $D[X]$ be the polynomial ring over D. For a polynomial $h \in K[X]$, we denote by $c(h)$ the fractional ideal of D generated by the coefficients of h.

As in [12], we say that D is an *almost GCD-domain* (AGCD-domain) if for each $0 \neq a, b \in D$, there is an integer $n \geq 1$ such that $a^nD \cap b^nD$ is principal. Clearly, GCD-domains are AGCD-domains, but not vice versa (for example, if $\mathbb{F}$ is a field of characteristic 2, then $\mathbb{F}[X^2, X^3]$ is an AGCD-domains but not a GCD-domain (cf. [6, Lemma 3.2])). An *upper to zero* in $D[X]$ is a nonzero prime ideal Q of $D[X]$ with $Q \cap D = (0)$,

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while D is called a *UMT-domain* if each upper to zero in $D[X]$ is a maximal t-ideal of $D[X]$. (Definitions related to the t-operation will be reviewed in the sequel.) D is a *Prüfer v-multiplication domain* (PvMD) if each nonzero finitely generated ideal of D is t-invertible. It is known that AGCD-domains are UMT-domains with torsion class group [4, Lemma 3.1], and D is a PvMD if and only if D is an integrally closed UMT-domain [10, Proposition 3.2]; so D is an integrally closed AGCD-domain if and only if D is a PvMD with torsion class group. We say that a multiplicative subset S of D is an *almost splitting set* of D if for each $0 \neq d \in D$, there is an integer $n \geq 1$ such that $d^n = sa$ for some $s \in S$ and $a \in N(S)$, where $N(S) = \{0 \neq x \in D | (x, s')_t = D \text{ for all } s' \in S\}$. It is known that D^* is an almost splitting set of $D[X]$ if and only if D is a UMT-domain and $Cl(D[X])$ is torsion [4, Theorem 2.4]; which also implies that if D is integrally closed, then D^* is an almost splitting set of $D[X]$ if and only if D is an AGCD-domain.

In this paper, we show that if D_S is a principal ideal domain (PID), then S is an almost splitting set of D if and only if each nonzero prime ideal of D disjoint from S contains a primary element. (A nonzero element $a \in D$ is said to be *primary* if aD is a primary ideal.) We use this result to recover [4, Theorem 2.4] that D^* is an almost splitting set of $D[X]$ if and only if D is a UMT-domain and $Cl(D[X])$ is torsion, if and only if each upper to zero in $D[X]$ contains a primary element. We also show that $D[X]$ is an AGCD-domain if and only if $D[X]_{N_v}$ is an AGCD-domain and D^* is an almost splitting set of $D[X]$, where $N_v = \{f \in D[X] \mid c(f)_v = D\}$.

We first review some definitions related to the v- and t-operations. Let $\mathbf{F}(D)$ be the set of nonzero fractional ideals of D. For each $I \in \mathbf{F}(D)$, let $I^{-1} = \{x \in K \mid xI \subseteq D\}$, $I_v = (I^{-1})^{-1}$ and $I_t = \bigcup\{J_v \mid J \subseteq I$ and J is a nonzero finitely generated fractional ideal of $D\}$. Clearly, if I is finitely generated, then $I_v = I_t$. An $I \in \mathbf{F}(D)$ is called a *t-ideal* if $I_t = I$, and an integral ideal is a maximal t-ideal if it is maximal among proper integral t-ideals. Let t-Max(D) be the set of maximal t-ideals of D. It is well known that t-Max$(D) \neq \emptyset$ if D is not a field; a prime ideal minimal over a t-ideal is a t-ideal (hence an upper to zero in $D[X]$ is a t-ideal); each proper integral t-ideal is contained in a maximal t-ideal; and each maximal t-ideal is a prime ideal.

We say that an $I \in \mathbf{F}(D)$ is *t-invertible* if $(II^{-1})_t = D$; equivalently, if $II^{-1} \nsubseteq P$ for all $P \in t\text{-Max}(D)$. Let $T(D)$ be the group of t-invertible fractional t-ideals of D under the t-multiplication $A * B = (AB)_t$, and let $Prin(D)$ be its subgroup of principal fractional ideals. The (t-)class group of D is an abelian group $Cl(D) = T(D)/Prin(D)$. It is well known that D is a GCD-domain if and only if D is a PvMD and $Cl(D) = 0$ [5, Corollary 1.5]. The readers can refer to [9] for any undefined notation or terminology.

2. Results

Let D be an integral domain with quotient field K, $D^* = D \setminus \{0\}$, X be an indeterminate over D, and $D[X]$ be the polynomial ring over D.

We begin this section with a nice characterization of almost splitting sets, which appears in [2, Proposition 2.7].

LEMMA 1. *Let S be a multiplicative subset of D. Then S is an almost splitting set of D if and only if, for each $0 \neq d \in D$, there is a positive integer $n = n(d)$ such that $d^n D_S \cap D$ is principal.*

As in [1], we say that a multiplicative subset S of D is a *t-splitting set* if each $0 \neq d \in D$, we have $dD = (AB)_t$ for some integral ideals A, B of D, where $A_t \cap sD = sA_t$ for all $s \in S$ and $B_t \cap S \neq \emptyset$. An almost splitting set is t-splitting [6, Proposition 2.3], and if $Cl(D)$ is torsion, a t-splitting set is almost splitting [6, Corollary 2.4]. It is known that if D_S is a PID, then S is a t-splitting set of D if and only if each nonzero prime ideal of D disjoint from S is t-invertible [7, Theorem 2.8], which was used to show that D^* is a t-splitting set in $D[X]$ if and only if D is a UMT-domain [7, Corollary 2.9]. Our next result, which is the main result of this paper, is an almost splitting set analog of [7, Theorem 2.8].

Theorem 2. *Let S be a multiplicative subset of D such that D_S is a PID. Then S is an almost splitting set in D if and only if every nonzero prime ideal of D disjoint from S contains a primary element.*

Proof. ($\Rightarrow$) Assume that S is an almost splitting set of D, and let P be a nonzero prime ideal of D disjoint from S. Then $PD_S = pD_S$ for some $p \in P$, because D_S is a PID. By Lemma 1, there is a positive integer n such that $P = PD_S \cap D \supseteq P^n D_S \cap D = p^n D_S \cap D = qD$ for some $q \in D$. Note that q is a primary element, because $p^n D_S$ is primary. Thus, P contains a primary element q.

($\Leftarrow$) Let $0 \neq d \in D$. Then since D_S is a PID, we have $dD_S = p_1^{e_1} \cdots p_k^{e_k} D_S$ for some $p_i \in D$ and positive integers e_i such that p_i's are distinct prime elements in D_S. Let P_i be the prime ideal of D such that $P_i D_S = p_i D_S$. Since $p_i D_S$ is minimal over dD_S, P_i is minimal over dD. Moreover, $P_i \cap S = \emptyset$, and so P_i contains a primary element q_i. Since $P_i D_S = p_i D_S$, there is a positive integer n_i for which $q_i D_S = p_i^{n_i} D_S$. Let $n = n_1 \cdots n_k$ and $m_i = \frac{n}{n_i} e_i$. Then $d^n D_S = (p_1^{ne_1} \cdots p_k^{ne_k}) D_S = (p_1^{ne_1} D_S) \cap \cdots \cap (p_k^{ne_k} D_S) = (q_1^{m_1} D_S) \cap \cdots \cap (q_k^{m_k} D_S)$, whence

$$
\begin{aligned}
& d^n D_S \cap D \\
& = ((q_1^{m_1} D_S) \cap \cdots \cap (q_k^{m_k} D_S)) \cap D \\
& = (q_1^{m_1} D_S \cap D) \cap \cdots \cap (q_k^{m_k} D_S \cap D) = (q_1^{m_1} D) \cap \cdots \cap (q_k^{m_k} D) \\
& = (q_1^{m_1} \cdots q_k^{m_k}) D,
\end{aligned}
$$

where the last equality follows from the fact that each $q_i^{m_i}$ is a primary element, so [3, Corollary 2] applies. Therefore, S is an almost splitting set by Lemma 1. □

Let $N_v = \{f \in D[X] \mid c(f)_v = D\}$ and $N(D^*) = \{f \in D[X] \mid f \neq 0$ and $(f, d)_v = D[X]$ for all $d \in D^*\}$. Obviously, $N_v = N(D^*)$, and thus $Cl(D[X]_{N(D^*)}) = 0$ [11, Theorems 2.4 and 2.14]. The next result is already known, but we use Theorem 2 to give another simple proof.

Corollary 3. ([4, Theorem 2.4]) *The following statements are equivalent.*

(1) *D^* is an almost splitting set in $D[X]$.*

(2) *D is a UMT-domain and $Cl(D[X])$ is torsion.*
(3) *Each upper to zero in $D[X]$ contains a primary element.*

Proof. (1) $\Rightarrow$ (2) Suppose that D^* is an almost splitting set in $D[X]$. Then $Cl(D[X]_{D^*}) = Cl((D[X])_{N(D^*)}) = 0$, and thus $Cl(D[X])$ is torsion [4, Lemma 2.3]. Also, if Q is an upper to zero in $D[X]$, then $Q \cap D^* = \emptyset$, and hence Q contains a primary element f by Theorem 2. For $g \in D[X] \setminus Q$, if $u \in (g, f)^{-1}$, then $uf \cdot g = ug \cdot f \in fD[X]$, and since $g \notin Q$, we have $uf \in fD[X]$. Hence, $u \in D[X]$, which means that $(f, g)^{-1} = (f, g)_v = D[X]$. Thus, Q is a maximal t-ideal.

(2) $\Rightarrow$ (3) Assume that D is a UMT-domain and $Cl(D[X])$ is torsion, and let Q be an upper to zero in $D[X]$. Then Q is a maximal t-ideal of $D[X]$, and hence Q is t-invertible [10, Theorem 1.4]. Also, since $Cl(D[X])$ is torsion, there is an integer $n \geq 1$ such that $(Q^n)_t = fD[X]$ for some $f \in D[X]$. If $g, h \in D[X]$ such that $gh \in fD[X]$ and $g \notin Q$, then $(Q^n, g)_t = D[X]$, because Q is a maximal t-ideal. Hence $Q \supseteq fD[X] \supseteq h(Q^n, g)_t = hD[X] \ni h$. Thus, f is a primary element such that $f \in Q$.

(3) $\Rightarrow$ (1) This is an immediate consequence of Theorem 2, because $D[X]_{D^*}$ is a PID and each nonzero prime ideal of $D[X]$ disjoint from D^* is an upper to zero in $D[X]$. □

It is known that $D[X]$ is an AGCD-domain if and only if D is an AGCD-domain and $\bar{D}[X]$ is a root extension of $D[X]$, where $\bar{D}$ is the integral closure of D [2, Theorem 3.4]. (Let $A \subseteq B$ be an extension of integral domains. Then B is said to be a root extension of A if for each $b \in B$, $b^n \in A$ for some integer $n \geq 1$.) We next give another characterization of $D[X]$ being an AGCD-domain.

Corollary 4. *$D[X]$ is an AGCD-domain if and only if $D[X]_{N_v}$ is an AGCD-domain and D^* is an almost splitting set of $D[X]$.*

Proof. Assume that $D[X]$ is an AGCD-domain. Then $D[X]_{N_v}$ is an AGCD-domain [6, Corollary 2.12], and since an AGCD-domain is a UMT-domain with torsion class group, D^* is an almost splitting set of $D[X]$ by Corollary 3. Conversely, assume that $D[X]_{N_v}$ is an AGCD-domain and D^* is an almost splitting set of $D[X]$. Note that $N(D^*) = N_v$ and $D[X]_{D^*} = K[X]$ is a PID (hence an AGCD-domain). Thus, $D[X]$ is an AGCD-domain [6, Corollary 2.12]. □

COROLLARY 5. *If D is integrally closed, the following statements are equivalent.*

(1) *D^* is an almost splitting set in $D[X]$.*
(2) *D is an AGCD-domain.*
(3) *D is a PvMD and $Cl(D)$ is torsion.*
(4) *$D[X]$ is an AGCD-domain.*
(5) *Each upper to zero in $D[X]$ contains a primary element.*

Proof. (1) ⇔ (2) [6, Proposition 2.6]. (1) ⇔ (3) If D is integrally closed, then $Cl(D[X]) = Cl(D)$ [8, Theorem 3.6], and D is a UMT-domain if and only if D is a PvMD [10, Proposition 3.2]. Thus, the result follows from Corollary 3. (3) ⇒ (4) This follows, because $D[X]$ is a PvMD and $Cl(D[X]) = Cl(D)$. (4) ⇒ (1) Corollary 4. (1) ⇔ (5) Corollary 3. □

We end this paper with an example of non-integrally closed AGCD-domain. Let S be a multiplicative subset of D, and let $R = D+XD_S[X]$. It is known that R is an AGCD-domain if and only if D is an AGCD-domain and $\bar{D}_S[X]$ is a root extension of $D_S[X]$ [2, Theorems 3.4 and 3.12]. Clearly, D^* is an almost splitting set of D. Thus, $D + XK[X]$ is an AGCD-domain if and only if D is an AGCD-domain. For example, let $\mathbb{F}$ be a field of characteristic > 0, Z be an indeterminate over $\mathbb{F}$, and $D = \mathbb{F}[Z^2, Z^3]$. Then $D + XK[X]$ is a non-integrally closed AGCD-domain.

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